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REMARKS ON METALLIC MAPS BETWEEN METALLIC RIEMANNIAN MANIFOLDS AND CONSTANCY OF CERTAIN MAPS

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Abstract. In this paper, we introduce metallic maps between metallic Riemannian manifolds, provide an example and obtain certain conditions for such maps to be totally geodesic. We also give a sufficient condition for a map between metallic Riemannian manifolds to be harmonic map. Then we investigate the constancy of certain maps between metallic Riemannian manifolds and various manifolds by imposing the holomorphic-like condition. Moreover, we check the reverse case and show that some such maps are constant if there is a condition for this.

1. Introduction

In differential geometry, it is desirable to introduce and use suitable types of maps between Riemannian manifolds. Such maps may help to compare geometric properties of manifolds. Indeed, almost complex manifolds, almost contact manifolds, almost product manifolds and almost para-contact metric manifolds and maps between such manifolds have been studied extensively by many authors.

The number $\phi = (1+\sqrt{5})/2 = 1,618...$ which is a solution of the equation $x^2 - x - 1 = 0$, represents the golden ratio. Being inspired by the Golden ratio, the notion of Golden manifold M was defined in [6] by a tensor field Φ on M satisfying $\Phi^2 = \Phi + I$. The authors studied properties of Golden manifolds and they showed that Φ is an automorphism of the tangent bundle TM and its eigenvalues are ϕ and $1-\phi$. Also see: [7], [11], [16], [18], [19], [20], [28].

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In [29], we introduced a new map between Golden Riemannian manifolds by imposing a holomorphic-like condition. We show that such map is a harmonic map, and then we obtain a certain condition for such maps to be totally geodesic.

As a generalization of the golden mean was introduced in 1997 by vera W. de Spinadel in [23]-[27] and called *metallic means family* or *metallic proportions*. More precisely, fix two positive integers p and q. The positive solution of the equation $x^2 - px - q = 0$ is named member of the metallic means family [23]-[27]. These numbers, denoted by:

$$\sigma_{p,q} = \frac{p + \sqrt{p^2 + 4q}}{2},$$

are also called (p, q)-metallic numbers. The members of the metallic means family have the property of carrying the name of a metal, like the golden mean and its relatives: the silver mean, the bronze mean, the copper mean and many others. The notion of metallic manifold M was defined in [5] by a tensor field J on M satisfying $J^2 = pJ + qI$, where p and q positive integers. The authors studied properties of metallic manifolds and they showed that J is an automorphism of the tangent bundle TM and its eigenvalues are $\sigma_{p,q}$ and $p - \sigma_{p,q}$. They also find conditions for this kind of submanifold to be also a metallic Riemannian manifold in terms of invariance. (For metallic structures, see also [1], [12], [17], [21]).

In this paper, we study a new map between metallic Riemannian manifolds by imposing a holomorphic-like condition for the first time as far as we know. We give an example, obtain certain conditions for such maps to be totally geodesic and give a sufficient condition for a map between metallic Riemannian manifolds to be harmonic map. Moreover, we also check the existence of such maps between metallic Riemannian manifolds and another manifold equipped with a differentiable structure (Golden, almost product, almost complex, almost contact, almost para contact).

2. Preliminaries

In this section, we give a brief information for almost complex manifolds, almost contact metric manifolds, almost product manifolds, almost para-contact metric manifolds, Golden Riemannian manifolds and metallic Riemannian manifolds. We note that throughout this paper all manifolds and bundles, along with sections and connections, are assumed to be of class C^∞. A map is always a C^∞ map between manifolds.

Let M' be a $2n$-dimensional real manifold. An almost complex structure J' on M' is a tensor field $J' : TM' \to TM'$ such that

$$J'^2 = -I, \tag{1}$$

where I is the identity transformation. Then (M', J') is called almost complex manifold [30].

A smooth map $\varphi : M_1' \to M_2'$ between almost complex manifolds (M_1', J_1') and (M_2', J_2') is called an almost complex (or holomorphic) map if $d\varphi(J_1'X) = J_2' d\varphi(X)$ for $X \in \Gamma(TM_1')$, where J_1' and J_2' are complex structures of M_1' and M_2', respectively.

An $(2n+1)-$dimensional differentiable manifold M is said to have an almost contact structure (ϕ, ξ, η) if it carries a tensor field ϕ of type $(1,1)$, a vector field ξ and $1-$ form η on M respectively such that

$$\phi^2 = -I + \eta \otimes \xi,\ \phi\xi = 0,\ \eta \circ \phi = 0,\ \eta(\xi) = 1, \tag{2}$$

where I is the identity transformation. The almost contact structure is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi). \tag{3}$$

Then (M, ϕ, ξ, η, g) is called an almost contact metric manifold [4].

Let N be an $n-$dimensional manifold with a tensor of type $(1,1)$ such that

$$\varphi^2 = I, \tag{4}$$

where I is the identity transformation. Then we say that N is an almost product manifold with almost product structure φ. We put

$$Q = \frac{1}{2}(I + \varphi), \quad Q' = \frac{1}{2}(I - \varphi). \tag{5}$$

Then we have

$$Q + Q' = I,\ Q^2 = Q,\ Q'^2 = Q',\ QQ' = Q'Q = 0 \tag{6}$$

and

$$\varphi = Q - Q'. \tag{7}$$

If an almost product manifold N admits a Riemannian metric g such that

$$g(\varphi X, \varphi Y) = g(X, Y)$$

for any vector fields X and Y on N, then N is called an almost product Riemannian manifold [30].

Let N' be a $(2n+1)-$dimensional smooth manifold, ϕ' a (1, 1)-tensor field called the structure endomorphism, ξ' a vector field called the characteristic vector field, η' a 1 -form called the paracontact form, and g' a pseudo Riemannian metric on N' of signature $(n+1, n)$. In this case, we say that (ϕ', ξ', η', g') defines an almost paracontact metric structure on N' if:

$$\phi'^2 = I - \eta' \otimes \xi', \ \eta'(\xi') = 1, \ g'(\phi' X, \phi' Y) = g'(X, Y) - \eta'(X)\eta'(Y). \tag{8}$$

From the definition it follows $\phi'(\xi') = 0, \ \eta' \circ \phi' = 0, \ \eta'(X) = g'(X, \xi')$, $g'(\xi', \xi') = 1$ and the fact that ϕ' is $g'-$skewsymmetric: $g'(\phi' X, Y) = -g'(\phi' Y, X)$. The associated 2-form $\omega(X, Y) = g'(X, \phi' Y)$ is skew-symmetric and is called the fundamental form of the almost metric paracontact manifold $(N', \phi', \xi', \eta', g')$[3, 8, 9, 15, 22, 31].

Let $(\bar{M}, g)$ be a Riemannian manifold with a tensor of type $(1, 1)$ such that

$$P^2 = P + I, \tag{9}$$

where I is the identity transformation. We say that the metric g is P compatible if the equality

$$g(PX, Y) = g(X, PY) \tag{10}$$

for all $X, Y \in \Gamma(TM)$. If we substitute PX into X in (10) the equation (10) may also written as

$$g(PX, Y) = g(P^2 X, Y) = g((P+I)X, Y) = g(PX, Y) + g(X, Y).$$

The Riemannian metric (10) is called $P-$compatible and $(\bar{M}, P, g)$ is named a Golden Riemannian manifold [6].

Let $(\tilde{M}, g)$ be a Riemannian manifold with a tensor of type $(1, 1)$ such that

$$J^2 = pJ + qI, \tag{11}$$

where p, q are positive integers and I is the identity operator on the Lie algebra $\chi(M)$ of the vector fields on $\tilde{M}$. We say that the metric g is $J-compatible$ if:

$$g(JX, Y) = g(X, JY) \tag{12}$$

for every $X, Y \in \chi(M)$, which means that J is a self-adjoint operator with respect to g. This condition is equivalent in our framework with:

$$g(JX, JY) = p.g(X, JY) + q.g(X, Y).$$

The Riemannian metric (12) is called $J-$compatible and $(\tilde{M}, J, g)$ is named a metallic Riemannian manifold [5].

Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $\pi : M \longrightarrow N$ is a smooth map between them. Then the differential of π_* of π can be viewed a section of the bundle $Hom(TM, \pi^{-1}TN) \longrightarrow M$, where $\pi^{-1}TN$ is the pullback bundle which has fibres $(\pi^{-1}TN)_p = T_{\pi(p)}N$, $p \in M$. $Hom(TM, \pi^{-1}TN)$ has a connection ∇ induced from the Levi-Civita connection ∇^M and the pullback connection. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla^{\pi}_X \pi_*(Y) - \pi_*(\nabla^M_X Y) \tag{13}$$

for $X, Y \in \Gamma(TM)$, where ∇^{π} is the pullback connection. It is known that the second fundamental form is symmetric. A smooth map $\pi : (M, g_M) \longrightarrow (N, g_N)$ is said to be harmonic if $trace(\nabla\pi_*) = 0$. On the other hand, the tension field of π is the section $\tau(\pi)$ of $\Gamma(\pi^{-1}TN)$ defined by

$$\tau(\pi) = div\pi_* = \sum_{i=1}^{m} (\nabla\pi_*)(e_i, e_i), \tag{14}$$

where $\{e_1, ..., e_m\}$ is the orthonormal frame on M. Then it follows that π is harmonic if and only if $\tau(\pi) = 0$. For more information, see [2].

3. Metallic maps between metallic Riemannian manifolds

In this section, we give a new notion, namely a metallic map, and give a sufficient condition for a map between metallic Riemannian manifolds to be harmonic. We also investigate conditions for a metallic map to be totally geodesic.

Definition 3.1. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to a metallic Riemannian manifold (M_2, g_2, J_2). Then F is called a metallic map if and only if following condition is satisfied.*

$$F_* J_1 = J_2 F_*. \tag{15}$$

We provide the following elementary example.

Example 3.2. *Consider the following map defined by*

$$\begin{aligned} F : R^4 &\rightarrow R^2 \\ (x_1, x_1, x_3, x_4) &\rightarrow \left(\frac{x_1 + x_2}{2}, \frac{x_3 + x_4}{2}\right). \end{aligned}$$

Then, the kernel of F_* *is*

$$\mathcal{V} = KerF_* = Span\{Z_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, Z_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\}$$

and the horizontal distribution is spanned by

$$\mathcal{H} = (KerF_*)^\perp = Span\{H_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, H_2 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\}.$$

Then considering metallic structures on R^4 *and* R^2 *defined by*

$$J_1(a_1, a_2, a_3, a_4) = (\sigma_{p,q}a_1, \sigma_{p,q}a_2, (p - \sigma_{p,q})a_3, (p - \sigma_{p,q})a_4)$$

and

$$J_2(a_1, a_2) = (\sigma_{p,q}a_1, (p - \sigma_{p,q})a_2),$$

where $\sigma_{p,q}$ *and* $p - \sigma_{p,q}$ *are eigenvalues of metallic structures* [5]. *Also by direct computations* $F_*(J_1H_1) = J_2F_*(H_1) = (\sigma_{p,q}, 0)$ *and* $F_*(J_1H_2) = J_2F_*(H_2) = (0, p - \sigma_{p,q})$. *Thus* F *is a metallic map.*

From now on, when we mention a metallic Riemannian manifold, we will assume that its almost metallic structure is integrable. It means that $\nabla J = 0$.

We now give a necessary and sufficient condition for a map F to be totally geodesic. We recall that a map F is totally geodesic if $\nabla F_* = 0$. A geometric interpretation of a totally geodesic map is that it maps every geodesic in the total manifold into a geodesic in the base manifold in proportion to arc lengths.

Theorem 3.3. *Let* F *be a metallic map from a metallic Riemannian manifold* (M_1, g_1, J_1) *to a metallic Riemannian manifold* (M_2, g_2, J_2). *Then* F *is totally geodesic if and only if*

$$(16) \quad (\nabla F_*)(X, Y) = \frac{1}{q_1}\{(p_2 - p_1)J_2(\nabla F_*)(X, Y) + q_2(\nabla F_*)(X, Y)\}$$

for $X, Y \in \Gamma(TM_1)$.

Proof. For $X, Y \in \Gamma(TM_1)$, (13) and (11) we have

$$(\nabla F_*)(X, Y) = \frac{1}{q_1}\{\overset{F}{\nabla}_X F_*(J_1^2Y - p_1J_1Y) - F_*(\nabla_X J_1^2Y - p_1J_1Y)\}.$$

Then using (15) we get

$$\begin{aligned}(\nabla F_*)(X, Y) &= \frac{1}{q_1}\{\overset{F}{\nabla}_X J_2F_*(J_1Y) - p_1\overset{F}{\nabla}_X F_*(J_1Y) - J_2F_*(\nabla_X J_1Y) \\ &+ p_1F_*(\nabla_X J_1Y)\}.\end{aligned}$$

Since J_1 integrable, using (15) we obtain

$$\begin{aligned}(\nabla F_*)(X,Y) &= \frac{1}{q_1}\{\overset{F}{\nabla}_X J_2^2 F_* Y - p_1 \overset{F}{\nabla}_X J_2 F_* Y - J_2^2 F_*(\nabla_X Y) \\ &+ p_1 J_2 F_*(\nabla_X Y)\}.\end{aligned}$$

Then using (11), we get

$$\begin{aligned}(\nabla F_*)(X,Y) =& \frac{1}{q_1}\{(p_2-p_1)\overset{F}{\nabla}_X J_2 F_* Y + q_2 \overset{F}{\nabla}_X F_* Y - (p_2-p_1) J_2 F_*(\nabla_X Y) \\ & - q_2 F_*(\nabla_X Y)\}.\end{aligned}$$

Since J_2 integrable, from (13) we obtain (16). □

We now give a sufficient condition for a map between metallic Riemannian manifold to be harmonic.

Theorem 3.4. *Let F be a metallic map from a metallic Riemannian manifold (M_1, g_1, J_1) to a metallic Riemannian manifold (M_2, g_2, J_2). Then F is a harmonic map if $q_1 - q_2 \neq (p_2 - p_1)\sigma(p_2, q_2)$.*

Proof. Let $\{e_1, e_2, ..., e_m\}$ be a basis of $T_x M_1$, $x \in M_1$. Then from (14) and (16), we have

$$\begin{aligned} q_1 \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) &= (p_2 - p_1) J_2 \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) + q_2 \sum_{i=1}^{m} (\nabla F_*)(e_i, e_i) \\ q_1 \tau(F) &= (p_2 - p_1) J_2 \tau(F) + q_2 \tau(F) \end{aligned}$$

$$(q_1 - q_2)\tau(F) = (p_2 - p_1) J_2 \tau(F). \tag{17}$$

Applying J_2 to (17), we obtain

$$(q_1 - q_2) J_2 \tau(F) = (p_2 - p_1) J_2^2 \tau(F). \tag{18}$$

Using (11), we get

$$(q_1 - q_2) J_2 \tau(F) = (p_2 - p_1) p_2 J_2 \tau(F) + (p_2 - p_1) q_2 \tau(F). \tag{19}$$

From above equation, we obtain

$$\{(q_1 - q_2) - p_2(p_2 - p_1)\} J_2 \tau(F) = (p_2 - p_1) q_2 \tau(F). \tag{20}$$

Now, taking inner product with $(p_2 - p_1)q_2$ in (18) we get

$$(p_2 - p_1)(q_1 - q_2) q_2 \tau(F) = (p_2 - p_1)^2 q_2 J_2 \tau(F). \tag{21}$$

In a similar way, taking inner product with $(q_1 - q_2)$ in (20) we obtain

$$(q_1 - q_2)\{(q_1 - q_2) - p_2(p_2 - p_1)\} J_2 \tau(F) = (q_1 - q_2)(p_2 - p_1) q_2 \tau(F). \tag{22}$$

Substracting (21) and (22), we get

$$\{(p_2 - p_1)^2 q_2 - (q_1 - q_2)^2 + p_2(p_2 - p_1)(q_1 - q_2)\} J_2 \tau(F) = 0. \tag{23}$$

From (23), since J_2 is a isomorphism, F is harmonic map if $(q_1 - q_2) \neq (p_2 - p_1)\sigma_{(p_2,q_2)}$. This proof is complete. □

One can see that contrary to the Golden case, metallic map is not always harmonic.

Remark 3.5. *We note that for any C^2 real valued function f defined on an open subset of a Riemannian manifold M, the equation $\triangle f = 0$ is called Laplace's equation and solutions are called harmonic functions on U. Let $F : M \to N$ be a smooth map between Riemannian manifolds. Then F is called a harmonic morphism if, for every harmonic function $f : V \to R$ defined an open subset V of N with $F^{-1}(V)$ non-empty, the composition $f \circ F$ is harmonic on $F^{-1}(V)$. A smooth map $F : M \to N$ between Riemannian manifolds is harmonic morphism if and only if F is both harmonic and horizontally weakly conformal* [10], [13] *and* [14]. *In this respect, from above theorem a metallic map is candidate for a harmonic morphism.*

4. Certain constancy conditions for maps between metallic Riemannian manifolds and manifolds equipped with other differential structures

In this section we investigate constancy of certain maps between metallic Riemannian manifolds and manifolds equipped with other differential structures by imposing holomorphic-like conditions. We first check the situation for a map between metallic Riemannian manifolds and Golden Riemannian manifolds.

Theorem 4.1. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to a Golden Riemannian manifold $(\bar{M}, P, g)$ such that $F_*J_1 = PF_*$ is satisfied. Then F is a constant map if $p \neq q(\phi) + (1 - \phi)$ and $p \neq q(1 - \phi) + \phi$.*

Proof. Let (M_1, g_1, J_1) be a metallic Riemannian manifold and $(\bar{M}, P, g)$ a Golden Riemannian manifold. Suppose that $F : M_1 \to \bar{M}$ satisfies

$$F_*(J_1X) = PF_*(X), \quad X \in \Gamma(TM_1). \tag{24}$$

Then apply P to the above equation and using (9) and (11), we have

$$pF_*(J_1X) + qF_*(X) = PF_*(X) + F_*(X), \quad X \in \Gamma(TM_1). \tag{25}$$

Using (24), we get

$$(p - 1)F_*(J_1X) = (1 - q)F_*(X), \quad X \in \Gamma(TM_1). \tag{26}$$

Applying P to (26) again and using (24), we have

(27) $p^2F_*(J_1X)+pqF_*(X)+qF_*(J_1X) = pF_*(J_1X)+qF_*(X)+F_*(J_1X)$

for $X \in \Gamma(TM_1)$. From above equation, we get

(28) $$(p^2+q-p-1)F_*(J_1X) = (q-pq)F_*(X), \;\; X \in \Gamma(TM_1).$$

From (26) and (28) we obtain

(29) $$(p^2-q^2+3q-pq-p-1)F_*(X) = 0, \;\; X \in \Gamma(TM_1).$$

From (29), F is a constant map if $p \neq q(\phi)+(1-\phi)$ and $p \neq q(1-\phi)+\phi$. □

In a similar way, we have the following result.

Theorem 4.2. *Let F be a smooth map from a Golden Riemannian manifold $(\bar{M}, P, g)$ to a metallic Riemannian manifold (M_1, g_1, J_1) such that $F_*P = J_1F_*$ is satisfied. Then F is a constant map if $p \neq q(\phi)+(1-\phi)$ and $p \neq q(1-\phi)+\phi$.*

We now check a similar situation for a map between metallic Riemannian manifolds and almost product manifolds.

Theorem 4.3. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to an almost product manifold (N, g, φ) such that $F_*J_1 = \varphi F_*$ is satisfied. Then F is a constant map if $p \neq \mp\sqrt{q-1}$.*

Proof. Let (M_1, g_1, J_1) be a metallic Riemannian manifold and (N, g, φ) an almost product manifold. Suppose that $F : M_1 \to N$ satisfies

(30) $$F_*(J_1X) = \varphi F_*(X), \;\; X \in \Gamma(TM_1).$$

Then apply φ to the above equation and using (4) and (11), we have

(31) $$pF_*(J_1X)+qF_*(X) = F_*(X), \;\; X \in \Gamma(TM_1).$$

From above equation, we get

(32) $$pF_*(J_1X) = (1-q)F_*(X), \;\; X \in \Gamma(TM_1).$$

Applying φ to (31) again and using (30), we have

(33) $$p^2F_*(J_1X)+pqF_*(X)+qF_*(J_1X) = F_*(J_1X), \;\; X \in \Gamma(TM_1).$$

From above equation, we get

(34) $$(p^2+q-1)F_*(J_1X) = -pqF_*(X), \;\; X \in \Gamma(TM_1).$$

From (32) and (34) we obtain

(35) $$(p^2-q^2+2q-1)F_*(X) = 0, \;\; X \in \Gamma(TM_1).$$

From (35), F is a constant map if $p \neq \mp\sqrt{q-1}$. □

In a similar way, we have the following result.

Theorem 4.4. *Let F be a smooth map from an almost product manifold (N, g, φ) to a metallic Riemannian manifold (M_1, g_1, J_1) such that $F_*\varphi = J_1 F_*$ is satisfied. Then F is a constant map if and only if $p \neq \mp\sqrt{q-1}$.*

Remark 4.5. *The equality condition of Theorem 4.4 is satisfies for copper case due to $p = 1$ and $q = 2$. So there may be non-constants map between metallic Riemannian manifold and almost product manifold. This situation is quite different from Golden case.*

We check a similar situation for a map between metallic Riemannian manifolds and almost complex manifolds.

Theorem 4.6. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to an almost complex manifold (M', J') such that $F_*J_1 = J'F_*$ is satisfied. Then F is a constant map.*

Proof. Let (M_1, g_1, J_1) be a metallic Riemannian manifold and (M', J') an almost complex manifold. Suppose that $F : M_1 \to M'$ satisfies

$$F_*(J_1X) = J'F_*(X), \quad X \in \Gamma(TM_1). \tag{36}$$

Then apply J' to the above equation and using (1) and (11), we have

$$pF_*(J_1X) + qF_*(X) = -F_*(X), \quad X \in \Gamma(TM_1). \tag{37}$$

From above equation, we get

$$pF_*(J_1X) = -(q+1)F_*(X), \quad X \in \Gamma(TM_1). \tag{38}$$

Applying J' to (37) again and using (36), we have

$$p^2F_*(J_1X) + pqF_*(X) = -(q+1)F_*(J_1X), \quad X \in \Gamma(TM_1). \tag{39}$$

From above equation, we get

$$(p^2 + q + 1)F_*(J_1X) = -pqF_*(X), \quad X \in \Gamma(TM_1). \tag{40}$$

From (38) and (40) we obtain

$$(p^2 + (q+1)^2)F_*(X) = 0, \quad X \in \Gamma(TM_1). \tag{41}$$

which shows that F is constant. □

In a similar way, we have the following result.

Theorem 4.7. *Let F be a smooth map from an almost complex manifold (M', J') to a metallic Riemannian manifold (M_1, g_1, J_1) such that $F_* J' = J_1 F_*$ is satisfied. Then F is a constant map.*

The following result shows that a smooth map satisfying a compatible condition between metallic Riemannian manifolds and almost contact metric manifolds is also constant.

Theorem 4.8. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to an almost contact metric manifold (M, ϕ, ξ, η, g) such that $F_* J_1 = \phi F_*$ is satisfied. Then F is a constant map.*

Proof. Let (M_1, g_1, J_1) be a metallic Riemannian manifold and (M, ϕ, ξ, η, g) an almost contact metric manifold. Suppose that $F : M_1 \to M$ satisfies

$$F_*(J_1 X) = \phi F_*(X), \quad X \in \Gamma(TM_1). \tag{42}$$

Then apply ϕ to the above equation and using (2) and (11), we have

$$pF_*(J_1 X) + (q+1)F_*(X) = \eta(F_*(X))\xi, \quad X \in \Gamma(TM_1). \tag{43}$$

Then applying ϕ to (43) again and using (42) and (2), we have

$$(p^2 + q + 1)F_*(J_1 X) = -pqF_*(X), \quad X \in \Gamma(TM_1). \tag{44}$$

From (43) and (44), we obtain

$$F_*(X)\left(\frac{p^2 + (q+1)^2}{p^2 + q + 1}\right) = \eta(F_*(X))\xi, \quad X \in \Gamma(TM_1). \tag{45}$$

Again applying ϕ to (45), we get

$$\phi F_*(X)\left(\frac{p^2 + (q+1)^2}{p^2 + q + 1}\right) = 0, \quad X \in \Gamma(TM_1). \tag{46}$$

Then applying ϕ to (46), we get

$$-F_*(X)\left(\frac{p^2 + (q+1)^2}{p^2 + q + 1}\right) + \eta\left(F_*(X)\frac{p^2 + (q+1)^2}{p^2 + q + 1}\right)\xi = 0, \quad X \in \Gamma(TM_1). \tag{47}$$

Again applying ϕ to (47), we get

$$F_*(X)\left(\frac{p^2 + (q+1)^2}{p^2 + q + 1}\right) = 0, \quad X \in \Gamma(TM_1). \tag{48}$$

From (48), F is a constant map. □

In a similar way, we have the following result.

Theorem 4.9. *Let F be a smooth map from an almost contact metric manifold (M, ϕ, ξ, η, g) to a metallic Riemannian manifold (M_1, g_1, J_1) such that $F_*\phi = J_1 F_*$ is satisfied. Then F is a constant map.*

Finally, we check the same problem for almost para-contact metric manifolds.

Theorem 4.10. *Let F be a smooth map from a metallic Riemannian manifold (M_1, g_1, J_1) to an almost para-contact metric manifold $(N', \phi', \xi', \eta', g')$ such that $F_*J_1 = \phi' F_*$ is satisfied. Then F is a constant map if $p^2 \neq (q-1)^2$.*

Proof. Let (M_1, g_1, J_1) be a metallic Riemannian manifold and $(N', \phi', \xi', \eta', g')$ an almost para-contact metric manifold. Suppose that $F : M_1 \to N'$ satisfies

$$F_*(J_1 X) = \phi' F_*(X), \quad X \in \Gamma(TM_1). \tag{49}$$

Then apply ϕ' to the above equation and using (8) and (11), we have

$$pF_*(J_1 X) + (q-1)F_*(X) = -\eta(F_*(X))\xi, \quad X \in \Gamma(TM_1). \tag{50}$$

Then applying ϕ' to (50) again and using (49) and (8), we have

$$(p^2 + q - 1)F_*(J_1 X) = -pqF_*(X), \quad X \in \Gamma(TM_1). \tag{51}$$

From (50) and (51), we obtain

$$F_*(X)\left(\frac{(q-1)^2 - p^2}{p^2 + q - 1}\right) = -\eta(F_*(X))\xi, \quad X \in \Gamma(TM_1). \tag{52}$$

Again applying ϕ' to (52), we get

$$\phi' F_*(X)\left(\frac{(q-1)^2 - p^2}{p^2 + q - 1}\right) = 0, \quad X \in \Gamma(TM_1). \tag{53}$$

Then applying ϕ' to (53), we get

$$F_*(X)\left(\frac{(q-1)^2 - p^2}{p^2 + q - 1}\right) - \eta\left(F_*(X)\frac{(q-1)^2 - p^2}{p^2 + q - 1}\right)\xi = 0 \tag{54}$$

for $X \in \Gamma(TM_1)$. Again applying ϕ' to (54), we get

$$F_*(X)\left(\frac{(q-1)^2 - p^2}{p^2 + q - 1}\right) = 0, \quad X \in \Gamma(TM_1). \tag{55}$$

From (55), F is a constant map if $p^2 \neq (q-1)^2$. □

In a similar way, we have the following result.

Theorem 4.11. *Let F be a smooth map from an almost paracontact metric manifold $(N', \phi', \xi', \eta', g')$ to a metallic Riemannian manifold (M_1, g_1, J_1) such that $F_*\phi' = J_1 F_*$ is satisfied. Then F is a constant map if $p^2 \neq (q-1)^2$.*

Remark 4.12. *The equality condition of Theorem 4.9 is possible for copper case due to $p = 1$ and $q = 2$. Thus it is possible to find non-constant map between such manifolds.*

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