Honam Mathematical J. **41** (2019), No. 2, pp. 369–380
https://doi.org/10.5831/HMJ.2019.41.2.369

HYPERELASTIC LIE QUADRATICS

Gözde Özkan Tükel*, Tunahan Turhan, and Ahmet Yücesan

Abstract. Inspired by the problem of finding hyperelastic curves in a Riemannian manifold, we present a study on the variational problem of a hyperelastic curve in Lie group. In a Riemannian manifold, we reorganize the characterization of the hyperelastic curve with appropriate constraints. By using this equilibrium equation, we derive an Euler-Lagrange equation for the hyperelastic energy functional defined in a Lie group G equipped with bi-invariant Riemannian metric. Then, we give a solution of this equation for a null hyperelastic Lie quadratic when Lie group G is $SO(3)$.

1. Introduction

Elastic curves proposed by Bernoulli in 1740 and determined by Euler in 1744 are critical points of the bending energy functional $\int \left(\kappa^2 + \lambda\right) ds$, where κ and λ are respectively the curvature of a curve and the Lagrange multiplier, acting on suitable space of curves [4, 10]. This problem is generalized to finding extremals (called as hyperelastic curves or free hyperelastic curves when $\lambda = 0$) of the functional $\int (\kappa^r + \lambda)\, ds$ for any natural number $r \geq 2$ [1, 2]. Clearly that this is the functional of classical elastic curves when $r = 2$ [4, 10]. In [1], authors characterized the free hyperelastic curves in a Riemannian manifold. Motivated by this study, we consider the problem of finding hyperelastic curves in Lie Groups equipped with a bi-invariant Riemannian metric. On the other hand, a curve defined in the Lie group corresponds to the Lie reduction in its Lie algebra. Here, we characterize the hyperelastic curve in the Lie groups with aid of the corresponding hyperelastic Lie quadratic in the Lie algebra. Then, we describe hyperelastic Lie quadratures in $\mathfrak{so}(3)$ which is the Lie algebras of $SO(3)$ by using the same method in [8]. Finally,

Received October 25, 2018. Revised April 30, 2019. Accepted May 5, 2019.
2010 Mathematics Subject Classification. 49K99, 49Q99, 74B05.
Key words and phrases. Hyperelastic curve, Hyperelastic Lie quadratic, Lie groups.
*Corresponding author

we give a solution of the differential equation for a null hyperelastic Lie quadratic in $\mathfrak{so}(3)$.

2. Preliminaries

In this section, we recall some notions in Riemannian manifold. We reorganize the characterization of hyperelastic curves in a Riemannian manifold [1]. Also, we give basic facts for the structure of Lie groups and a geometrical construction that will be needed in Section 3.

Let M be a $n-$dimensional Riemannian manifold with Riemannian metric $<,>$, Levi-Civita connection $\triangledown$ and Riemannian curvature tensor R. We consider the family of C^{∞} curves as follows

$$\begin{aligned}\Omega_{v_0,v_1} = \{ \gamma | \, \gamma : [0,\ell] \subset \mathbb{R} \to M \\ \gamma\left(i\ell\right) = p_i, \quad p_i \in M, \quad \dot{\gamma}\left(i\ell\right) = v_i, \quad v_i \in T_{p_i}M \\ \left\|\dot{\gamma}\left(t\right)\right\|^2 = 1, \quad i = 0,1\}.\end{aligned}$$

For a curve $\gamma \in \Omega_{v_0,v_1}$, $\left\|\nabla_{\frac{d}{dt}}\dot{\gamma}\left(t\right)\right\|$ is the geodesic curvature of γ. Then a hyperelastic curve is critical points of the following functional

$$\begin{array}{rccl} \mathcal{F}: & \Omega_{v_0,v_1} & \to & [0,\infty) \\ & \gamma & \to & F\left(\gamma\right) = \int\limits_0^{\ell} \left(\kappa^r + \lambda\right) dt, \end{array} \tag{1}$$

where λ is the Lagrange multiplier and $r \geq 2$ is a natural number. Critical points of the functional (1) are characterized by the Euler-Lagrange equation

$$\begin{aligned} &\triangledown^2_{\frac{d}{dt}}\left(< \triangledown_{\frac{d}{dt}}\dot{\gamma}(t), \triangledown_{\frac{d}{dt}}\dot{\gamma}(t) >^{\frac{r}{2}-1} \triangledown_{\frac{d}{dt}}\dot{\gamma}(t)\right) \\ &+< \triangledown_{\frac{d}{dt}}\dot{\gamma}(t), \triangledown_{\frac{d}{dt}}\dot{\gamma}(t) >^{\frac{r}{2}-1} R(\triangledown_{\frac{d}{dt}}\dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t) \\ &+\triangledown_{\frac{d}{dt}}(\lambda\dot{\gamma}(t))=0, \end{aligned} \tag{2}$$

where for some constant $\widetilde{b} \in \mathbb{R}$,

$$\lambda = \frac{2r-1}{r} < \triangledown_{\frac{d}{dt}}\dot{\gamma}(t), \triangledown_{\frac{d}{dt}}\dot{\gamma}(t) >^{\frac{r}{2}} + \widetilde{b}$$

(see [1]). In the following proposition, we reconsider Eq. (2) as solutions of an unconstrained differential equation with initial conditions of a particular form.

Proposition 2.1. *Any C^∞ curve $\gamma : [0,\ell] \to M$ is a hyperelastic curve if and only if the Euler Lagrange equation* (2) *for all $t \in [0,\ell]$ and following equalities are satisfied*

$$\begin{aligned} 1 &= \|\dot{\gamma}(t_0)\|, \\ 0 &= < \nabla_{\frac{d}{dt}}\dot{\gamma}\Big|_{t=t_0}, \dot{\gamma}(t_0) >, \\ 0 &= < \nabla^2_{\frac{d}{dt}}\dot{\gamma}\Big|_{t=t_0}, \dot{\gamma}(t_0) > + \left\| \nabla_{\frac{d}{dt}}\dot{\gamma}\Big|_{t=t_0} \right\|^2, \end{aligned}$$

for some $t_0 \in [0,\ell]$.

A Lie group G is a C^∞ manifold that is also a group with smooth group operations, that is,

$$\begin{array}{rlcl} \mu : & G \times G & \to & G \\ & (x,y) & \to & \mu(x,y) = xy \end{array} \qquad \text{and} \qquad \begin{array}{rlcl} \imath : & G & \to & G \\ & x & \to & \imath(x) = x^{-1} \end{array}$$

are both smooth. The left and right multiplications by $x \in G$ are the diffeomorphisms L_x, $R_x : G \to G$ defined by $L_x(y) := xy$ and $R_x(y) := yx$, respectively. If a Riemannian metric $<,>$ satisfies for all $x, y \in G$ and $u, v \in T_yG$

$$< u, v >_y = < d\left(L_x\right)_y(u), d\left(L_x\right)_y(v) >_{L_x(y)}, \tag{3}$$

then the metric $<,>$ is called left-invariant. Similarly, if it satisfies for the same conditions

$$< u, v >_y = < d\left(R_x\right)_y(u), d\left(R_x\right)_y(v) >_{R_x(y)},$$

then it is called right-invariant. A Riemannian metric $<,>$ is known bi-invariant if it is invariant both left and right invariance [3, 7, 9]. Throughout this paper, we suppose that the manifold M is a Lie group G equipped with bi-invariant Riemannian metric $<,>$.

For a differentiable vector field X in a Lie group G, if $dL_xX = X$ for all $x \in G$, then X is a left-invariant vector field. Left invariant vector fields in G admits defining a Lie algebra $\mathfrak{g} = T_eG$ in identity element of G. Bi-invariance of a left-invariant metric $<,>$ for all $X, Y, Z \in \mathfrak{g}$ is equivalent

$$< [X,Y], Z > = < [Z,X], Y >, \tag{4}$$

where $[,]$ is the Lie bracket [3, 8]. In addition, $\|.\|$ will denote the norm corresponding to the restriction of $<,>$ to $\mathfrak{g}$.

Now, we suppose that $\gamma : I \subset \mathbb{R} \to G$ be a differentiable curve on G. Then we define $V : I \subset \mathbb{R} \to \mathfrak{g}$ by

$$V(t) = \left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \dot{\gamma}(t). \tag{5}$$

The curve V is known the Lie reduction corresponding to γ. (5) is also equivalent to the first order differential equation

$$\dot{\gamma}(t) = \left(dL_{\gamma(t)}\right)_e V(t)$$

[8].

Lemma 2.2. *Let $\gamma : I \to G$ be a differentiable curve. Suppose that the Lie reduction of γ is given by $V : I \to \mathfrak{g}$. Then we have for all $t \in I$ in the following equations:*

i) $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) = \dot{V}(t)$,

ii) $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla^2_{\frac{d}{dt}} \left(< \nabla_{\frac{d}{dt}} \dot{\gamma}(t), \nabla_{\frac{d}{dt}} \dot{\gamma}(t) >^{\frac{r}{2}-1} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right)$

$= \frac{d^2}{dt^2} \left(< \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}-1} \right) \dot{V}(t)$

$+2\frac{d}{dt} \left(< \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}-1} \right) \left(\ddot{V}(t) + \frac{1}{2}[V(t), \dot{V}(t)] \right)$

$+ < \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}-1} \left(\frac{d^3 V}{dt^3} + [V(t), \ddot{V}(t)] + \frac{1}{4}[V(t), [V(t), \dot{V}(t)]] \right)$,

iii) $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} < \nabla_{\frac{d}{dt}} \dot{\gamma}(t), \nabla_{\frac{d}{dt}} \dot{\gamma}(t) >^{\frac{r}{2}-1} R(\nabla_{\frac{d}{dt}} \dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t)$

$= -\frac{1}{4} < \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}-1} [V(t), [V(t), \dot{V}(t)]]$,

iv) $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \left[\left(< \nabla_{\frac{d}{dt}} \dot{\gamma}(t), \nabla_{\frac{d}{dt}} \dot{\gamma}(t) >^{\frac{r}{2}} \right) \dot{\gamma}(t) \right]$

$= \frac{d}{dt} \left(< \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}} \right) V(t) + \left(< \dot{V}(t), \dot{V}(t) >^{\frac{r}{2}} \right) \dot{V}(t)$.

Proof. Let $\{E_1, E_2, ..., E_n\}$ be an orthonormal basis of the Lie algebra $\mathfrak{g}$. We define for $1 \leq i, j \leq n$

$$\begin{array}{rccl} \left(dL_{\gamma(t)}\right)_e : & T_{\gamma(t)}G & \to & T_{\gamma(t)}G \\ & E_i & \to & \left(dL_{\gamma(t)}\right)_e E_i = \bar{E}_i(\gamma(t)). \end{array}$$

From left-invariance of the metric $<,>$, we have

$$< \bar{E}_i(\gamma), \bar{E}_j(\gamma) > = < \left(dL_{\gamma(t)}\right)_e E_i, \left(dL_{\gamma(t)}\right)_e E_j > = \langle E_i, E_j \rangle = \delta_{ij}.$$

Thus $\left\{\bar{E}_1(\gamma), \bar{E}_2(\gamma), ..., \bar{E}_n(\gamma)\right\}$ is an orthonormal basis for $T_{\gamma(t)}G$. By using the fact

$$\nabla_{\bar{E}_i}\bar{E}_j = \frac{1}{2}[\bar{E}_j, \bar{E}_i] \tag{6}$$

(see [5]), we have

$$\begin{aligned}\left(\nabla_{\bar{E}_i}\bar{E}_j\right)_\gamma &= \frac{1}{2}[\bar{E}_j, \bar{E}_i]_\gamma = \frac{1}{2}[\left(dL_{\gamma(t)}\right)_e E_j, \left(dL_{\gamma(t)}\right)_e E_i] \\ &= \frac{1}{2}\left(dL_{\gamma(t)}\right)_e [E_j, E_i].\end{aligned}$$

Then, from the following equation

$$\begin{aligned}\dot{\gamma}(t) &= \left(dL_{\gamma(t)}\right)_e V(t) = \left(dL_{\gamma(t)}\right)_e \sum_i v_i E_i \\ &= \sum_i v_i \left(dL_{\gamma(t)}\right)_e E_i = \sum_i v_i \bar{E}_i(\gamma),\end{aligned}$$

we can write

$$\dot{\gamma}(t) = \sum_i v_i \bar{E}_i(\gamma). \tag{7}$$

i) By using (6) and (7), we obtain

$$\begin{aligned}\nabla_{\frac{d}{dt}}\dot{\gamma}(t) &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) + \sum_i v_i \nabla_{\frac{d}{dt}}\bar{E}_i(\gamma) \\ &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) + \sum_{i,j} v_i v_j \left(\nabla_{\bar{E}_i}\bar{E}_j\right)_\gamma \\ &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) + \frac{1}{2}\sum_{i,j} v_i v_j [\bar{E}_j, \bar{E}_i]_\gamma \\ &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) + \frac{1}{2}[\sum_j v_j \bar{E}_j, \sum_i v_i \bar{E}_i]_\gamma \\ &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) + \frac{1}{2}[\dot{\gamma}(t), \dot{\gamma}(t)] \\ &= \sum_i \frac{dv_i}{dt}\bar{E}_i(\gamma) = \sum_i \frac{dv_i}{dt}\left(dL_{\gamma(t)}\right)_e E_i \\ &= \left(dL_{\gamma(t)}\right)_e \sum_i \frac{dv_i}{dt} E_i = \left(dL_{\gamma(t)}\right)_e \dot{V}(t).\end{aligned}$$

ii) Firstly, we can easily get from (3) and (*i*) of Lemma 2.2

$$\kappa = \left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\| = \left\| \left(dL_{\gamma(t)^{-1}} \right)_{\gamma(t)} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\| = \left\| \dot{V}(t) \right\|. \tag{8}$$

Then, by using (*i*) of Lemma 2.2 and equations $(6-8)$,we have

$$\begin{aligned}
&\nabla_{\frac{d}{dt}} \left(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|^{r-2} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right) \\
&= \frac{d}{dt} \left(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|^{r-2} \right) \left(dL_{\gamma(t)} \right)_e \sum_i \frac{dv_i}{dt} E_i \\
&\quad + \left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|^{r-2} \left[\left(dL_{\gamma(t)} \right)_e \sum_i \frac{d^2 v_i}{dt^2} E_i + \frac{dv_i}{dt} \left(v_j \nabla_{\bar{E}_i} \bar{E}_j \right) \right] \\
&= \left(dL_{\gamma(t)} \right)_e \left[\tfrac{d}{dt} \left(\left\| \dot{V}(t) \right\|^{r-2} \right) \dot{V}(t) \right. \\
&\quad \left. + \left\| \dot{V}(t) \right\|^{r-2} \left(\ddot{V}(t) + \tfrac{1}{2} [V(t), \dot{V}(t)] \right) \right]
\end{aligned}$$

and

$$\begin{aligned}
&\nabla^2_{\frac{d}{dt}} \left(\left\| \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right\|^{r-2} \nabla_{\frac{d}{dt}} \dot{\gamma}(t) \right) \\
&= \tfrac{d^2}{dt^2} \left(\left\| \dot{V}(t) \right\|^{r-2} \right) \dot{V}(t) + 2 \tfrac{d}{dt} \left(\left\| \dot{V}(t) \right\|^{r-2} \right) (\ddot{V}(t) + \frac{1}{2} [V(t), \dot{V}(t)]) \\
&\quad + \left\| \dot{V}(t) \right\|^{r-2} \left(\tfrac{d^3 V}{dt^3} + [V(t), \ddot{V}(t)] + \frac{1}{4} [V(t), [V(t), \dot{V}(t)]] \right)
\end{aligned}$$

The proof of (*iii*) can be easily seen from [5] and the proof of (*iv*) is a result of (*i*) and (*ii*) of Lemma 2.2. □

3. The Euler-Lagrange Equation

In this section, we give the following theorem which is the main result of this paper.

Theorem 3.1. *Any differentiable curve $\gamma : I \to G$ in the Lie group G is called a hyperelastic curve if and only if the curve $V : I \to \mathfrak{g}$, which is the Lie reduction of γ, in the Lie algebra $\mathfrak{g}$ satisfies*

$$\|V(t)\|^2 = 1, \tag{9}$$

$$\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t)\right)=\left\|\dot{V}(t)\right\|^{r-2}[\dot{V}(t),V(t)] \\ -(<V(t),C>+\left\|\dot{V}(t)\right\|^{r})V(t)+C \tag{10}$$

for some constant $C\in\mathfrak{g}$ and all $t\in I$.

Proof. Assume that $\gamma:I\to G$ is a hyperelastic curve in G. Then we have from (3) and (5)

$$1=\left\|\dot{\gamma(t)}\right\|^{2}=\left\|\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)}\dot{\gamma(t)}\right\|^{2}=\|V(t)\|^{2}.$$

If γ is a hyperelastic curve, then γ satisfies the Euler-Lagrange equation (2). Applying $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)}$ to (2) and using Lemma 2.2, we obtain

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\right)\dot{V(t)}\right)=-\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\ddot{V}(t)\right) \\ +\frac{d}{dt}\left(\left(\left\|\dot{V}(t)\right\|^{r-2}\right)[\dot{V}(t),V(t)]\right)-\frac{d}{dt}((\left(\tfrac{(2r-1)}{r}\left\|\dot{V}(t)\right\|^{r}+\widetilde{b}\right)V(t)).$$

Integrating once, we have

$$\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V(t)}\right)=\left\|\dot{V}(t)\right\|^{r-2}[\dot{V}(t),V(t)] \\ -\left(\tfrac{(2r-1)}{r}\left\|\dot{V}(t)\right\|^{r}+\widetilde{b}\right)V(t)+C \tag{11}$$

where $C\in\mathfrak{g}$ is a constant. The first and second derivative of (9) are found as follows

$$<\dot{V}(t),V(t)>=0, \tag{12}$$

$$<\ddot{V}(t),V(t)>+\left\|\dot{V}(t)\right\|^{2}=0. \tag{13}$$

Taking inner product of (11) with $\dot{V}(t)$, applying (12) and using (4), we get

$$\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\right)\left\|\dot{V}(t)\right\|^{2} \\ +\left\|\dot{V}(t)\right\|^{r-2}<\ddot{V}(t),\dot{V}(t)>=<\dot{V}(t),C>. \tag{14}$$

Integrating (14), yields

$$\left\|\dot{V}(t)\right\|^{r}=\tfrac{r}{r-1}<V(t),C>+b \tag{15}$$

for some constant $b \in \mathbb{R}$. If we take inner product of (11) with $V(t)$ and using (13), then we have

$$\tag{16} \tfrac{r-1}{r}\left\|\dot{V}(t)\right\|^{r} =< V(t), C > -\widetilde{b}.$$

Substituting (15) in (16), we obtain $\widetilde{b} = \frac{1-r}{r}b$. If we combine (16) and (11), we get (10).

Conversely, let $V : I \to \mathfrak{g}$ corresponds the Lie reduction of a curve $\gamma : I \to G$. Suppose that $\|V(t)\|^2 = 1$ and Eq. (10) is satisfied. From Lemma 2.2, we have

$$\|V(t)\|^2 = \left\|\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)} \dot{\gamma(t)}\right\|^2 = \left\|\dot{\gamma(t)}\right\|^2 = 1.$$

Then it remains the show that γ satisfies (2). From (16) and the derivative of (10), we obtain

$$\begin{aligned}\nabla^2_{\frac{d}{dt}}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t)\right) &= \nabla_{\frac{d}{dt}}(\left\|\dot{V}(t)\right\|^{r-2}[\dot{V}(t), V(t)]) \\ &\quad - \nabla_{\frac{d}{dt}}\left((\tfrac{2r-1}{r}\left\|\dot{V}(t)\right\|^{r} + \widetilde{b})V(t)\right).\end{aligned}$$

Applying (5) and using Lemma 2.2, we have for all $t \in I$

$$\begin{aligned}&\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)}(\nabla^2_{\frac{d}{dt}}\left(\left\|\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right\|^{r-2}\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right) \\ &\qquad + \left\|\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right\|^{r-2} R(\nabla_{\frac{d}{dt}}\dot{\gamma}(t), \dot{\gamma}(t))\dot{\gamma}(t) \\ &\qquad\qquad + \nabla_{\frac{d}{dt}}(\left(\frac{2r-1}{r}\left\|\nabla_{\frac{d}{dt}}\dot{\gamma}(t)\right\|^{r} + \widetilde{b}\right)\dot{\gamma}(t)) = 0.\end{aligned}$$

Since $\left(dL_{\gamma(t)^{-1}}\right)_{\gamma(t)}$ is an isomorphism, γ satisfies (2). □

Definition 3.2. *Any curve $V : I \to \mathfrak{g}$ satisfying* (9) *and* (10) *for some $C \in \mathfrak{g}$ and all $t \in I$ is called a hyperelastic Lie quadratic with constant C. Also, V defined by* (5) *is called a hyperelastic Lie quadratic associated with γ, if γ is a hyperelastic curve.*

Corollary 3.3. *Let $V : I \to \mathfrak{g}$ be a hyperelastic Lie quadratic. If we define $W : I \to \mathfrak{g}$ by*

$$\tag{17} W(t) = \frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t)\right) + (< V(t), C > + \left\|\dot{V}(t)\right\|^{r})V(t),$$

then we have for all $t \in I$

$$\dot{W}(t) = [W(t), V(t)] \tag{18}$$

and $\|W(t)\|$ is a constant.

Proof. Substituting (17) in (10), we obtain

$$W(t) = \left\|\dot{V}(t)\right\|^{r-2} [\dot{V}(t), V(t)] + C. \tag{19}$$

Differentiating (19), we calculate

$$\dot{W}(t) = [\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2} \dot{V}(t)\right), V(t)]. \tag{20}$$

Combining (17) and (20), we find

$$\dot{W}(t) = [W(t) - (< V(t), C > + \left\|\dot{V}(t)\right\|^{r})V(t), V(t)] = [W(t), V(t)].$$

So, we have

$$\dot{W}\,(t) = [W\,(t)\,, V\,(t)].$$

Finally, we calculate in the following result

$$\begin{aligned}\frac{d}{dt}\|W(t)\|^2 &= \frac{d}{dt} < W(t), W(t) > \\ &= 2 < \dot{W}(t), W(t) >= 2 < [W(t), V(t)], W(t) >= 0.\end{aligned}$$

This implies that $\|W(t)\|$ is a constant. □

Differential equations of the form (18) are called Lax equations. The Lax equation (18) is crucial to solution of (5) or equivalently $\dot{\gamma}(t) = \left(dL_{\gamma(t)}\right)_e V(t)$ for a hyperelastic curve γ in term of its hyperelastic Lie quadratic V. In [8], Popiel and Noakes prove that the differential equation that gives the elastic curve can expand the whole real axis by Picard's theorem and Lax equations. Then by the Theorem 3.1 in [8] and the Proposition 2.1, all hyperelastic curves in G extend uniquely to $\mathbb{R}$ when G is compact.

4. Hyperelastic Curves in SO(3)

In this section, we suppose $G = SO\,(3)$ which is the group of rotations of Euclidean 3−space. Then the Lie algebra of G is $\mathfrak{g} = \mathfrak{so}(3)$ which is the set off all skew symmetric real 3×3 matrices. Recall that $\mathfrak{so}(3)$ is a Lie algebra with the Lie bracket

$$\begin{aligned}[,]: \quad \mathfrak{so}(3) \times \mathfrak{so}(3) &\rightarrow \mathfrak{so}(3) \\ (A, B) &\rightarrow [A, B] = AB - BA\end{aligned}$$

and E^3 is a Lie algebra with the Lie bracket the cross product $\times$. The map
$B : E^3 \to \mathfrak{so}(3)$ defined by

$$B(v)w = v \times w$$

is a Lie algebra isomorphism.
Let $\gamma : \mathbb{R} \to SO(3)$ be a hyperelastic curve and $\tilde{V} : \mathbb{R} \to \mathfrak{so}(3)$ the associated hyperelastic Lie quadratic with the constant $\tilde{C}$. Define the inverse function as follows:

$$V = B^{-1}(\tilde{V}) : \mathbb{R} \to E^3 \tag{21}$$

and $C = B^{-1}(\tilde{C})$ for convenience. Since B is a Lie algebra isomorphism and isometry, V satisfies for all $t \in I$

$$\|V(t)\|^2 = \left\|B^{-1}(\tilde{V}(t))\right\|^2 = \left\|\tilde{V}(t)\right\|^2 = 1 \tag{22}$$

and from (10)

$$\begin{aligned} \tfrac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t)\right) &= \left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t) \times V(t) \\ &\quad - (< V(t), C > + \left\|\dot{V}(t)\right\|^{r})V(t) + C. \end{aligned} \tag{23}$$

This implies V is a hyperelastic Lie quadratic with constant C in the Lie algebra $(E^3, \times)$. We study with V rather than $\tilde{V}$, solving (23) with (22). So, we can say that for any $A \in SO(3)$ and $t_0 \in \mathbb{R}$, $t \to A(V(t))$ is a hyperelastic Lie quadratic in E^3 with constant $A(C)$ and $t \to V(t-t_0)$ is a hyperelastic Lie quadratic in E^3 with constant C by local uniqueness in Picard theorem.

Now, we may suppose without loss of generality that

$$C = [0\ 0\ c]^T \text{ for some } c \in \mathbb{R}, \text{ and } V_1(0) = 0,$$

where we write

$$V(t) = \begin{bmatrix} V_1(t) & V_2(t) & V_3(t) \end{bmatrix}^T \tag{24}$$

in the next part of the paper. If V is a hyperelastic Lie quadratic in E^3 with constant $C = 0$, then we call that V is null hyperelastic Lie quadratic (see [6] and [8]). Then Eq. (23) reduces to

$$\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t)\right) = \left\|\dot{V}(t)\right\|^{r-2}\dot{V}(t) \times V(t) - \left\|\dot{V}(t)\right\|^{r} V(t). \tag{25}$$

Now we suppose that $\left\|\dot{V}(t)\right\| = const.$ Then we have from the first derivative of $\left\|\dot{V}(t)\right\|$

$$< \ddot{V}(t), \dot{V}(t) >= 0.$$

This implies that

$$\frac{d}{dt}\left(\left\|\dot{V}(t)\right\|^{r-2}\right) = (r-2)\left\|\dot{V}\,(t)\right\|^{r-4} < \ddot{V}(t), \dot{V}(t) >= 0. \tag{26}$$

From (26), (25) reduces to

$$\ddot{V}(t) = \dot{V}(t) \times V(t) - \left\|\dot{V}(t)\right\|^{2} V(t).$$

Then we can give the following proposition:

Proposition 4.1. *If V is a null hyperelastic Lie quadratic and satisfies* (24), *then we have for all $t \in \mathbb{R}$,*

$$V(t) = \begin{bmatrix} a\sin(wt) & a\cos(wt) & \sqrt{1-a^2} \end{bmatrix}^T$$

where $a = \sqrt{b/b+1}$ and $w = 1/\sqrt{1-a^2}$.

Conclusion

We present a variational study of the hyperelastic curve which is a critical point of geometric energy functional related to the curvature of a curve, subject to suitable boundary conditions in Lie group with bi-invariant metric. We derive the Euler-Lagrange equations for a hyperelastic curve with regard to the Lie reduction of the curve in a Lie group G equipped with bi-invariant Riemannian metric. In this way, we have defined a new type of curve, the "hyperelastic Lie quadratic". We give a special solution for a null hyperelastic Lie quadratic for $G = SO(3)$. A general solution of this type curves is an open problem and can be studied future works.

Acknowledgement

The authors want to express her/his thanks to the referees for her/his valuable comments and suggestions.

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Gözde ÖZKAN TÜKEL
Isparta University of Applied Sciences,
Isparta, Turkey.
E-mail: gozdetukel@isparta.edu.tr

Tunahan TURHAN
Isparta University of Applied Sciences,
Isparta, Turkey.
E-mail: tunahanturhan@isparta.edu.tr

Ahmet YÜCESAN
Süleyman Demirel University,
Isparta, Turkey.
E-mail: ahmetyucesan@sdu.edu.tr