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GENERALIZED BIPOLAR FUZZY INTERIOR IDEALS IN ORDERED SEMIGROUPS

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Abstract. This research focuses on the characterization of an ordered semigroups (OS) in the frame work of generalized bipolar fuzzy interior ideals (BFII). Different classes namely regular, intra-regular, simple and semi-simple ordered semigroups were characterized in term of (α, β)-BFII (resp (α, β)-bipolar fuzzy ideals (BFI)). It has been proved that the notion of $(\in, \in \curlyvee \mathrm{q})$-BFII and $(\in, \in \curlyvee \mathrm{q})$-BFI overlap in semi-simple, regular and intra-regular ordered semigroups. The upper and lower part of $(\in, \in \curlyvee \mathrm{q})$-BFII are discussed.

1. Introduction

The membership degree of elements in traditional fuzzy sets ranges over the intervals [0,1], which defines relationship between scale of belongingness to elements of a fuzzy set, where, 1 represents complete belonging of an element to its corresponding set and 0 represents non-belongingness to the fuzzy set. Partial membership of an element to a fuzzy set is represented by the membership degree on the intervals (0,1). The membership degree also aptly explains the satisfaction level of an elements to some important property related to fuzzy set (see [1], [2]). When dealing with satisfaction level, the membership scale 0 is sanction to an element which does not fulfil some property. So the element with membership scale 0 is usually meant for containing the same features in the fuzzy set representation. Incidentally, among these elements, some possesses unrelated features to the property related to a fuzzy set and the some have opposite features to the property. The weakness with usual fuzzy set representation is that they cannot tell apart opposite

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and unrelated elements. Similarly, with membership degree ranged on the intervals [0,1], its hard to comprehend the difference of the unrelated elements from the opposite elements in fuzzy sets. In order to overcome these difficulties, Lee [3] established the idea of Bipolar fuzzy sets (BFS).

Bhakat and Das [4] firstly gave the idea of (α, β)-fuzzy subgroup and use the combine concept of belongingness and quasi-coincidence of fuzzy points and fuzzy sets [5, 6] which is a commodious generalization of Rosenfeld's fuzzy subgroups [21]. Later Davvaz bring-in the idea of $(\in, \in \curlyvee q)$-fuzzy sub-near-rings of a near ring and came up with some attractive features [22]. Additionally Jun and Song [24] talk about the generalized form of fuzzu interior ideal (FII) of a semigroup. The idea of $(\in, \in \curlyvee q)$-fuzzy generalized bi-ideals of a semigroup was first initiated by Kazanci and Yamak [25] and gave some features of fuzzy bi-ideals in terms $(\in, \in \curlyvee q)$-fuzzy bi-ideals. Khan, Jun and Shabir [14] introduced the notion of fuzzy generalized ideals in OS. Other researchers also added their part by using the concept of generalized fuzzy sets and gave tremendous results in different branches of algebra (e.g. see [18, 26, 27, 28, 38, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42]).

This research work is divided into the following sections, Section 1 explains the literature behind our hypothesis. Section 2 deals with some basic definitions and lemmas of BFII (resp bipolar fuzzy ideals) of an OS. In sections 3, we proved some basic theorems related to the concept of (α, β)-BFII. In section 4, we proved that the notion of $(\in, \in \curlyvee q)$-BFII and $(\in, \in \curlyvee q)$-BFI coincide in regular, semi-simple and intra-regular OS. The lower and upper part of $(\in, \in \curlyvee q)$-BFIIs are discussed in section 5 and gave some important results.

2. Preliminaries

An OS is a semigroup $(S, .)$ as well as poset $(S, \leq)$ such that for all $a, b, x \in S$, we have $a \preceq b \Longrightarrow ax \preceq bx$ and $xa \preceq xb$.

An OS S is said to be regular if for every $a \in S$, there exists $x \in S$ such that $a \preceq axa$. An OS S is called intra-regular if for every $a \in S$ there exist $x, y \in S$ such that $a \preceq xa^2y$. An OS S is called semisimple if for every $a \in S$ there exist $x, y, z \in S$ such that $a \preceq xayaz$.

Definition 2.1. *A BFS $\mu = (S; \mu_n, \mu_p)$ defined over S is a mapping of the type*

$$\mu = \{(x, \mu_n(x), \mu_p(x)) : x \in S\}$$
$$\mu_n(x) \colon S \longrightarrow [-1,0]$$
$$\mu_p(x) \colon S \longrightarrow [0,1]$$

where $\mu_p(x)$ indicates that how much x satisfy some property and $\mu_n(x)$ indicates that how much x satisfy some implicit counter-property.

For a BFS $\mu = (S; \mu_n, \mu_p)$ in S and $(s,t) \in [-1,0) \times (0,1]$, we define

$N(\mu; s) := \{x \in S \mid \mu_n(x) \preceq s\}$ and $P(\mu; t) := \{x \in S \mid \mu_p(x) \succeq t\}$,

which are called the negative s-cut and the positive t-cut of $\mu = (S; \mu_n, \mu_p)$ respectively.

The (s,t)-cut of $\mu = (S; \mu_n, \mu_p)$ is denoted by $C(\mu; (s,t))$ and is defined by $C(\mu; (s,t)) = N(\mu; s) \cap P(\mu; t)$.

If $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ are two BFSs in S. Then

$\mu \preceq \nu$ if and only if $(\forall\ x \in S)(\mu_n(x) \succeq \nu_n(x)$ and $\mu_p(x) \preceq \nu_p(x))$,

$\mu = \nu$ if and only if $\mu \preceq \nu$ and $\nu \preceq \mu$,

$\mu \cap \nu = (S; \mu_n \curlyvee \nu_n,\ \mu_p \curlywedge \nu_p)$,

$\mu \cup \nu = (S; \mu_n \curlywedge \nu_n, \mu_p \curlyvee \nu_p)$.

For an OS S, bipolar fuzzy subsets $0 = (S; 0_n, 0_p)$ and $1 = (S; 1_n, 1_p)$ are defined as follows:

$0_n(x) = 0 = 0_p(x)$, $1_n(x) = -1$ and $1_p(x) = 1$ for all $x \in S$. For $x \in S$, define $A_x = \{(y,z) \in S \times S \mid x \preceq yz\}$.

For bipolar fuzzy subsets $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ of S, define the product as $\mu \circ \nu = (S; \mu_n \circ \nu_n, \mu_p \circ \nu_p)$, where

$$(\mu_n \circ \nu_n)(x) = \begin{cases} \bigwedge\{\mu_n(y) \curlyvee \nu_n(z)\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset \end{cases}$$

and

$$(\mu_p \circ \nu_p)(x) = \begin{cases} \bigvee\{\mu_p(y) \curlywedge \nu_p(z)\} & \text{if } A_x \neq \emptyset \\ 0 & \text{if } A_x = \emptyset. \end{cases}.$$

Definition 2.2. *A BFS $\mu = (S; \mu_n, \mu_p)$ is called BFII of an OS S, if*

(I_1) $(\forall x_1, x_2 \in S)(x_1 \preceq x_2 \Rightarrow \mu_n(x_1) \preceq \mu_n(x_2)$ and $\mu_p(x_1) \succeq \mu_p(x_2))$,

(I_2) $(\forall x_1, x_2 \in S)(\mu_n(x_1 x_2) \preceq \bigvee\{\mu_n(x_1), \mu_n(x_2)\}$ and $\mu_p(x_1 x_2) \succeq \bigwedge\{\mu_p(x_1), \mu_p(x_2)\})$,

(I_3) $(\forall x_1, a, x_2 \in S)(\mu_n(x_1 a x_2) \preceq \mu_n(a)$ and $\mu_p(x_1 a x_2) \succeq \mu_p(a))$.

Definition 2.3. [10] *A BFS $\mu = (S, \mu_n, \mu_p)$ is called a bipolar fuzzy left (resp. right) ideal of an OS S if*

(1) $(\forall x, y \in S)(x \preceq y \Longrightarrow \mu_n(x) \preceq \mu_n(y)$ *and* $\mu_p(x) \succeq \mu_p(y)$,
(2) $(\forall x, y \in S)(\mu_n(xy) \preceq \mu_n(y)$*(resp.* $\mu_n(xy) \preceq \mu_n(x))$ *and* $\mu_p(xy) \succeq \mu_p(y)$*(resp.* $\mu_p(xy) \succeq \mu_p(x)))$.

In regular (intra-regular and semisimple) OS, the concept of BFI and BFII coincide.

Theorem 2.4. *Let* $\mu=(S;\mu_n,\mu_p)$ *be a BFS in* S*. Then* $\mu=(S;\mu_n,\mu_p)$ *is a BFII of an OS* S *if and only if the non-empty* (s,t)*-cut* $C(\mu;(s,t))$ *of* $\mu=(S;\mu_n,\mu_p)$ *is an interior ideal of an OS* S $\forall$ $(s,t) \in [-1,0) \times (0,1]$.

If $\emptyset \neq I \subseteq S$, then the characteristic function of I denoted by $\chi_I = (S, \chi_{nI}, \chi_{pI})$ where χ_{nI} and χ_{pI} are defined as

$$\chi_{nI}(x) = \begin{cases} -1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases} \text{ and } \chi_{pI}(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}.$$

3. (α, β)-bipolar fuzzy interior ideals in ordered semigroup

Let $(s,t) \in [-1,0) \times (0,1]$, then an ordered bipolar fuzzy point is defined by

$$\mu_n(y) := \begin{cases} s \text{ if } y \in (x], \\ 0 \text{ if } y \notin (x], \end{cases} \quad \mu_p(y) := \begin{cases} t \text{ if } y \in (x], \\ 0 \text{ if } y \notin (x]. \end{cases}$$

and is represented by $\frac{x}{(s,t)}$.

For a BFS $\mu = (S;\mu_n,\mu_p)$ in S and an ordered bipolar fuzzy point $\frac{x}{(s,t)}$, we say that

1) $\frac{x}{(s,t)} \in \mu$ if $\mu_n(x) \preceq s$ and $\mu_p(x) \succeq t$,
2) $\frac{x}{(s,t)}$ q μ if $\mu_n(x) + s \prec -1$ and $\mu_p(x) + t \succ 1$,
3) $\frac{x}{(s,t)} \in \curlyvee q$ μ if $\frac{x}{(s,t)} \in \mu$ or $\frac{x}{(s,t)}$ q μ,
4) $\frac{x}{(s,t)} \in \curlywedge q$ μ if $\frac{x}{(s,t)} \in \mu$ and $\frac{x}{(s,t)}$ q μ,

Let $\mu = (S;\mu_n,\mu_p)$ be a BFS in S such that $\mu_n(x) \succeq -0.5$ and $\mu_p(x) \preceq 0.5$ $\forall$ $x \in S$. Let $\frac{x}{(s,t)} \in \curlywedge q$ μ, then $\frac{x}{(s,t)} \in \mu$ and $\frac{x}{(s,t)}$ q μ, that is, $\mu_n(x) \preceq s$, $\mu_p(x) \succeq t$, $\mu_n(x) + s \prec -1$ and $\mu_p(x) + t \succ 1$. So $-1 \succ \mu_n(x)+s \succeq 2\mu_n(x)$ and $1 \prec \mu_p(x)+t \preceq 2\mu_p(x)$, thus $\mu_n(x) \prec -0.5$ and $\mu_p(x) \succ 0.5$. This mean that $\{\frac{x}{(s,t)} \mid \frac{x}{(s,t)} \in \curlywedge q\} = \emptyset$. Hence we skip the case $\alpha = \in \curlywedge q$.

Definition 3.1. *A BFS* $\mu = (S;\mu_n,\mu_p)$ *in* S *is said to be* (α,β)*-bipolar fuzzy left ideal (BFLI) (resp.* (α,β)*-BFRI) of an OS* S *where* $\alpha \neq \in \curlywedge q$ *if it fulfil the following conditions. For* $(s,t) \in [-1,0) \times (0,1]$

(1) $(\forall x_1, x_2 \in S)(x_1 \preceq x_2, \frac{x_2}{(s,t)}\alpha\mu \Longrightarrow \frac{x_1}{(s,t)}\beta\mu)$ (2)
(2) $(\forall x_1, x_2 \in S)(\frac{x_2}{(s,t)}\alpha\mu \Longrightarrow \frac{x_1x_2}{(s,t)}\beta\mu)(\text{resp.}\frac{x_1}{(s,t)}\alpha\ \mu \Longrightarrow \frac{x_1x_2}{(s,t)}\beta\mu)$.

Definition 3.2. *A BFS $\mu = (S; \mu_n, \mu_p)$ in S is said to be (α, β)-BFII of an OS S where $\alpha \neq\in \curlywedge q$ if it fulfil the following conditions. For $(s,t) \in [-1,0) \times (0,1]$*

(1) $(\forall x_1, x_2 \in S)(x_1 \preceq x_2, \frac{x_2}{(s,t)}\alpha\mu \Longrightarrow \frac{x_1}{(s,t)}\beta\mu)$,
(2) $(\forall x_1, x_2 \in S)(\frac{x_1}{(s_1,t_1)}\alpha\mu \text{ and } \frac{x_2}{(s_2,t_2)}\alpha\mu \Longrightarrow \frac{x_1x_2}{(\curlyvee\{s_1,s_2\},\curlywedge\{t_1,t_2\})}\beta\mu)$,
(3) $(\forall x_1, a, x_2 \in S)(\frac{a}{(s,t)}\alpha\mu \Longrightarrow \frac{x_1ax_2}{(s,t)}\beta\mu)$.

Theorem 3.3. *Let $\mu = (S; \mu_n, \mu_p)$ be a BFS in S. Then $\mu = (S; \mu_n, \mu_p)$ is a BFII of an OS S if and only if for all $(s,t) \in [-1,0)\times(0,1]$, we have*

(1) $(\forall x_1, x_2 \in S)(x_1 \preceq x_2, \frac{x_2}{(s,t)} \in \mu \Longrightarrow \frac{x_1}{(s,t)} \in \mu)$,
(2) $(\forall x_1, x_2 \in S)(\frac{x_1}{(s_1,t_1)} \in \mu \text{ and } \frac{x_2}{(s_2,t_2)} \in \mu \Longrightarrow \frac{x_1x_2}{(\curlyvee\{s_1,s_2\},\curlywedge\{t_1,t_2\})} \in \mu)$,
(3) $(\forall x_1, a, x_2 \in S)(\frac{a}{(s,t)} \in \mu \Longrightarrow \frac{x_1ax_2}{(s,t)} \in \mu)$.

Proof. Suppose $\mu = (S; \mu_n, \mu_p)$ is a BFII of an OS S. Let $x_1, x_2 \in S$ with $x_1 \preceq x_2$ and $\frac{x_2}{(s,t)} \in \mu$, then $\mu_n(x_2) \preceq s$ and $\mu_p(x_2) \succeq t$. Since $\mu = (S; \mu_n, \mu_p)$ is a BFII of an OS S, we have $\mu_n(x_1) \preceq \mu_n(x_2)$ and $\mu_p(x_1) \succeq \mu_p(x_2)$. This implies that $\mu_n(x_1) \preceq s$ and $\mu_p(x_1) \succeq t$ and so $\frac{x_1}{(s,t)} \in \mu$. Let $x_1, x_2 \in S$ such that $\frac{x_1}{(s_1,t_1)} \in \mu$ and $\frac{x_2}{(s_2,t_2)} \in \mu$. Then $\mu_n(x_1) \preceq s_1$, $\mu_n(x_2) \preceq s_2$, $\mu_p(x_1) \succeq t_1$ and $\mu_p(x_2) \succeq t_2$. From definition of BFII, we have $\mu_n(x_1x_2) \preceq \bigvee\{\mu_n(x_1), \mu_n(x_2) \preceq \bigvee\{s_1, s_2\}$ and $\mu_p(x_1x_2) \succeq \bigwedge\{\mu_p(x_1), \mu_p(x_2)\} \succeq \bigwedge\{t_1, t_2\}$, so that $\frac{x_1x_2}{(\curlyvee\{s_1,s_2\},\curlywedge\{t_1,t_2\})} \in \mu$. Let $x_1, a, x_2 \in S$ such that $\frac{a}{(s,t)} \in \mu$. Then $\mu_n(a) \preceq s$ and $\mu_p(a) \succeq t$. This implies that $\mu_n(x_1ax_2) \preceq \mu_n(a) \preceq s$ and $\mu_p(x_1ax_2) \succeq \mu_p(a) \succeq t$, so $\frac{x_1ax_2}{(s,t)} \in \mu$.

Conversely, let $x_1, x_2 \in S$ with $x_1 \preceq x_2$. Let $\mu_n(x_2) = s$ and $\mu_p(x_2) = t$. Then $\frac{x_2}{(s,t)} \in \mu$, this implies that $\frac{x_1}{(s,t)} \in \mu$ and so $\mu_n(x_1) \preceq s$ and $\mu_p(x_1) \succeq t$. Therefore $\mu_n(x_1) \preceq \mu_n(x_2)$ and $\mu_p(x_1) \succeq \mu_p(x_2)$. Since $\frac{x_1}{(\mu_n(x_1),\mu_p(x_1))} \in \mu$ and $\frac{x_2}{(\mu_n(x_2),\mu_p(x_2))} \in \mu$ for all $x_1, x_2 \in S$, so by (2), we have $\frac{x_1x_2}{(\curlyvee\{\mu_n(x_1),\mu_n(x_2)\},\curlywedge\{\mu_p(x_1),\mu_p(x_2)\})} \in \mu$. This implies that, $\mu_n(x_1x_2) \preceq \bigvee\{\mu_n(x_1), \mu_n(x_2)$ and $\mu_p(x_1x_2) \succeq \bigwedge\{\mu_p(x_1), \mu_p(x_2)\}$. Let $x_1, a, x_2 \in S$. Since $\frac{a}{(\mu_n(a),\mu_p(a))} \in \mu$ for all $a \in S$, so by (3) we have $\frac{x_1ax_2}{(\mu_n(a),\mu_p(a))} \in \mu$. This implies that $\mu_n(x_1ax_2) \preceq \mu_n(a)$ and $\mu_p(x_1ax_2) \succeq \mu_p(a)$. This complete the proof. □

Clearly a BFS $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in)$-BFII of S if and only if the non-empty (s,t)-cut $C(\mu; (s,t))$ of $\mu = (S; \mu_n, \mu_p)$ is an interior ideal of an OS S for all $(s,t) \in [-1, 0) \times (0, 1]$.

4. $(\in, \in \curlyvee q)$-BFII

Lemma 4.1. *A is an interior ideal of an OS S if and only if* $\chi_A = (S, \chi_{nA}, \chi_{pA})$ *is* $(\in, \in \curlyvee q)$*-BFII of an OS S.*

Theorem 4.2. [8]*A BFS* $\mu = (S; \mu_n, \mu_p)$ *in S is an* $(\in, \in \curlyvee q)$*-bipolar fuzzy generalized bi-ideal of S if and only if it satisfies*

(1) $(\forall x_1, x_2 \in S)$ $(x_1 \preceq x_2 \Longrightarrow \mu_n(x_1) \preceq \bigvee\{\mu_n(x_2), -0.5\}$ and $\mu_p(x_1) \succeq \bigwedge\{\mu_p(x_2), 0.5\})$,
(2) $(\forall x_1, x_2, x_3 \in S)$ $(\mu_n(x_1x_2x_3) \preceq \bigvee\{\mu_n(x_1), \mu_n(x_3), -0.5\}$ and $\mu_p(x_1x_2x_3) \succeq \bigwedge\{\mu_p(x_1), \mu_p(x_3), 0.5\})$.

Theorem 4.3. *A BFS* $\mu = (S; \mu_n, \mu_p)$ *in S is an* $(\in, \in \curlyvee q)$*-BFII of an OS S if and only if it satisfies:*

(1) $(\forall x_1, x_2 \in S)$ $(x_1 \preceq x_2 \Longrightarrow \mu_n(x_1) \preceq \bigvee\{\mu_n(x_2), -0.5\}$ and $\mu_p(x_1) \succeq \bigwedge\{\mu_p(x_2), 0.5\})$,
(2) $(\forall x_1, x_2 \in S)$ $(\mu_n(x_1x_2) \preceq \bigvee\{\mu_n(x_1), \mu_n(x_2), -0.5\}$ and $\mu_p(x_1x_2) \succeq \bigwedge\{\mu_p(x_1), \mu_p(x_2), 0.5\})$,
(3) $(\forall x_1, a, x_2 \in S)$ $(\mu_n(x_1ax_2) \preceq \bigvee\{\mu_n(a), -0.5\}$ and $\mu_p(x_1ax_2) \succeq \bigwedge\{\mu_p(a), 0.5\})$.

Proof. The proof follows from theorem 4.2. □

Proposition 4.4. *Every* $(\in, \in \curlyvee q)$*-BFI of an OS S is an* $(\in, \in \curlyvee q)$*-BFII of S.*

Proof. Proof is straight forward. □

From example 4.5, it is clear that the converse of proposition 4.4 is not true.

Example 4.5. *Consider an OS* $S = \{0, 1, 2, 3\}$ *whose multiplication table and order relation are*

Table 1

·	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$\preceq := \{(0,0),(1,1),(2,2),(3,3),(0,1)\}$. *Let* $\mu = (S; \mu_n, \mu_p)$ *be a BFS in* S *defined by*

Table 2

S	0	1	2	3
μ_n	-0.8	-0.3	-0.5	-0.1
μ_p	0.7	0.4	0.6	0.2

Then clearly $\mu = (S; \mu_n, \mu_p)$ *is an* $(\in, \in \curlyvee q)$*-BFII of* S*, but* $\mu = (S; \mu_n, \mu_p)$ *is not an* $(\in, \in \curlyvee q)$*-BFI of* S*. Because if* $xyz = 0$*, then* $\mu_n(xyz) = \mu_n(0) = -0.8 \preceq \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(xyz) = \mu_p(0) = 0.7 \succeq \mu_p(y) \curlywedge 0.5$*. If* $xyz = 1$*, then* $\mu_n(xyz) = \mu_n(1) = -0.3 \preceq -0.1 = \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(xyz) = \mu_p(1) = 0.4 \succeq 0.2 = \mu_p(y) \curlywedge 0.5$*. If* $xy = 0$*, then* $\mu_n(xy) = \mu_n(0) = -0.8 \preceq \mu_n(x) \curlyvee \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(xy) = \mu_p(0) = 0.7 \succeq \mu_p(x) \curlywedge \mu_p(y) \curlywedge 0.5$*. If* $xy = 1$*, then* $\mu_n(xy) = \mu_n(1) = -0.3 \preceq -0.1 = \mu_n(x) \curlyvee \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(xy) = \mu_p(1) = 0.4 \succeq 0.2 = \mu_p(x) \curlywedge \mu_p(y) \curlywedge 0.5$*. If* $xy = 2$*, then* $\mu_n(xy) = \mu_n(2) = -0.5 \preceq -0.1 = \mu_n(x) \curlyvee \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(xy) = \mu_p(2) = 0.6 \succeq 0.2 = \mu_p(x) \curlywedge \mu_p(y) \curlywedge 0.5$*. Let* $x, y \in S$ *with* $x \preceq y$*, then* $\mu_n(x) \preceq \mu_n(y) \curlyvee -0.5$ *and* $\mu_p(x) \succeq \mu_p(y) \curlywedge 0.5$*. But* $\mu_p(2 \cdot 3) = \mu_p(1) = 0.4 \prec \mu_p(2) \curlywedge 0.5$*. Therefore,* $\mu = (S, \mu_n, \mu_p)$ *is not an* $(\in, \in \curlyvee q)$*-BFLI of an OS* S*. Hence* $\mu = (S; \mu_n, \mu_p)$ *is not an* $(\in, \in \curlyvee q)$*-BFI of an OS* S*.*

Proposition 4.6. *Let* S *be regular OS, then every* $(\in, \in \curlyvee q)$*-BFII is an* $(\in, \in \curlyvee q)$*-BFI of* S*.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee \mathrm{q})$-BFII and $a, b \in S$. Then $a \preceq axa$ for some $x \in S$, so

$$\begin{aligned}\mu_n(ab) &\preceq \mu_n((axa)b) \curlyvee -0.5 = \mu_n((ax)ab) \curlyvee -0.5\\ &\preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5\end{aligned}$$

and

$$\begin{aligned}\mu_p(ab) &\succeq \mu_p((axa)b) \curlywedge 0.5 = \mu_p((ax)ab) \curlywedge 0.5\\ &\succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5.\end{aligned}$$

Similarly, we can show that $\mu_n(ab) \preceq \mu_n(b) \curlyvee -0.5$ and $\mu_p(ab) \succeq \mu_p(b) \curlywedge 0.5$. Thus $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee \mathrm{q})$-BFI of an OS S. □

The following corollary is obtained from proposition 4.4 and 4.6.

Corollary 4.7. *The concept of an* $(\in,\in \curlyvee q)$*-BFI and* $(\in,\in \curlyvee q)$*-BFII coincide in regular OS.*

Proposition 4.8. *Let* S *be an intra-regular OS, then every* $(\in, \in \curlyvee q)$*-BFII is* $(\in, \in \curlyvee q)$*-BFI of an OS* S*.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFII and $a, b \in S$. Then $a \preceq xa^2y$ for some $x, y \in S$. So

$$\begin{aligned}\mu_n(ab) &\preceq \mu_n((xa^2y)b) \curlyvee -0.5 = \mu_n((xa)a(yb)) \curlyvee -0.5 \\ &\preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5\end{aligned}$$

and

$$\begin{aligned}\mu_p(ab) &\succeq \mu_p((xa^2y)b) \curlywedge 0.5 = \mu_p((xa)a(yb)) \curlywedge 0.5 \\ &\succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5.\end{aligned}$$

Similarly, we can show that $\mu_n(ab) \preceq \mu_n(b) \curlyvee -0.5$ and $\mu_p(ab) \succeq \mu_p(b) \curlywedge 0.5$ for every $a, b \in S$. Thus $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee q)$-BFI of an OS S. □

The following corollary is obtained from proposition 4.4 and 4.8.

Corollary 4.9. *The concept of* $(\in,\in \curlyvee q)$*-BFI and* $(\in,\in \curlyvee q)$*-BFII coincide in intra-regular OS.*

Proposition 4.10. *Let S be a semipimple OS, then every* $(\in, \in \curlyvee q)$*-BFII is* $(\in,\in \curlyvee q)$*-BFI of an OS S.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFII and $a, b \in S$. Then $a \preceq xayaz$ for some $x, y, z \in S$. So

$$\begin{aligned}\mu_n(ab) &\preceq \mu_n((xayaz)b) \curlyvee -0.5 = \mu_n((xay)a(zb)) \curlyvee -0.5 \\ &\preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5\end{aligned}$$

and

$$\begin{aligned}\mu_p(ab) &\succeq \mu_p((xayaz)b) \curlywedge 0.5 = \mu_p((xay)a(zb)) \curlywedge 0.5 \\ &\succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5.\end{aligned}$$

Similarly, $\mu_n(ab) \preceq \mu_n(b) \curlyvee -0.5$ and $\mu_p(ab) \succeq \mu_p(b) \curlywedge 0.5$ for every $a, b \in S$. Thus $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee q)$-BFI of an OS S. □

The following corollary is obtained rom proposition 4.4 and 4.10.

Corollary 4.11. *The concept of* $(\in,\in \curlyvee q)$*-BFI and* $(\in,\in \curlyvee q)$*-BFII coincide in semismple OS.*

Let $\mu = (S; \mu_n, \mu_p)$ be a BFS of an OS S, then $I_a \subseteq S$ and its definition is

$I_a = \{b \in S \mid \mu_n(b) \preceq \mu_n(a) \curlyvee -0.5$ and $\mu_p(b) \succeq \mu_p(a) \curlywedge 0.5\}$.

Proposition 4.12. *Let $\mu = (S; \mu_n, \mu_p)$ be an* $(\in, \in \curlyvee q)$*-BFRI of S. Then for every $a \in S$, I_a is right ideal of an OS S.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFRI of S and $a \in S$. Then $I_a \neq \emptyset$, because $a \in I_a$. Let $b \in I_a$ and $x \in S$. Since $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee q)$-BFRI of S, we have $\mu_n(bx) \preceq \mu_n(b) \curlyvee -0.5$ and $\mu_p(bx) \succeq \mu_p(b) \curlywedge 0.5$. Since $b \in I_a$, we have $\mu_n(b) \preceq \mu_n(a) \curlyvee -0.5$ and $\mu_p(b) \succeq \mu_p(a) \curlywedge 0.5$. Thus $\mu_n(bx) \preceq \mu_n(b) \curlyvee -0.5 \preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5$ and $\mu_p(bx) \succeq \mu_p(b) \curlywedge 0.5 \succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5$. Hence $bx \in I_a$.

Let $y \in I_a$ and $x \preceq y$. Since μ is $(\in, \in \curlyvee q)$-BFRI of S, we have $\mu_n(x) \preceq \mu_n(y) \curlyvee -0.5$ and $\mu_p(x) \succeq \mu_p(y) \curlywedge 0.5$. Thus $\mu_n(x) \preceq \mu_n(y) \curlyvee -0.5 \preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5$ and $\mu_p(x) \succeq \mu_p(y) \curlywedge 0.5 \succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5$, therefore $x \in I_a$. Hence I_a is right ideal of an OS S for every $a \in S$. □

Proposition 4.13. *Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFLI of an OS S. Then I_a is a left ideal of an OS S for every $a \in S$.*

From Proposition 4.12 and 4.13, we have:

Proposition 4.14. *Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFI of an OS S. Then I_a is an ideal of an OS S for every $a \in S$*

5. Lower and upper part of $(\in, \in \curlyvee q)$-BFII

Definition 5.1. *Let S be an OS and $\mu = (S; \mu_n, \mu_p)$ be BFS in S. We define the upper part $\mu^+ = (S; \mu_n^+, \mu_p^+)$ of $\mu = (S; \mu_n, \mu_p)$ as follows; $\mu_n^+(x) = \mu_n(x) \curlywedge -0.5$ and $\mu_p^+(x) = \mu_p(x) \curlyvee 0.5$. Similarly we define the lower part $\mu^- = (S; \mu_n^-, \mu_p^-)$ of $\mu = (S; \mu_n, \mu_p)$ as follows; $\mu_n^-(x) = \mu_n(x) \curlyvee -0.5$ and $\mu_p^-(x) = \mu_p(x) \curlywedge 0.5$.*

Lemma 5.2. [8]*Let $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ be BFSs of an OS S. Then*

(1) $(\mu \curlywedge \nu)^- = (\mu^- \curlywedge \nu^-)$.
(2) $(\mu \curlyvee \nu)^- = (\mu^- \curlyvee \nu^-)$.
(3) $(\mu \circ \nu)^- = (\mu^- \circ \nu^-)$.

Lemma 5.3. [8]*Let $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ be BFSs of an OS S. Then the following hold:*

(1) $(\mu \curlywedge \nu)^+ = (\mu^+ \curlywedge \nu^+)$.
(2) $(\mu \curlyvee \nu)^+ = (\mu^+ \curlyvee \nu^+)$.
(3) $(\mu \circ \nu)^+ \succeq (\mu^+ \circ \nu^+)$.
If $A_x \neq \emptyset$ then $(\mu \circ \nu)^+ = (\mu^+ \circ \nu^+)$

Definition 5.4. *Let A be a nonempty subsets of an OS S. Then the upper part $\chi_A^+ = (S; \chi_{nA}^+, \chi_{pA}^+)$ of the characteristic function $\chi_A = (S; \chi_{nA}, \chi_{pA})$ of A is defined by*

$$\chi_{nA}^+(x) = \begin{cases} -1 \text{ if } x \in A, \\ -0.5 \text{ if } x \notin A, \end{cases} \text{ and } \chi_{pA}^+(x) = \begin{cases} 1 \text{ if } x \in A, \\ 0.5 \text{ if } x \notin A. \end{cases}$$

Similarly we can define the lower part $\chi_A^- = (S; \chi_{nA}^-, \chi_{pA}^-)$ of the characteristic function $\chi_A = (S; \chi_{nA}, \chi_{pA})$ of A by

$$\chi_{nA}^-(x) = \begin{cases} -0.5 \text{ if } x \in A, \\ 0 \text{ if } x \notin A, \end{cases} \text{ and } \chi_{pA}^-(x) = \begin{cases} 0.5 \text{ if } x \in A, \\ 0 \text{ if } x \notin A. \end{cases}$$

Lemma 5.5. [8]*Let S be an OS and A, B are nonempty subsets of S. Then*
(1) $(\chi_A \curlywedge \chi_B)^- = \chi_{A\cap B}^-$.
(2) $(\chi_A \curlyvee \chi_B)^- = \chi_{A\cup B}^-$.
(3) $(\chi_A \circ \chi_B)^- = \chi_{(AB]}^-$.

Lemma 5.6. *The lower part of χ_A^- is an $(\in, \in \curlyvee q)$-BFII of an OS S if and only if A is an interior ideal of an OS S.*

Proof. The proof follows from Lemma 4.1. □

Lemma 5.7. *The lower part of χ_A^- is an $(\in, \in \curlyvee q)$-BFI of an OS S if and only if A is an ideal of an OS S.*

Proposition 5.8. *Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee q)$-BFII of an OS S, then $\mu^- = (S; \mu_n^-, \mu_p^-)$ is a BFII of an OS S.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee$q)-BFII of S. Then $\forall\, x_1, x_2 \in S$, we get $\mu_n(x_1x_2) \preceq (\mu_n(x_1) \curlyvee \mu_n(x_2) \curlyvee -0.5)$ and $\mu_p(x_1x_2) \succeq (\mu_p(x_1) \curlywedge \mu_p(x_2) \curlywedge 0.5)$. So $\mu_n(x_1x_2) \curlyvee -0.5 \preceq (\mu_n(x_1) \curlyvee \mu_n(x_2) \curlyvee -0.5) \curlyvee -0.5 = (\mu_n(x_1) \curlyvee -0.5) \curlyvee (\mu_n(x_2) \curlyvee -0.5)$ and $\mu_p(x_1x_2) \curlywedge 0.5 \preceq (\mu_p(x_1) \curlywedge \mu_p(x_2) \curlywedge 0.5) \curlywedge 0.5 = (\mu_p(x_1) \curlywedge 0.5) \curlywedge (\mu_p(x_2) \curlywedge 0.5)$. Hence $\mu_n^-(x_1x_2) \preceq \mu_n^-(x_1) \curlyvee \mu_n^-(x_2)$ and $\mu_p^-(x_1x_2) \succeq \mu_p^-(x_1) \curlywedge \mu_p^-(x_2)$. Since $\mu_n(x_1ax_2) \preceq \mu_n(a) \curlyvee -0.5$ and $\mu_p(x_1ax_2) \succeq \mu_p(a) \curlywedge 0.5$. So $\mu_n(x_1ax_2) \curlyvee -0.5 \preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5$ and $\mu_p(x_1ax_2) \curlywedge 0.5 \succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5$. Hence $\mu_n^-(x_1ax_2) \preceq \mu_n^-(a)$ and $\mu_p^-(x_1ax_2) \succeq \mu_p^-(a)$. Let $x_1, x_2 \in S$ with $x_1 \preceq x_2$. Then $\mu_n(x_1) \preceq \mu_n(x_2) \curlyvee -0.5$ and $\mu_n(x_1) \succeq \mu_p(x_2) \curlywedge 0.5$. Thus $\mu_n(x_1) \curlyvee -0.5 \preceq (\mu_n(x_2) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(x_2) \curlyvee -0.5$ and $\mu_n(x_1) \curlywedge 0.5 \succeq (\mu_p(x_2) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(x_2) \curlywedge 0.5$. This implies that $\mu_n^-(x_1) \preceq \mu_n^-(x_2)$ and $\mu_p^-(x_1) \succeq \mu_p^-(x_2)$. Hence $\mu^- = (S; \mu_n^-, \mu_p^-)$ is a BFII of an OS S. □

Definition 5.9. *If every $(\in, \in \curlyvee q)$-BFI in an OS S is constant, that is for every $x, y \in S$, we get $\mu^-(x) = \mu^-(y)$ then such OS is called $(\in, \in \curlyvee q)$-bipolar fuzzy simple.*

Theorem 5.10. *An OS S is simple if and only if it is $(\in, \in \curlyvee q)$-bipolar fuzzy simple.*

Proof. Suppose $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee \text{q})$-BFI of an OS S and $x, y \in S$. Then I_x is an ideal of an OS S by proposition 4.14. As we are given that an OS S is simple, so $I_x = S$ thus $y \in I_x$. Hence $\mu_n(y) \preceq \mu_n(x) \curlyvee -0.5$ and $\mu_p(y) \succeq \mu_p(x) \curlywedge 0.5$, so $\mu_n^-(y) = \mu_n(y) \curlyvee -0.5 \preceq \mu_n(x) \curlyvee -0.5 = \mu_n^-(x)$ and $\mu_p^-(y) = \mu_p(y) \curlywedge 0.5 \succeq \mu_p(x) \curlywedge 0.5 = \mu_p^-(x)$. Thus $\mu^-(y) \succeq \mu^-(x)$. Similarly we get $\mu^-(y) \preceq \mu^-(x)$. Hence $\mu^-(y) = \mu^-(x)$ and S is an $(\in, \in \curlyvee \text{q})$-bipolar fuzzy simple.

Conversely, assume that S has a proper ideal I. Then χ_I^- is $(\in, \in \curlyvee \text{q})$-BFI of an OS S by lemma 5.7. Let $a \in S$ then χ_I^- is constant function because an OS S is $(\in, \in \curlyvee \text{q})$-bipolar fuzzy simple, So that is, $\chi_I^-(a) = \chi_I^-(b)$ for every $b \in S$. Thus, for any $x \in I$, we have $\chi_{nI}^-(a) = \chi_{nI}^-(x) = -0.5$ and $\chi_{pI}^-(a) = \chi_{pI}^-(x) = 0.5$ and so $a \in I$. We get $S = I$, which is a contradiction. Hence, an OS S is simple. □

Lemma 5.11. [19] *For every $x \in S$, we have $S = (SxS]$ if and only if S is simple OS.*

Theorem 5.12. *An OS S is simple if and only if for every $(\in, \in \curlyvee q)$-BFII of an OS S, we have $\mu^-(x_1) = \mu^-(x_2)$ for every $x_1, x_2 \in S$.*

Proof. Let an OS S is simple and $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee \text{q})$-BFII of S. Now by Lemma 5.11, $S = (Sx_2S]$ because $x_2 \in S$. Also as $x_1 \in S$, so $x_1 \in (Sx_2S]$. Therefore $x_1 \preceq ax_2b$ for $a, b \in S$. As we are given that $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee \text{q})$-BFII of S, so we have $\mu_n(x_1) \preceq \mu_n(ax_2b) \curlyvee -0.5 \preceq \mu_n(x_2) \curlyvee -0.5$ and $\mu_p(x_1) \succeq \mu_p(ax_2b) \curlywedge 0.5 \succeq \mu_p(x_2) \curlywedge 0.5$. Hence $\mu_n^-(x_1) = \mu_n(x_1) \curlyvee -0.5 \preceq \mu_n(x_2) \curlyvee -0.5 = \mu_n^-(x_2)$ and $\mu_p^-(x_1) = \mu_p(x_1) \curlywedge 0.5 \succeq \mu_p(x_2) \curlywedge 0.5 = \mu_p^-(x_2)$. Thus $\mu^-(x_2) \preceq \mu^-(x_1)$. In similar way, we can show easily that $\mu^-(x_2) \succeq \mu^-(x_1)$. Therefore $\mu^-(x_1) = \mu^-(x_2)$ for every $x_1, x_2 \in S$.

Conversely, assume that $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee \text{q})$-BFII of S, then by Proposition 4.10, $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee \text{q})$-BFI of an OS S. As we are given that, $\mu^-(x_1) = \mu^-(x_2)$ for every $x_1, x_2 \in S$. So by definition 5.9, S is an $(\in, \in \curlyvee \text{q})$-bipolar fuzzy simple and hence S is simple. □

Theorem 5.13. *For every $(\in, \in \curlyvee q)$-BFII $\mu = (S; \mu_n, \mu_p)$ of an OS S, we get $\mu^-(a) = \mu^-(a^2)$ for all $a \in S$ if and only if S be an intra-regular OS.*

Proof. Let $\mu = (S; \mu_n, \mu_p)$ be an $(\in, \in \curlyvee$q)-BFI of S and let $a \in S$. Then there exist $x, y \in S$ such that $a \preceq xa^2y$. Therefore

$$\begin{aligned}\mu_n(a) &\preceq (\mu_n(xa^2y) \curlyvee -0.5) = \mu_n(x(a^2y)) \curlyvee -0.5 \\ &\preceq (\mu_n(a^2y) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a^2) \curlyvee -0.5.\end{aligned}$$

So $\mu_n^-(a) = \mu_n(a) \curlyvee -0.5 \preceq (\mu_n(a^2) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a^2) \curlyvee -0.5 = \mu_n^-(a^2)$ and $\mu_n^-(a^2) = (\mu_n(a^2) \curlyvee -0.5) \preceq (\mu_n(a) \curlyvee -0.5) \curlyvee -0.5 = \mu_n(a) \curlyvee -0.5 = \mu_n^-(a)$. Thus $\mu_n^-(a) = \mu_n^-(a^2)$. Also

$$\begin{aligned}\mu_p(a) &\succeq (\mu_p(xa^2y) \curlywedge 0.5) = \mu_p(x(a^2y)) \curlywedge 0.5 \\ &\succeq (\mu_p(a^2y) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a^2) \curlywedge 0.5.\end{aligned}$$

So $\mu_p^-(a) = \mu_n(a) \curlywedge 0.5 \succeq (\mu_p(a^2) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a^2) \curlywedge 0.5 = \mu_p^-(a^2)$ and $\mu_p^-(a^2) = (\mu_p(a^2) \curlywedge 0.5) \succeq (\mu_p(a) \curlywedge 0.5) \curlywedge 0.5 = \mu_p(a) \curlywedge 0.5 = \mu_p^-(a)$. Thus $\mu_p^-(a) = \mu_p^-(a^2)$. Hence $\mu^-(a) = \mu^-(a^2)$.

Conversely, let us assume an ideal generated by x^2, that is $I(x^2) = (x^2 \cup Sx^2 \cup x^2S \cup Sx^2S]$. Then $\chi^-_{I(x^2)}$ is an $(\in, \in \curlyvee$q)-BFI of an OS S, by Lemma 5.7. Also by given information, we have $\chi^-_{I(x^2)}(x) = \chi^-_{I(x^2)}(x^2)$. Therefore we have $x \in I(x^2) = (x^2 \cup Sx^2 \cup x^2S \cup Sx^2S]$. This implies that if $a, b \in S$ then we have $x \preceq x^2$ or $x \preceq ax^2$ or $x \preceq x^2a$ or $x \preceq ax^2b$. In all cases we concluded that $x \in (Sx^2S]$. Hence S is intra-regular OS. □

Theorem 5.14. *For an intraregular OS S, we have $\mu^-(xy) = \mu^-(yx)$ for every $x, y \in S$ where $\mu = (S; \mu_n, \mu_p)$ is an $(\in, \in \curlyvee q)$-BFI of S.*

Proof. Let $x, y \in S$. Then we have

$$\mu^-(xy) = \mu^-((xy)^2) = \mu^-((xy)(xy)) = \mu^-(x(yx)y) \succeq \mu^-(ya)$$

by Theorem 5.13. By the symmetry, we have $\mu^-(yx) \succeq \mu^-(xy)$. Thus $\mu^-(xy) = \mu^-(yx)$ for every $x, y \in S$. □

Theorem 5.15. *If S is semisimple OS and $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ are $(\in, \in \curlyvee q)$-BFIIs of S. Then $(\mu \circ \nu)^- \preceq (\mu \cap \nu)^-$.*

Proof. If $A_a = \emptyset$, then $(\mu_n \circ \nu_n)^-(a) = (\mu_n \circ \nu_n)(a) \curlyvee -0.5 = 0 \curlyvee -0.5 = 0 \succeq (\mu_n \curlyvee \nu_n)(a) \curlyvee -0.5 = (\mu_n \curlyvee \nu_n)^-(a)$ and $(\mu_p \circ \nu_p)^-(a) = (\mu_p \circ \nu_p)(a) \curlywedge 0.5 = 0 \curlywedge 0.5 = 0 \preceq (\mu_p \curlywedge \nu_p)(a) \curlywedge 0.5 = (\mu_p \curlywedge \nu_p)^-(a)$. Thus $(\mu \circ \nu)^- \preceq (\mu \cap \nu)^-$.

Let $A_a \neq \emptyset$, then

$$\begin{aligned}(\mu_n \circ \nu_n)^-(a) &= (\mu_n \circ \nu_n)(a) \curlyvee -0.5 \\ &= \left[\bigwedge_{(y,z)\in A_a} (\mu_n(y) \curlyvee \nu_n(z))\right] \curlyvee -0.5 \\ &= \bigwedge_{(y,z)\in A_a} (\mu_n(y) \curlyvee \nu_n(z) \curlyvee -0.5) \\ &= \bigwedge_{(y,z)\in A_a} ((\mu_n(y) \curlyvee -0.5) \curlyvee (\nu_n(z) \curlyvee -0.5) \curlyvee -0.5).\end{aligned}$$

Since S is semisimple and $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ are $(\in, \in \curlyvee q)$-BFIIs of S, so by Proposition 4.10, we have μ and ν are $(\in, \in \curlyvee q)$-BFIs of S. Since $a \preceq yz$, we have $\mu_n(a) \preceq \mu_n(yz) \curlyvee -0.5 \preceq \mu_n(y) \curlyvee -0.5$ and $\nu_n(a) \preceq \nu_n(yz) \curlyvee -0.5 \preceq \nu_n(z) \curlyvee -0.5$. Thus

$$\begin{aligned}(\mu_n \circ \nu_n)^-(a) &= \bigwedge_{(y,z)\in A_a} ((\mu_n(y) \curlyvee -0.5) \curlyvee (\nu_n(z) \curlyvee -0.5) \curlyvee -0.5) \\ &\succeq (\mu_n(a) \curlyvee \nu_n(a) \curlyvee -0.5) = (\mu_n \curlyvee \nu_n)(a) \curlyvee -0.5 \\ &= (\mu_n \curlyvee \nu_n)^-(a).\end{aligned}$$

In similar way, we can easily show that $(\mu_p \circ \nu_p)^-(a) \preceq (\mu_p \curlywedge \nu_p)^-(a)$. Thus we conclude that $(\mu \circ \nu)^- \preceq (\mu \cap \nu)^-$. □

Theorem 5.16. *An OS S is semisimple if and only if for every $(\in, \in \curlyvee q)$-BFIIs $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ of S, we have $(\mu \cap \nu)^- = (\mu \circ \nu)^-$.*

Proof. Let $(\mu \cap \nu)^- = (\mu \circ \nu)^-$ for every $(\in, \in \curlyvee q)$-BFIIs $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ of S. Let us assume that A be an interior ideal of an OS S, then χ_A^- is an $(\in, \in \curlyvee q)$-BFII of an OS S, by Lemma 5.6. We have $\chi_A^- = \chi_A^- \cap \chi_A^- = \chi_A^- \circ \chi_A^- = \chi_{(A^2]}^-$ using Lemma 5.5 (3). Thus $A = (A^2]$ and S is semisimple.

Conversely, let S is semisimple and $\mu = (S; \mu_n, \mu_p)$ and $\nu = (S; \nu_n, \nu_p)$ are $(\in, \in \curlyvee q)$-BFIIs of S. So for each $a \in S$, there exist $x, y, z \in S$ such that $a \preceq xayaz \preceq (xay))(xayaz^2)$. Thus $(xay, xayaz^2) \in A_a$ and $A_a \neq \emptyset$. So

$$\begin{aligned}(\mu_n \circ \nu_n)^-(a) = (\mu_n \circ \nu_n)(a) \curlyvee -0.5 &= \bigwedge_{(m,n)\in A_a} (\mu_n(m) \curlyvee \nu_n(n) \curlyvee -0.5) \\ &\preceq \mu_n(xay) \curlyvee \nu_n((xay)a(z)^2) \curlyvee -0.5 \\ &\preceq (\mu_n(a) \curlyvee -0.5) \curlyvee (\nu_n(a) \curlyvee -0.5) \curlyvee -0.5 \\ &\preceq (\mu_n \curlyvee \nu_n)(a) \curlyvee -0.5 = (\mu_n \curlyvee \nu_n)^-(a).\end{aligned}$$

Also

$$\begin{aligned}(\mu_p \circ \nu_p)^-(a) = (\mu_p \circ \nu_p)(a) \curlywedge 0.5 &= \bigvee_{(m,n)\in A_a} (\mu_p(m) \curlywedge \nu_p(n) \curlywedge 0.5) \\ &\succeq \mu_p(xay) \curlywedge \nu_p((xay)a(z)^2) \curlywedge 0.5 \\ &\succeq (\mu_p(a) \curlywedge 0.5) \curlywedge (\nu_p(a) \curlywedge 0.5) \curlywedge 0.5 \\ &\succeq (\mu_p \curlywedge \nu_p)(a) \curlywedge 0.5 = (\mu_p \curlywedge \nu_p)^-(a).\end{aligned}$$

Thus $(\mu \cap \nu)^- \preceq (\mu \circ \nu)^-$. Also by Proposition 5.15, we get $(\mu \cap \nu)^- \succeq (\mu \circ \nu)^-$ and so we get the require result $(\mu \cap \nu)^- = (\mu \circ \nu)^-$. □

By combining the result of Proposition 5.17 and remark 5.16, we get the following:

Theorem 5.17. *An OS S is semisimple if and only if for every $(\in, \in \curlyvee q)$-BFII $\mu = (S; \mu_n, \mu_p)$ of S, we have $(\mu \circ \mu)^- = \mu^-$.*

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