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RINGS WHOSE ELEMENTS ARE SUMS OF FOUR COMMUTING IDEMPOTENTS

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Abstract. We completely characterize the isomorphic class of those associative unitary rings whose elements are sums of four commuting idempotents. Our main theorem enlarges results due to Hirano-Tominaga (Bull. Austral. Math. Soc., 1988), Tang et al. (Lin. & Multilin. Algebra, 2019), Ying et al. (Can. Math. Bull., 2016) as well as results due to the author in (Alban. J. Math., 2018), (Gulf J. Math., 2018), (Bull. Iran. Math. Soc., 2018) and (Boll. Un. Mat. Ital., 2019).

1. Introduction and Background

Everywhere in the text of the present paper, all our rings R are assumed to be associative, containing the identity element 1, which in general differs from the zero element 0 of R, and all subrings are unital (i.e., containing the same identity as that of the former ring). Our notions and notations are mainly in agreement with [12]. For instance, to be more precise, $U(R)$ denotes the set of all units in R, $Id(R)$ the set of all idempotents in R, $Nil(R)$ the set of all nilpotents in R, and $J(R)$ the Jacobson radical of R. All other conventions, which are not explicitly defined herein, will be stated below in detail.

Imitating the terminology from [8], we shall consider here rings R from the class C^{4+}, that is, $R = Id(R) + Id(R) + Id(R) + Id(R)$ in which equality the existing idempotents are commuting each to other. Specifically, one can state the following concept:

Definition 1.1. *We shall say that a ring R is of the type C^{4+} if each element of R is a sum of (at most) four commuting idempotents.*

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Obvious constructions of such rings are the finite rings $\mathbb{Z}_k$, where $k = 2, 3, 4, 5, 6$. Likewise, in contrast to the given examples in the cited below literature, the direct product $\mathbb{Z}_5 \times \mathbb{Z}_5$ lies in the class C^{4+} of rings, whereas the field $\mathbb{Z}_7$ is not so. Generally, if for any fixed $n \in \mathbb{N}$ we define by analogy the ring class C^{n+} consisting of elements represented as the sum of n commuting idempotents, and if $n+1 = p$ is a prime number, it follows that the field $\mathbb{Z}_p$ satisfies this property (compare with [10], too).

Despite of the techniques developed in [6]-[9], [11], [13] and [14], we here need somewhat of another methodology for the development of the case $n = 4$. To that aim, we will use some properties of the polynomial ring (see, for more account, the proof of Proposition 2.2 listed below).

A brief history of the principally known results in the topic is as follows: Boolean rings, that are rings in which every element is an idempotent, were described in the past as the subdirect products of copies of the field $\mathbb{Z}_2$. Generalizing this famous classical result, Hirano and Tominaga studied in [11] those rings whose elements are sums of two commuting idempotents and, especially, they proved that these rings are subdirect products of copies of the fields $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Further intensive work is done in [14] where rings in which the elements are sums or differences of two commuting idempotents were classified. In this direction, in [9] the rings whose elements are sums or minus sums of two commuting idempotents were characterized, too. After that, in [13] and [8] were independently described those rings in which the elements are sums of three commuting idempotents. Various other aspects of rings with combinations of at most three idempotents are considered in [6] and [7], respectively, as sums of idempotents in some other but closely related variations are explored in [10], [4] and [5] as well.

That is why, it is quite natural to continue these investigations for sums of four idempotents and our basic motivation here is to promote this in a rather attractive way by diving up the complete characterization result up to an isomorphism.

2. Main Results

We start with one useful technicality.

Lemma 2.1. *Suppose R is a C^{4+} ring. Then R can be decomposed as the direct product $R \cong R_1 \times R_2 \times R_3$ for some C^{4+} rings R_1, R_2, R_3 with $4 = 2^2 = 0$ in R_1, $3 = 0$ in R_2 and $5 = 0$ in R_3.*

Proof. We assert that the equality $4 \times 3 \times 5 = 0$ holds in such a ring R. In fact, writing $-1 = e_1 + e_2 + e_3 + e_4$ for some commuting idempotents e_1, e_2, e_3, e_4, whence $-2 = e_1 + e_2 + e_3 - (1 - e_4) = e_1 + e_2 + e_3 - e'_4$. We may with no harm of generality assume that $e_1 e'_4 = e_2 e'_4 = e_3 e'_4 = 0$; indeed, $e_3 - e'_4 = e_3(1 - e'_4) - e'_4(1 - e_3)$ is a difference of two orthogonal idempotents. Further, using the formula $a - b = a(1 - b) - b(1 - a)$ for any elements a, b with $ab = ba$, we deduce that $e_2 - e'_4(1 - e_3) = e_2(1 - e'_4(1 - e_3)) - e'_4(1 - e_3)(1 - e_2)$ is also a difference of two commuting idempotents as $e_3(1 - e'_4).e'_4(1 - e_3)(1 - e_2) = 0$ and etc., we can proceed by analogy with e_1 to verify our assumption. So, under these circumstances, multiplying $-2 = e_1 + e_2 + e_3 - e'_4$ by e'_4, we obtain that $e'_4 = 0$ and thus $-2 = e_1 + e_2 + e_3$. Writing $-3 = e_1 + e_2 - (1 - e_3) = e_1 + e_2 - e'_3$, we may exploit the same idea to get that $e_1 e'_3 = e_2 e'_3 = 0$. Therefore, a multiplication of $-3 = e_1 + e_2 - e'_3$ by e'_3 assures that $2e'_3 = 0$. However, multiplying this with $1 - e$ and next, squaring the outcome, will ensure by simple manipulation that $12(1 - e_1) = 0$, i.e., $12e_1 = 12$ as $e'_3 = -e'_3$. Similarly, $12e_2 = 12$. Consequently, one has that $-36 = 12 + 12$, that is, $60 = 2^2 \times 3 \times 5 = 0$, as asserted.

Finally, the Chinese Remainder Theorem yields our initial claim on the decomposition of R into direct factors, as stated. □

The next statement sheds some light on the idempotent sum property in subdirect products of fields of prime characteristic.

Proposition 2.2. *Let R be a ring of characteristic 5. Then $R \in C^{4+}$ if, and only if, $x^5 = x$ for all $x \in R$.*

Proof. "$\Rightarrow$". If $r \in R$ is written as $r = e + f + g + h$ for some four commuting idempotents, it is easily checked with the standard binomial formula at hand that $r^5 = (e + f + g + h)^5 = e + f + g + h = r$, taking into account that $5 = 0$ in R.

"$\Leftarrow$". Let x be an arbitrary non-identity element in R. Then the subring, say S, generated by 1 and x has the same property: its characteristic is equal to 5 and $y^5 = y$ for every $y \in S$. So, without loss of generality, we may replace R by this subring S and it suffices to prove the wanted decomposition property there. Indeed, if we can find 4 idempotents in S whose sum is exactly x, then they are also in R.

To that purpose, we first claim that the subring S is commutative and is isomorphic to a quotient of the factor-ring $\mathbb{Z}_5[X]/(X^5 - X)$. In fact, that is trivial by considering the surjective ring homomorphism $\mathbb{Z}_5[X] \to S$ defined by mapping X to x. That is why, it is enough to show the desired idempotent property for the ring $\mathbb{Z}_5[X]/(X^5 - X)$

which is obviously a direct product of precisely 5 copies of the field $\mathbb{Z}_5$. However, since the existing there idempotents are the five-sized vectors with coordinates only 0 and/or 1 and since the elements of $\mathbb{Z}_5$ are exactly $\{0, 1, 2, 3, 4\}$ as $2 = 1+1$, $3 = 1+1+1$ and $4 = 1+1+1+1$, this is pretty easy, which technical details we leave to the interested readers. $\square$

Remark 2.3. *Certainly, it is worthy to mention that if all elements of the ring R of characteristic 5 satisfy the equality $x^5 = x$, then R is a subdirect product (finite or infinite) of a family of copies of the field $\mathbb{Z}_5$ (see, e.g., [12]) and that, in this direct product, it readily could be verified that all elements are sums of exactly four idempotents. Nevertheless, it is not at all obvious that the same relation remains true and in R, while our idea presented above unambiguously shows the truthfulness of this fact, thus increasing the assertion.*

Likewise, by the same token as in Proposition 2.2, it could be successfully proved that $R \in C^{2+} \iff x^3 = x\ \forall x$, thus giving up a new more transparent proof of the chief result from [11].

We now have all the ingredients necessary to proceed by proving the following chief result.

Theorem 2.4. *A ring R is from the class C^{4+} if, and only if, it is decomposable as $R \cong R_1 \times R_2 \times R_3$, where $R_1 = \{0\}$ or R_1 is a commutative ring such that $J(R_1) = 2Id(R_1)$ is nil of nilpotence index less than or equal to 2 and the quotient $R_1/J(R_1)$ is a Boolean ring, $R_2 = \{0\}$ or R_2 is a subdirect product of copies of the field $\mathbb{Z}_3$, and $R_3 = \{0\}$ or R_3 is a subdirect product of copies of the field $\mathbb{Z}_5$.*

Proof. "**Necessity.**" According to Lemma 2.1 one writes that $R \cong R_1 \times R_2 \times R_3$, where R_1, R_2, R_3 are rings in which $4 = 0$, $3 = 0$ and $5 = 0$, respectively. Moreover, with this at hand, it is self-evident that all of the three direct factors inherit the property to be C^{4+} rings. We next differ the description of these three factors separately.

Describing R_1: Here $4 = 0$. We intend to prove that $R_1/J(R_1)$ is a Boolean factor-ring and that $J(R_1) = 2Id(R_1)$. To this purpose, we foremost see that the elements in R_1 are solutions of the equations $x^2 = x^4$ and $2x = 2x^2$. In fact, writing $x = e_1+e_2+e_3+e_4$ for some four commuting each to other idempotents e_1, e_2, e_3, e_4, one sees that $x^2 = x+2(e_1e_2+e_1e_3+e_1e_4+e_2e_3+e_2e_4+e_3e_4)$ which immediately implies the validity of the two equations quoted above, taking into account that $4 = 0$.

Now, since $2^2 = 0$ yields that $2 \in Nil(R_1)$ and so $2 \in J(R_1)$, it readily follows that the quotient $R_1/J(R_1)$ is a ring of characteristic 2 whose elements are sums of four commuting idempotents, and thus it is necessarily Boolean.

As for $J(R_1)$, choosing an arbitrary element $z \in J(R_1)$, by what we have shown so far, $z^4 = z^2$ so that $z^2(z^2 - 1) = 0$ meaning that $z^2 = 0$ because $z^2 - 1 \in U(R_1)$. Also, $2z^2 = 2z$ and therefore $2z(z - 1) = 0$ leading to $2z = 0$ since $z - 1 \in U(R_1)$. With these two equalities at hand, we will establish now that $J(R_1) = 2Id(R_1)$. In fact, as $z + 1$ is representable as a sum of four commuting idempotents, we may freely write as in Lemma 2.1 that $z = f_1 + f_2 + f_3 - f_4$ for some four commuting idempotents f_1, f_2, f_3, f_4, where f_1, f_2, f_3 are orthogonal with f_4. Furthermore, squaring the current equality for z, we arrive at $z = 2(f_4 + f_1f_2 + f_2f_3 + f_1f_3)$. On the other hand, multiplying $z = f_1 + f_2 + f_3 - f_4$ by $f_1f_2f_3$, we deduce that $zf_1f_2f_3 = 3f_1f_2f_3 = -f_1f_2f_3$ because $4 = 0$. So, $(z + 1)f_1f_2f_3 = 0$ allowing us to derive that $f_1f_2f_3 = 0$ since $z + 1$ inverts in R_1. But this enables us that $f_4, f_1f_2, f_2f_3, f_1f_3$ are para-wise orthogonal idempotents, whence their sum is surely an idempotent, as expected. Consequently, one concludes that $z \in 2Id(R_1)$, as promised, and thereby the relation sustained.

Describing R_2: Here $3 = 0$. We intend to prove that R_2 is a subdirect product of family of copies of the field $\mathbb{Z}_3$. To this aim, we simple observe that all elements in R_2 are solutions of the equation $x^3 = x$ and further employ [11] to infer the wanted assertion.

Describing R_3: Here $5 = 0$. We intend to prove that R_3 is a subdirect product of family of copies of the field $\mathbb{Z}_5$. To this goal, we just observe that all elements in R_3 are solutions of the equation $x^5 = x$ and then apply [12] to get the desired assertion.

"**Sufficiency.**" If we succeed to prove that each of the rings R_1, R_2 and R_3 is from the class C^{4+}, then it will follow directly by virtue of standard coordinate-wise arguments that $R_1 \times R_2 \times R_3 \cong R$ lies in C^{4+} too. To that strategy, we shall consider these three rings separately as follows:

Considering R_1: Since $R_1/J(R_1)$ is Boolean, it follows that for every $r \in R_1$ the relationship $r^2 - r \in J(R_1) = 2Id(R_1)$ is fulfilled. However, $J(R_1)$ is nil and, as it is well-known, there exists $f \in Id(R_1)$ such that $f - r \in 2Id(R_1)$. Finally, r must be a sum of three idempotents (as $4 = 0$) and, consequently, R_1 belongs to the class C^{3+} and so also to the class C^{4+}.

Considering R_2: In virtue of [11], it follows immediately that R_2 belongs to the class C^{2+} and thus it is in the class C^{4+} as well.

Considering R_3: In view of Proposition 2.2, it follows immediately that R_3 belongs to the class C^{4+}. □

It is worthwhile to mention that the commutative ring R_1 from the decomposition of R is also an invo-clean ring, that is, every its element is the sum of an idempotent and an involution (for more details, see [1], [2] and [3]), and vice versa when the index of nilpotence is at most 2. In fact, for any commutative ring K having the property $4 = 0$ and whose elements satisfy the polynomial identities $x^4 = x^2$ and $2x^2 = 2x$ for all $x \in K$, it must be that $x = (1 - x^2) + (x^2 + x - 1)$. A routine check shows that $1 - x^2$ is an idempotent, i.e., $(1 - x^2)^2 = 1 - x^2$, whereas $x^2 + x - 1$ is an involution, i.e., $(x^2 + x - 1)^2 = 1$.

Conversely, assuming K is a commutative invo-clean ring of even characteristic not exceeding 4 and index of nilpotence at most 2, and writing for any $x \in K$ that $x = e + v$, where $e \in K$ with $e^2 = e$ and $v \in K$ with $v^2 = 1$, we obtain that $x^2 = 1 + e + 2ev$, and hence $2x^2 = 2e + 2$ along with $2x = 2e + 2v$. But we claim that $2v = 2$, so that $x^2 = 1 + 3e = 1 - e = x^4$ accomplished with $2x^2 = 2x$, as required. Indeed, one observes that $(1-v)^2 = 2(1-v)$ whence, $(1-v)^4 = 0$ which, in the presence of our initial assumptions, leads to $(1 - v)^2 = 0$, i.e., to $2 = 2v$, as claimed. This substantiates our statement.

Besides, one more useful observation is that the proof of Theorem 2.4 illustrates the surprising fact that the classes C^{4+} and C^{3+} do coincide when $4 = 0$.

We shall now give a few more constructions of rings from the class C^{4+}.

Example 2.5. *As it was already established in Theorem 2.4, the rings from the class C^{4+} are necessarily commutative. As more specific constructions of such rings, could be viewed the following ones:*

- *The direct product $L = \prod_\lambda \mathbb{Z}_2$, where λ is a finite or infinite cardinal.*

- *The direct product $L = \prod_\mu \mathbb{Z}_4$, where μ is a finite or infinite cardinal.*

In fact, simple calculations show that $J(L) = 2Id(L)$ because $J(\mathbb{Z}_4) = \{0, 2\} = 2\mathbb{Z}_4 = 2Id(\mathbb{Z}_4)$ in conjunction with $J(L) = \prod_\mu 2\mathbb{Z}_4$, so that the quotient ring $L/J(L) \cong \prod_\mu \mathbb{Z}_2$ is obviously Boolean.

- *The direct product $L = \prod_\nu \mathbb{Z}_3$, where ν is a finite or infinite cardinal.*

• *The direct product* $L = \prod_{\alpha} \mathbb{Z}_5$, *where* α *is finite or infinite cardinal.*

as well as arbitrary combinations between the above direct products.

We end our work with the following:

Problem. Characterize up to an isomorphism the class of C^{n+} **rings, where** $n \geq 1$ **is an integer.**

In regard to this, one states the following:

Conjecture. For a fixed $n \in \mathbb{N}$, all C^{n+} rings are themselves commutative.

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