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MODULE AMENABILITY OF BANACH ALGEBRAS AND SEMIGROUP ALGEBRAS

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Abstract. We define the concepts of the first and the second module dual of a Banach space X. And also bring a new concept of module amenability for a Banach algebra $\mathcal{A}$. For inverse semigroup S, we will give a new action for $\ell^1(S)$ as a Banach $\ell^1(E_S)$-module and show that if S is amenable then $\ell^1(S)$ is $\ell^1(E_S)$-module amenable.

1. Introduction

The most important results in the theory of amenable groups is Johnson's theorem [3]. The auther states that a locally compact topological group G is amenable if and only if the Banach algebra $L^1(G)$ is amenable. But this result is not true for inverse semigroups. Inverse semigroup S is amenable if and only if the discrete group G_S is amenable, where G_S is the maximal group homomorphic image of S that is defined as $G_S := S/\backsim$ for each congeruence relation $\backsim$ on S in [8]. For more details, about amenability for C^*-algebras and Banach algebras, one can refer to be refrences [4], [5], [6] and [7].

The concept of module amenability for a class of Banach algebras that are modules over another Banach algebra has been introduced by Amini in [1]. He considered J as the closed ideal of $\mathcal{A}$ generated by $\{\alpha.(ab) - (ab).\alpha\}$ for $a, b \in \mathcal{A}$ and $\alpha \in \mathfrak{A}$. For an inverse semigroup S along with the set of idempotents E_S, $\ell^1(S)$ as $\ell^1(E_S)$-module with the right multiplication and left multiplication is trivial actions, that is $\delta_e.\delta_s = \delta_s$, $\delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e$, $(s \in S, e \in E_s)$, is module amenable if and only if S is amenable [1].

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In section two, we introduce a closed submodule ${J_X}^{\perp}$ of X^* for Banach $\mathcal{A}$-$\mathfrak{A}$-module X, afther that, the new concepts of module amenability, module virtual diagonal and module approximate diagonal for a Banach algebra $\mathcal{A}$, are given. Finally we show that the Banach algebra $\mathcal{A}$ is module amenable if and only if $\mathcal{A}$ has a module virtual diagonal.

We will give a new definition of Banach $\ell^1(E_S)$-module for $\ell^1(S)$ with no trivial left action. In fact, we will consider the semigroup algebra $\ell^1(S)$ as $\ell^1(E_S)$-module with the following as the right module action and the left multiplication

$$\delta_e.\delta_s = \delta_{s^*s} = \delta_{s^*} * \delta_s,\ \delta_s.\delta_e = \delta_{se} = \delta_s * \delta_e, \qquad (s \in S, e \in E_S).$$

With respect to the above definition, we will show that if inverse semigroup S is amenable then, semigroup algebra $\ell^1(S)$ is module amenable.

2. Main results

Let $\mathcal{A}$ and $\mathfrak{A}$ be Banach algebras and let $\mathcal{A}$ be a Banach $\mathfrak{A}$-module such that

$$(\alpha.a)b = \alpha.(ab)\ ,\ (ab).\alpha = a(b.\alpha) \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

If Y is Banach $\mathcal{A}$-module and Banach $\mathfrak{A}$-module with compatible actions, such that

$$\alpha.(a.y) = (\alpha.a).y, \quad (a.y).\alpha = a.(y.\alpha) \quad (a \in \mathcal{A}, y \in Y, \alpha \in \mathfrak{A}),$$

and with similar operations for right actions. Then Y is called an $\mathcal{A}$-$\mathfrak{A}$-module.
If moreover,

$$\alpha.y = y.\alpha \qquad (\alpha \in \mathfrak{A}, y \in Y),$$

then Y is called a *commutative* $\mathcal{A}$-$\mathfrak{A}$-module.

If Y is a (commutative) Banach $\mathcal{A}$-$\mathfrak{A}$-module so is Y^*, with the following actions:

$$\langle \alpha.f, y\rangle = \langle f, y.\alpha\rangle, \qquad \langle f.\alpha, y\rangle = \langle f, \alpha.y\rangle$$
$$\langle a.f, y\rangle = \langle f, y.a\rangle, \qquad \langle f.a, y\rangle = \langle f, a.y\rangle \quad (a \in \mathcal{A}, y \in Y, \alpha \in \mathfrak{A}, f \in Y^*).$$

Let Z and Y be $\mathcal{A}$-$\mathfrak{A}$-modules, and $\phi : Z \to Y$ satisfies the following conditions:

$$\phi(\alpha.z) = \alpha.\phi(z), \qquad \phi(z.\alpha) = \phi(z).\alpha$$
$$\phi(a.z) = a.\phi(z), \qquad \phi(z.a) = \phi(z).a \qquad (a \in \mathcal{A}, z \in Z, \alpha \in \mathfrak{A}).$$

Then ϕ is called an *module bihomomorphism.*

Let Y be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module, then the projective tensor product $\mathcal{A}\hat{\otimes}Y$ is a $\mathcal{A}$-$\mathfrak{A}$-module with the following actions:

$$a.(b\otimes y)=(ab)\otimes y,\quad (b\otimes y).a=b\otimes(y.a)$$
$$\alpha.(b\otimes y)=(\alpha.b)\otimes y,\quad (b\otimes y).\alpha=b\otimes(y.\alpha)\ (a,b\in\mathcal{A},y\in Y,\alpha\in\mathfrak{A}).$$

Now, define $\pi_X:\mathcal{A}\hat{\otimes}X\to X$ by

$$\pi_X(a\otimes x)=a.x\qquad (a\in\mathcal{A},x\in X).$$

It is clear that π_X is a $\mathcal{A}$-$\mathfrak{A}$-module bihomomorphism.

Let I_X be the closed $\mathcal{A}$-$\mathfrak{A}$-submodule of the projective tensor product $A\hat{\otimes}X$ generated by

$$\{(a.\alpha)\otimes x-a\otimes(\alpha.x):a\in\mathcal{A},\alpha\in\mathfrak{A},x\in X\}.$$

Let J_X be the closed submodule of X generated by $\pi(I_X)$, that is

$$J_X=\overline{\langle\pi_X(I_X)\rangle}.$$

In particular case, when $X=\mathcal{A}$, $J_{\mathcal{A}}$ is the closed ideal in $\mathcal{A}^*$ generated by $\{(a.\alpha)b-a(\alpha.b)\}$ for $a,b\in\mathcal{A},\alpha\in\mathfrak{A}$.

Definition 2.1. *The closed $\mathcal{A}$-$\mathfrak{A}$-module ${J_X}^{\perp}$ of X^* and $J^{\perp}_{{J_X}^{\perp}}$ of X^{**} are called respectively the* first and the second module dual *of X.*

In the case that $\mathcal{A}$ be a commutative $\mathfrak{A}$-module, then ${J_X}^{\perp}=X^*$ and $J^{\perp}_{{J_X}^{\perp}}=X^{**}$.

Remark 2.2. *Since $(\mathcal{A}/J_{\mathcal{A}})^*\simeq {J_{\mathcal{A}}}^{\perp}$, we have*

$$(1)\qquad \langle\tilde{f},a+J_{\mathcal{A}}\rangle=\langle f,a\rangle\qquad(a\in\mathcal{A}),$$

when $f\in {J_{\mathcal{A}}}^{\perp}$ is the corresponding element $\tilde{f}\in(\mathcal{A}/J_{\mathcal{A}})^*$. Since $(\mathcal{A}/J_{\mathcal{A}})^{**}\simeq\mathcal{A}^{**}/{J_{\mathcal{A}}}^{\perp\perp}$, we have

$$(2)\qquad \langle\tilde{F},\tilde{f}\rangle=\langle F,f\rangle\qquad(\tilde{f}\simeq f\in {J_{\mathcal{A}}}^{\perp}),$$

where $F+{J_{\mathcal{A}}}^{\perp\perp}\in\mathcal{A}^{**}/{J_{\mathcal{A}}}^{\perp\perp}$ is the corresponding element to $\tilde{F}\in(\mathcal{A}/J_{\mathcal{A}})^{**}$.
Note that $(\mathcal{A}/J_{\mathcal{A}})^*$ is an $\mathcal{A}$-module, where the actions $\mathcal{A}$ on $(\mathcal{A}/J_{\mathcal{A}})^*$ are defined by:

$$(3)\quad \langle\tilde{f}.a,b+J_{\mathcal{A}}\rangle=\langle\tilde{f},ab+J_{\mathcal{A}}\rangle,\quad \langle a.\tilde{f},b+J_{\mathcal{A}}\rangle=\langle\tilde{f},ba+J_{\mathcal{A}}\rangle,$$

for all $a,b\in\mathcal{A}$ and $\tilde{f}\in(\mathcal{A}/J_{\mathcal{A}})^*$.

Therefore the second module dual of X is a closed submodule of $X^{**}/{J_X}^{\perp\perp}$.

Definition 2.3. *Let $\mathcal{A}$ and $\mathfrak{A}$ be two Banach algebras and X be a Banach $\mathcal{A}$-$\mathfrak{A}$-module. A bounded linear map $D : \mathcal{A} \to X$ is a* module derivation *if D satisfies the following relations:*

$$D(ab) = D(a).b + a.D(b)$$
$$D(\alpha.a) = \alpha.D(a), \qquad D(a.\alpha) = D(a).\alpha \qquad (a, b \in \mathcal{A}, \alpha \in \mathfrak{A}).$$

Lemma 2.4. *Let X^* be a commutative Banach $\mathcal{A}$-$\mathfrak{A}$-module and $D : \mathcal{A} \to X^*$ be a module derivation, then $D(\mathcal{A}) \subseteq {J_X}^{\perp}$.*

Proof. For each $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$ *and* $x \in X$, we have $(a.\alpha).x - a.(\alpha.x) \in J_X$. Hence

$$\langle D(b), (\alpha.a).x - a.(\alpha.x)\rangle = \langle D(b).(\alpha.a) - (D(b).a).\alpha, x\rangle = 0.$$

□

Definition 2.5. *A Banach algebra $\mathcal{A}$ is called* module amenable *(as an $\mathfrak{A}$- module) if for every Banach $\mathcal{A}$-$\mathfrak{A}$-module X^* with commutative ${J_X}^{\perp}$ (as an $\mathfrak{A}$- module) and $a.(\alpha.y) = (a.\alpha).y \quad (a \in \mathcal{A}, \alpha \in \mathfrak{A}, y \in {J_X}^{\perp})$, for each module derivation $D : \mathcal{A} \to {J_X}^{\perp}$ there exist $y \in {J_X}^{\perp}$ such that $D(a) = a.y - y.a \quad (a \in \mathcal{A})$.*

Proposition 2.6. *Let $\mathcal{A}$ be a amenable Banach algebra, then $\mathcal{A}$ is module amenable.*

Proof. Since ${J_X}^{\perp} = (X/J_X)^*$, then the proof is trivial. □

Proposition 2.7. *Let $\mathcal{A}$ be a module amenable Banach algebra and let $\mathcal{A}/J_{\mathcal{A}}$ be a commutative Banach $\mathfrak{A}$-module, then $\mathcal{A}$ has an approximate identity.*

Proof. Put $X = {J_{\mathcal{A}}}^{\perp}$, then X is a Banach $\mathcal{A}$-module with modules actions $a.f = 0$ and $f.a$, with is a canonical action for each $a \in \mathcal{A}, f \in X$. Since $\mathcal{A}/J_{\mathcal{A}}$ is commutative $\mathfrak{A}$-module, for each $a, b \in \mathcal{A}, \alpha \in \mathfrak{A}$ and $f \in {J_{\mathcal{A}}}^{\perp}$, we have

$$\begin{aligned}\langle (a.\alpha).f - a.(\alpha.f), b\rangle &= -\langle a.(\alpha.f), b\rangle \\ &= -\langle a.(f.\alpha), b\rangle \\ &= -\langle f, \alpha.(b.a)\rangle = -\langle f, \alpha.(b.a) - b.(\alpha.a)\rangle = 0.\end{aligned}$$

So $J_{{J_{\mathcal{A}}}^{\perp}} = 0$ and $\mathcal{A}^{**}/{J_{\mathcal{A}}}^{\perp\perp} \simeq J^{\perp}_{{J_{\mathcal{A}}}^{\perp}}$. Define

$$\begin{aligned}\varphi : \mathcal{A} &\to \mathcal{A}^{**}/{J_{\mathcal{A}}}^{\perp\perp} \simeq J^{\perp}_{{J_{\mathcal{A}}}^{\perp}} \\ \varphi(a) &= \hat{a} + {J_{\mathcal{A}}}^{\perp\perp} \simeq \hat{a}.\end{aligned}$$

It is easy to check that φ is module derivation. Therefore there exists some $F \in J^{\perp}_{J_{\mathcal{A}}^{\perp}}$ such that $\varphi(a) = a.F - F.a = a.F$ for all $a \in \mathcal{A}$. Take a norm bounded net $\{a_\alpha\}$ in $\mathcal{A}$ such that $w^*\text{-}\lim \varphi(a_\alpha) = F$, then we have $w\text{-}\lim aa_\alpha = a$. By classical method, the Banach algebra $\mathcal{A}$ has a bounded right approximate identity. Similarly, $\mathcal{A}$ has a bounded left approximate identity. □

Let $\pi_{\mathcal{A}/J_{\mathcal{A}}} = \pi : \mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}} \to \mathcal{A}/J_{\mathcal{A}}$ defined by $\pi(a \otimes b + J_{\mathcal{A}}) := ab + J_{\mathcal{A}}$. Since π is a module homomorphism, then the second dual $\pi^{**} : (\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^{**} \to \mathcal{A}^{**}/{J_{\mathcal{A}}}^{\perp\perp}$ is module homomorphism.

Definition 2.8. *A bounded net* $\{m_\alpha\}_\alpha$ *in* $\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}}$ *is called* module approximate diagonal *if*
(i) $\{\pi(m_\alpha)\}$ *is a bounded approximate identity for the Banach algebra* $\mathcal{A}/J_{\mathcal{A}}$.
(ii) $m_\alpha.a - a.m_\alpha \to 0 \qquad (a \in \mathcal{A})$.
Also $M \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^{**}$ *is called* module virtual diagonal *if*
(i) $M.a = a.M$
(ii) $(\pi^{**}M).a = \hat{a} + {J_{\mathcal{A}}}^{\perp\perp} \qquad (a \in \mathcal{A})$.

Proposition 2.9. *For a Banach algebra* $\mathcal{A}$ *the following are equivalent:*
(i) $\mathcal{A}$ *has a module approximate diagonal.*
(ii) $\mathcal{A}$ *has a module virtual diagonal.*

Proof. $(i) \Rightarrow (ii)$ Let $\{m_\alpha\}$ be module approximate diagonal for $\mathcal{A}$, so $\{\pi(m_\alpha)\}$ is bounded approximate identity for Banach algebra $\mathcal{A}/J_{\mathcal{A}}$ and $m_\alpha.a - a.\alpha \to 0$ for each $a \in \mathcal{A}$. Since $\{m_\alpha\}$ is a bounded net in $\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}}$ and $\|m_\alpha\| = \|\hat{m}_\alpha\|$, then $\{\hat{m}_\alpha\}$ is a bounded net in $(\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^{**}$.

Let $M \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^{**}$ be a ω^*-accumulation point of $\{m_\alpha\}$, Therefore

$$\lim_\alpha \langle m_\alpha, f\rangle = \langle M, f\rangle \qquad (f \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^*).$$

Then for each $a \in \mathcal{A}$,

$$\lim_\alpha \langle \hat{m}_\alpha.a, f\rangle = \lim_\alpha \langle \hat{m}_\alpha, a.f\rangle = \langle M, a.f\rangle = \langle M.a, f\rangle.$$

Hence $w^*\text{-}\lim_\alpha m_\alpha.a = M.a$ and similarly $w^*\text{-}\lim_\alpha a.m_\alpha = a.M$, therefore $w^*\text{-}\lim_\alpha(\hat{m}_\alpha.a - a.\hat{m}_\alpha) = M.a - a.M$. Since $\{m_\alpha\}$ is module approximate diagonal, then $\lim_\alpha(m_\alpha.a - a.m_\alpha) = 0$, therefore $M.a = a.M$.

In order to show that $(\pi^{**}M).a = \hat{a} + {J_{\mathcal{A}}}^{\perp\perp}$, for all $a \in \mathcal{A}$ we have

$$w^*\text{-}\lim_\alpha \pi^{**}(\hat{m}_\alpha) = \pi^{**}(M) \Rightarrow w^*\text{-}\lim_\alpha \widehat{\pi(m_\alpha)} = \pi^{**}(M)$$
$$\Rightarrow w^*\text{-}\lim_\alpha \pi(m_\alpha).(a + J_{\mathcal{A}}) = \pi^{**}(M).a.$$

Also $\{\pi(m_\alpha)\}$ is approximate identity for $\mathcal{A}/J_\mathcal{A}$, hence

$$\lim_\alpha \pi(m_\alpha).(a+J_\mathcal{A}) = a + J_\mathcal{A} \Rightarrow \lim_\alpha \pi(m_\alpha).(a+J_\mathcal{A}) = \hat{a} + {J_\mathcal{A}}^{\perp\perp}.$$

Therefore $w^*\text{-}\lim_\alpha \pi(m_\alpha).(a+J_\mathcal{A}) = \hat{a} + {J_\mathcal{A}}^{\perp\perp}$, and so M is module virtual diagonal.
$(ii) \to (i)$ Let $M \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_\mathcal{A})^{**}$ be a module virtual diagonal for $\mathcal{A}$. We use Goldstein's theorem to obtain a bounded net $\{m_\alpha\}$ in $\mathcal{A}\hat{\otimes}\mathcal{A}/J_\mathcal{A}$ such that $w^*\text{-}\lim_\alpha \hat{m}_\alpha = M$. For each $a \in \mathcal{A}$ and $f \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_\mathcal{A})^*$,

$$\begin{aligned}\lim_\alpha \langle \hat{m_\alpha}.a - a.\hat{m_\alpha}, f\rangle = 0 &\Rightarrow \lim_\alpha \langle m_\alpha.a \hat{-} a.m_\alpha, f\rangle = 0\\ &\Rightarrow \lim_\alpha \langle f, m_\alpha.a - a.m_\alpha\rangle = 0\\ &\Rightarrow w - \lim_\alpha (m_\alpha.a - a.m_\alpha) = 0\\ &\Rightarrow \lim_\alpha (m_\alpha.a - a.m_\alpha) = 0.\end{aligned}$$

Since $w^*\text{-}\lim_\alpha \pi(\hat{m}_\alpha).a = \pi^{**}(M).a$, we have $w^*\text{-}\lim_\alpha \pi(m_\alpha).a = \hat{a} + {J_\mathcal{A}}^{\perp\perp}$. For each $\tilde{f} \in (\mathcal{A}/J_\mathcal{A})^*$,

$$\lim_\alpha \langle \pi(\hat{m_\alpha}).a, \tilde{f}\rangle = \langle \hat{a} + {J_\mathcal{A}}^{\perp\perp}, \tilde{f}\rangle.$$

Therefore

$$\begin{aligned}\lim_\alpha \langle \pi(\hat{m_\alpha}), a.\tilde{f}\rangle &= \lim_\alpha \langle a.\tilde{f}, \pi(m_\alpha)\rangle\\ &= \lim_\alpha \langle \tilde{f}, \pi(m_\alpha).a\rangle\\ &= \lim_\alpha \langle \tilde{f}, \pi(m_\alpha)(a+J_\mathcal{A})\rangle\\ &= \langle \hat{a} + {J_\mathcal{A}}^{\perp\perp}, \tilde{f}\rangle = \langle \hat{a}, f\rangle = \langle f, a\rangle = \langle \tilde{f}, a + J_\mathcal{A}\rangle.\end{aligned}$$

So $\lim\langle \tilde{f}, \pi(m_\alpha).(a+J_\mathcal{A})\rangle = \langle \tilde{f}, a + J_\mathcal{A}\rangle$. Hence $w\text{-}\lim \pi(m_\alpha).(a+J_\mathcal{A}) = a + J_\mathcal{A}$.

By a classical method, the Banach algebra $\mathcal{A}$ has a bounded right approximate identity then, $\lim \pi(m_\alpha).(a+J_\mathcal{A}) = a + J_\mathcal{A}$. □

Theorem 2.10. *Let $\mathcal{A}/J_\mathcal{A}$ be a commutative $\mathfrak{A}$-module. If $\mathcal{A}$ is a module amenable Banach algebra then $\mathcal{A}$ has a module virtual diagonal.*

Proof. By Proposition (2.7), $\mathcal{A}$ has a bounded approximate identity $\{m_\alpha\}$. Therefore $\{m_\alpha\hat{\otimes}(m_\alpha + J_\mathcal{A})\}$ is bounded net in $(\mathcal{A}\hat{\otimes}\mathcal{A}/J_\mathcal{A})^{**}$ and there exits $E \in (\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}})^{**}$ such that for each $a \in \mathcal{A}$,

$$\pi^{**}(\delta_E(a)) = \pi^{**}(a.E - E.a) =$$
$$\pi^{**}(w^*\text{-}\lim_\alpha(am_\alpha \otimes (m_\alpha + J_{\mathcal{A}}) - m_\alpha \otimes (m_\alpha a + J_{\mathcal{A}})))$$
$$= w^*\text{-}\lim_\alpha(\pi(am_\alpha \otimes (m_\alpha + J_{\mathcal{A}}) - m_\alpha \otimes (m_\alpha a + J_{\mathcal{A}}))$$
$$= w^*\text{-}\lim_\alpha((am_\alpha{}^2 - m_\alpha{}^2 a) + J_{\mathcal{A}}) = 0.$$

Then $\delta_E(a) \subseteq ker\pi^{**} = (ker\pi)^{**}$ for each $a \in \mathcal{A}$. Let $X = (ker\pi)^*$, by lemma 2.4, $\delta_E(\mathcal{A}) \subseteq J_X{}^\perp$ and hence $\delta_E : \mathcal{A} \to J_X{}^\perp$ is inner. Therefore three exists some $V \in J_X{}^\perp \leq ker\pi^{**}$, such that $\delta_E = \delta_V$. We put $F = E - V$, so

$$a.E - E.a = a.V - V.a \Rightarrow a.(E - V) = (E - V).a \Rightarrow a.F = F.a \quad (a \in \mathcal{A}).$$

Hence for all $\tilde{f} \in (\mathcal{A}/J_{\mathcal{A}})^*$

$$\begin{aligned}\langle \pi^{**}(F).a, \tilde{f}\rangle = \langle \pi^{**}(F), a.\tilde{f}\rangle &= \langle \pi^{**}(E - V), a.\tilde{f}\rangle \\ &= \langle \pi^{**}(E), a.\tilde{f}\rangle \\ &= \lim_\alpha \langle \pi^{**}(m_\alpha \otimes (m_\alpha + J_{\mathcal{A}})), a.\tilde{f}\rangle \\ &= \lim_\alpha \langle \pi(m_\alpha \otimes (m_\alpha + J_{\mathcal{A}})), a.\tilde{f}\rangle \\ &= \lim_\alpha \langle m_\alpha{}^2 + J_{\mathcal{A}}, a.\tilde{f}\rangle \\ &= \lim_\alpha \langle a.\tilde{f}, m_\alpha{}^2 + J_{\mathcal{A}}\rangle \\ &= \lim_\alpha \langle \tilde{f}, m_\alpha{}^2.a + J_{\mathcal{A}}\rangle \\ &= \langle f, a\rangle = \langle \hat{a}, f\rangle = \langle \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}, \tilde{f}\rangle,\end{aligned}$$

therefore $\pi^{**}(F).a = \hat{a} + J_{\mathcal{A}}{}^{\perp\perp}$. □

Theorem 2.11. *If $\mathcal{A}$ has a module approximate diagonal, then $\mathcal{A}$ is a module amenable.*

Proof. Let $\{m_\alpha\}_\alpha$ be a module approximate diagonal for $\mathcal{A}\hat{\otimes}\mathcal{A}/J_{\mathcal{A}}$, then $\{\pi(m_\alpha)\}_\alpha$ is a left bounded approximate identity for $\mathcal{A}/J_{\mathcal{A}}$. We show that for a Banach $\mathcal{A}$-$\mathfrak{A}$-module X with commutative $J_X{}^\perp$,

$$H^1(\mathcal{A}, J_X^\perp) = 0.$$

We may assume that X is pseudo-unital. Let $D : \mathcal{A} \to J_X{}^\perp$ be a module derivation and

$m_\alpha = \sum_{n=1}^\infty a_n^\alpha \otimes b_n^\alpha + J_{\mathcal{A}}$ that $\sum_{n=1}^\infty \|a_n^\alpha\|\|b_n^\alpha\| < \infty$, then $\sum_{n=1}^\infty a_n^\alpha.Db_n^\alpha$ is a bounded net in $J_X^\perp$, which has a w^*-accumulation point $\varphi \in J_X^\perp$

such that, w^*-$\lim \sum_{n=1}^{\infty} a_n^\alpha.Db_n^\alpha = \varphi \Rightarrow a\varphi = w^*$-$\lim(a \sum_{n=1}^{\infty} a_n^\alpha.Db_n^\alpha)$. Therefore

$$\begin{aligned}
\langle x, a.\varphi\rangle &= \lim_\alpha \langle x, \sum_{n=1}^{\infty} a.a_n^\alpha.Db_n^\alpha\rangle \\
&= \lim_\alpha \langle x, \sum_{n=1}^{\infty} a_n^\alpha.D(b_n^\alpha.a)\rangle \\
&= \lim_\alpha \langle x, \sum_{n=1}^{\infty} a_n^\alpha.b_n^\alpha D(a) + \sum_{n=1}^{\infty} a_n^\alpha.D(b_n^\alpha)a\rangle \\
&= \lim_\alpha \langle x, \sum_{n=1}^{\infty} a_n^\alpha b_n^\alpha.D(a)\rangle + \lim_\alpha \langle x, \sum_{n=1}^{\infty} a_n^\alpha D(b_n^\alpha).a\rangle \\
&= \langle x, D(a)\rangle + \langle x, \varphi.a\rangle,
\end{aligned}$$

for $x \in J_X$. Hence

$$\langle x, Da\rangle = \langle x, \varphi.a\rangle - \langle x, a.\varphi\rangle = \langle x, \varphi.a - a.\varphi\rangle \Longrightarrow D = ad_\varphi.$$

□

3. Semigroup algebra

In this section, we show that for the inverse semigroup S with a set of idempotents E_S, if inverse semigroup S is amenable, then $\ell^1(S)$ is $\ell^1(E_S)$-module amenable.

Recall that a discrete semigroup S is called *inverse semigroup* if for each $s \in S$ there is a unique element $s^* \in S$ such that $s^*ss^* = s$ and $ss^*s = s^*$. An element $e \in S$ is called *idempotent* if $e = e^* = e^2$. The set of *idempotent* elements in semigroup S is denote by E_S.

It is easy to see that $\ell^1(S)$ is a Banach algebra and a Banach $\ell^1(E_S)$-module with compatible right and left actions as

$$\delta_e.\delta_s = \delta_{s^*s}, \ \delta_s\delta_e = \delta_{se} \qquad (e \in E_S, s \in S).$$

These actions $\ell^1(S)$ makes a Banach $\ell^1(E_S)$-module. Therefore $J_{\ell^1(S)}$ is the closed submodule of $\ell^1(S)$ generated by

$$\Big\{\delta_{set} - \delta_{st^*t} : s, t \in S, e \in E_S\Big\}.$$

in [1], the trivial left action has been considered, but we did not consider this limitation on left action $\ell^1(E_S)$ on $\ell^1(S)$.

Lemma 3.1. *With the above notions,*

$$\ell^1(S) \otimes \ell^1(S)/J_{\ell^1(S)} \simeq \ell^1(S \times S)/J_{\ell^1(S\times S)}.$$

Proof. Let $e \in E_S$, consider the map $\psi : \ell^1(S) \to \ell^1(S \times S)$ by

$$\psi(\delta_x) = \delta_{(x,e)} \qquad (x \in S).$$

It is clear that ψ is one-to-one linear map and so $J_{\ell^1(S)}$ is embedding in $J_{\ell^1(S\times S)}$. Now consider the canonical embedding $T : \ell^1(S) \times \ell^1(S) \to \ell^1(S \times S)$ by

$$T(f,g)(x,y) = f(x)g(y) \qquad (f,g \in \ell^1(S), x,y \in S).$$

Trivially that T is a bounded bilinear mapping, so $T_1 : \ell^1(s) \times (\ell^1(s)/J_{\ell^1(S)}) \to \ell^1(S \times S)/J_{\ell^1(S\times S)}$ defined by

$$T_1(f, g + J_{\ell^1(S)}) = T(f,g) + J_{\ell^1(S\times S)} \qquad (f,g \in \ell^1(S)).$$

Therefore it can be extended to a bounded linear mapping $T_2 : \ell^1(s) \otimes \frac{\ell^1(s)}{J_{\ell^1(S)}} \to \frac{\ell^1(S\times S)}{J_{\ell^1(S\times S)}}$ defined by

$$T_2(\sum_{i=1}^{n}(f_i \otimes g_i + J_{\ell^1(S)})) = \sum_{i=1}^{n} T_1(f_i, g_i + J_{\ell^1(S)}).$$

Therefore T_2 is an isometry. □

Consider $\omega : \ell^1(S) \times \ell^1(S) \longrightarrow \ell^1(S)$ defined by $\omega(f \times g) = f * g$, for each $f, g \in \ell^1(S)$. Then ω and ω^{**} are $\ell^1(E_S)$-module homomorphism. Also if $\phi : \ell^1(S) \otimes (\ell^1(S)/J_{\ell^1(S)}) \longrightarrow \ell^1(S)/J_{\ell^1(S)}$ be defined by $\phi(f \times g + J_{\ell^1(S)}) := f * g + J_{\ell^1(S)}$, so we have

$$\omega^{**}(M) = \phi^{**}(M + {J_{\ell^1(S)}}^{\perp\perp}) \quad (M \in \ell^1(S) \otimes \ell^1(S)).$$

Proposition 3.2. *The following are equivalent:*
(i) *$\ell^1(S)$ has a module virtual diagonal;*
(ii) *There is $M \in \ell^1(S) \otimes \ell^1(S)^{**}$ such that*

$$\omega^{**}(M).s - s \in {J_{\ell^1(S)}}^{\perp\perp}, \quad M.s - s.M \in {J_{\ell^1(S)}}^{\perp\perp} \quad (s \in S).$$

Proof. $(i) \to (ii)$, we defined

$$N \in (\ell^1(S) \otimes (\ell^1(S)/J_{\ell^1(S)}))^{**} = (\ell^1(S \times S))^{**}/J_{\ell^1(S\times S)^{\perp\perp}}$$

by $N = M + J^{\perp\perp}$. Since $M.s - s.M \in {J_{\ell^1(S\times S)}}^{\perp\perp}$, clearly $N.s = s.N$ and since $\omega^{**}(M) = \phi^{**}(M + {J_{\ell^1(S)}}^{\perp\perp}) = \phi^{**}(N)$, therefore $\phi^{**}(N).s - \hat{s} = \omega^{**}(M).s - s \in {J_{\ell^1(S)}}^{\perp\perp}$.
$(ii) \to (i)$ Let $N \in \big(\ell^1(S) \otimes (\ell^1(S)/J_{\ell^1(S)})\big)^{**} = \big(l^1(S \times S)/J_l^1(S \times S)\big)^{**}$

is a module virtual diagonal, choose $M \in (\ell^1(S) \otimes \ell^1(S))^{**}$ such that $N = M + J^{\perp\perp}$. For each $s \in S$,

$$(M.s - s.M) + J^{\perp\perp} = N.s - s.N = 0 \in \ell^1(S) \otimes \ell^1(S)/J^{\perp\perp}.$$

Therefore $M.s - s.M \in J^{\perp\perp}$, now we have

$$\begin{aligned}\big(\omega^{**}(M).s - s\big) + J^{\perp\perp} &= \phi^{**}(M + J^{\perp\perp}).s - s \\ &= \phi^{**}(N).s - s = 0 \in \ell^1(S)^{**}/J^1_\ell(S)^{\perp\perp}.\end{aligned}$$

□

Remark 3.3. *Consider the congruence $\sim$ on S defined by $s \sim t$ if and only if there exist $e \in E_S$ such that $se = te$. It is clear that if $s \sim t$ and $f \in \ell^\infty(S)$, then $f(\delta_s) = f(\delta_t)$.*

Now we are ready to state the main result in this section.

Theorem 3.4. [1, Theorem 3.1] *Let S be an inverse semigroup. If S is amenable, then $\ell^1(S)$ is $\ell^1(E_S)$-module amenable.*

Proof. If μ is a right invariant mean on S and M is defined on $\ell^\infty(S \times S)$ by

$$M(f) = \int_S f(s^*, s) d\mu(s).$$

Then M is clearly a bounded linear functional and $M(1 \otimes 1) = \mu(1) = 1$. For each $s \in S$ and $f \in \ell^\infty(S \times S)$

$$\begin{aligned}s.M(f) = M(f.s) &= \int_S f(st^*, t) d\mu(t) = \int_S f(s(ts)^*, ts) d\mu(t) \\ &= \int_S f(ss^*t^*, ts) d\mu(t) = \int_S f((tss^*)^*, (tss^*)s) d\mu(t) \\ &= \int_S f((t^*, ts) d\mu(t) = M(s.f) = m.s(f).\end{aligned}$$

For each $s \in S$ and $f \in {J_{\ell^1(S\times S)}}^{\perp} \subseteq \ell^\infty(S \times S)$,

$$\begin{aligned}
\omega^{**}(M).s(f) &= \omega^{**}(M)(f.s) = M(\omega^*(f.s)) \\
&= \int_S \omega^*(f.s)(t^*,t)d\mu(t) = \int_S f.s(t^*t)d\mu(t) \\
&= \int_S f.s(t^*t)d\mu(t) = \int_S f(st^*t)d\mu(t) \\
&= f(s)\int_S d\mu(t) \qquad (\ \textit{f(se)= f(s) by Remark 3.3}\ ) \\
&= f(s).
\end{aligned}$$

Therefore M gives rise to a module virtual diagonal for $\ell^1(S)$ and so $\ell^1(S)$ is module amenable. □

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