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ON ALMOST ALPHA-COSYMPLECTIC MANIFOLDS WITH SOME NULLITY DISTRIBUTIONS

Hakan Öztürk

Abstract. The object of the paper is to investigate almost alpha-cosymplectic (κ, μ, ν) spaces. Some results on almost alpha-cosymplectic (κ, μ, ν) spaces with certain conditions are obtained. Finally, we give an example on 3-dimensional case.

1. Introduction

It is well known that there exist contact metric manifolds M^{2n+1} whose curvature tensor R and the direction of the characteristic vector field ξ holds $R(X,Y)\xi = 0$ for any vector fields on M^{2n+1}. Using a D-homothetic deformation to a contact metric manifold with $R(X,Y)\xi = 0$, we get a contact metric manifold satisfying the following special condition

$$R(X,Y)\xi = \eta(Y)(\kappa I + \mu h)X - \eta(X)(\kappa I + \mu h)Y, \tag{1}$$

where κ, μ are constants and h is the self-adjoint $(1,1)$-type tensor field. This condition is called (κ, μ)-nullity on M^{2n+1}. Contact metric manifolds with (κ, μ)-nullity condition studied for κ, μ =const. in [1], [2]. In [2], the author introduced contact metric manifold whose characteristic vector field belongs to the (κ, μ)-nullity condition and proved that non-Sasakian contact metric manifold was completely determined locally by its dimension for the constant values of κ and μ.

Koufogiorgos and Tsichlias found a new class of 3-dimensional contact metric manifolds that κ and μ are non-constant smooth functions. They

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generalized (κ, μ)-contact metric manifolds on non-Sasakian manifolds for $n > 1$, where the functions κ, μ are constants, see [12].

Also, Olszak and Dacko extensively studied almost cosymplectic (κ, μ, ν) manifolds. These manifolds whose almost cosymplectic structures (ϕ, ξ, η, g) holds the condition

$$(2) \quad R(X,Y)\xi = \eta(Y)(\kappa I + \mu h + \nu\phi h)X - \eta(X)(\kappa I + \mu h + \nu\phi h)Y,$$

for $\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1})$, where $\mathcal{R}_\eta(M^{2n+1})$ be the subring of the ring of smooth functions f on M^{2n+1} such that $df \wedge \eta = 0$, see [11]. Such manifolds are called almost cosymplectic (κ, μ, ν)-spaces. The condition (2) is invariant with respect to the D-homothetic deformations of these structures. The authors show that the integral submanifolds of the distribution $\mathcal{D}$ of such manifolds are locally flat Keahlerian manifolds and give a new characterization which is established up to a D-homothetic deformation of the almost cosymplectic manifolds. In [10], a complete local description of almost cosymplectic $(-1, \mu, 0)$-spaces via "model spaces" is given depending on the function μ. When μ is constant, the models are Lie groups with a left-invariant almost cosymplectic structure.

Furthermore, the curvature properties of almost Kenmotsu manifolds with special attention to (κ, μ)-nullity condition for $\kappa, \mu =$const. and $\nu = 0$ are studied by Dileo and Pastore, see [4], [3]. In [4], the authors prove that an almost Kenmotsu manifolds M^{2n+1} is locally a warped product of an almost Kaehler manifold and an open interval. If additionally M^{2n+1} is locally symmetric then it is locally isometric to the hyperbolic space H^{2n+1} of constant sectional curvature $c = -1$. We recall that model spaces for almost cosymplectic case are given in [11], however illustrative examples are not sufficiently available in the literature for an almost α-cosymplectic manifold satisfying (2) with non-constant smooth functions.

Section 2 is devoted to preliminaries on almost α-cosymplectic manifolds. In section 3 the notion of almost α-cosymplectic (κ, μ, ν)-spaces in terms of a specific curvature condition are studied. Finally, in section 4 we investigate the existence of almost α-cosymplectic (κ, μ, ν)-space in 3-dimensional case and construct an example on such 3-dimensional (κ, μ)-space.

2. Preliminaries

An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α being a non-zero real constant.

Geometrical properties and examples of almost α-Kenmotsu manifolds are studied in [14], [6], [15]. An almost Kenmotsu metric structure (ϕ, ξ, η, g) is given by the deformed structure

$$\eta^{'} = \frac{1}{\alpha}\eta,\ \xi^{'} = \alpha\xi,\ \phi^{'} = \phi,\ g^{'} = \frac{1}{\alpha^2}g,\ \alpha \neq 0,\ \alpha \in \mathbb{R},$$

where α is a non-zero real constant. So we get an almost α-Kenmotsu structure $(\phi^{'}, \xi^{'}, \eta^{'}, g^{'})$. This deformation is called a homothetic deformation, see [14], [15]. It is important to note that almost α-Kenmotsu structures are related to some special local conformal deformations of almost cosymplectic structures, see [6].

If we combine these two classes, we obtain a new notion defined by $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$ called almost α-cosymplectic manifold for any real number α, see [14]. Obviously, a normal almost α-cosymplectic manifold is an α-cosymplectic manifold. An α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu $(\alpha \neq 0)$ for $\alpha \in \mathbb{R}$.

Let M^{2n+1} be an almost α-cosymplectic manifold and

$$\mathcal{D} = \{X : \eta(X) = 0\}.$$

Since the 1-form is closed, we have $\mathcal{L}_\xi \eta = 0$ and $[X, \xi] \in \mathcal{D}$ for any $X \in \mathcal{D}$. The Levi-Civita connection satisfies $\nabla_\xi \xi = 0$ and $\nabla_\xi \phi \in \mathcal{D}$, which implies that $\nabla_\xi X \in \mathcal{D}$ for any $X \in \mathcal{D}$.

Now, we set $A = -\nabla\xi$ and $h = \frac{1}{2}\mathcal{L}_\xi \phi$ for any vector fields X on M^{2n+1} where α is a smooth function such that $d\alpha \wedge \eta = 0$. Obviously, $A(\xi) = 0$ and $h(\xi) = 0$. Moreover, the following relations are held

$$\nabla_X \xi = -\alpha\phi^2 X - \phi h X, \tag{3}$$

$$(\phi \circ h)X + (h \circ \phi)X = 0, \quad (\phi \circ A)X + (A \circ \phi)X = -2\alpha\phi, \tag{4}$$

$$(\nabla_X \eta)Y = \alpha\left[g(X,Y) - \eta(X)\eta(Y)\right] + g(\phi Y, hX), \tag{5}$$

$$\delta\eta = -2\alpha n, \quad tr(h) = 0, \tag{6}$$

Besides , we have

$$(\nabla_X \phi)Y + (\nabla_{\phi X}\phi)\phi Y = -\alpha\eta(Y)\phi X + 2\alpha g(\phi X, Y)\xi - \eta(Y)hX, \tag{7}$$

for any vector fields X, Y on M^{2n+1}, see [14].

Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. We denote the curvature tensor and Ricci tensor of g by R and S respectively. We define a self adjoint operator $l = R(., \xi)\xi$ called the Jacobi operator with respect to ξ. Then we have the curvatures relations

$$(8) \quad R(X,Y)\xi = (\nabla_Y \phi h)X - (\nabla_X \phi h)Y - \alpha\left[\eta(X)\phi hY - \eta(Y)\phi hX\right] + \left[\alpha^2 + \xi(\alpha)\right]\left[\eta(X)Y - \eta(Y)X\right],$$

$$(9) \quad lX = \left[\alpha^2 + \xi(\alpha)\right]\phi^2 X + 2\alpha\phi hX - h^2 X + \phi(\nabla_\xi h)X,$$

$$(10) \quad lX - \phi l\phi X = 2\left[(\alpha^2 + \xi(\alpha))\phi^2 X - h^2 X\right],$$

$$(11) \quad (\nabla_\xi h)X = -\phi lX - \left[\alpha^2 + \xi(\alpha)\right]\phi X - 2\alpha hX - \phi h^2 X,$$

$$(12) \quad S(X,\xi) = -2n\left[\alpha^2 + \xi(\alpha)\right]\eta(X) - (\mathrm{div}(\phi h))X,$$

$$(13) \quad S(\xi,\xi) = -\left[2n(\alpha^2 + \xi(\alpha)) + tr(h^2)\right],$$

(14)

$$g(R_{\xi X}Y,Z) - g(R_{\xi X}\phi Y,\phi Z) + g(R_{\xi\phi X}Y,\phi Z) + g(R_{\xi\phi X}\phi Y,Z) = 2(\nabla_{hX}\Phi)(Y,Z) + 2(\alpha^2 + \xi(\alpha))\left[\eta(Y)g(X,Z) - \eta(Z)g(X,Y)\right] - 2\alpha\eta(Y)g(\phi hX,Z) + 2\alpha\eta(Z)g(\phi hX,Y),$$

for any vector fields X, Y, Z on M^{2n+1} where α is a smooth function such that $d\alpha \wedge \eta = 0$, see [9].

Corollary 2.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. If the equation $\xi(\alpha) = 0$ holds then α is a constant function such that $d\alpha \wedge \eta = 0$. It follows that α is parallel along the characteristic vector field ξ. Thus throughout the paper, we will accept it in this context.*

3. (κ, μ, ν)-Spaces

In this section, we are especially interested in almost α-cosymplectic manifolds whose almost α-cosymplectic structure (ϕ, ξ, η, g) satisfies the condition (2) for $\kappa, \mu, \nu \in \mathcal{R}_\eta(M^{2n+1})$. Such manifolds are said to be almost α-cosymplectic (κ, μ, ν)-spaces.

Proposition 3.1. *The following relations are held on almost α-cosymplectic (κ, μ, ν)-space*

$$(15) \quad l = -\kappa\phi^2 + \mu h + \nu\phi h,$$

$$(16) \quad l\phi - \phi l = 2\mu h\phi + 2\nu h,$$

$$(17) \quad h^2 = (\kappa + \alpha^2)\phi^2, \ \kappa \leq -\alpha^2,$$

$$(\nabla_\xi h) = -\mu\phi h + (\nu - 2\alpha)h, \tag{18}$$

$$\nabla_\xi h^2 = 2(\nu - 2\alpha)(\kappa + \alpha^2)\phi^2, \tag{19}$$

$$\xi(\kappa) = 2(\nu - 2\alpha)(\kappa + \alpha^2), \tag{20}$$

$$\begin{aligned} R(\xi, X)Y &= \kappa(g(Y,X)\xi - \eta(Y)X) + \mu(g(hY,X)\xi \\ &\quad -\eta(Y)hX) + \nu(g(\phi hY, X)\xi - \eta(Y)\phi hX), \end{aligned} \tag{21}$$

$$Q\xi = 2n\kappa\xi, \tag{22}$$

$$(\nabla_X \phi)Y = g(\alpha\phi X + hX, Y)\xi - \eta(Y)(\alpha\phi X + hX), \tag{23}$$

$$\begin{aligned} (\nabla_X \phi h)Y - (\nabla_Y \phi h)X &= -(\kappa + \alpha^2)(\eta(Y)X - \eta(X)Y) - \mu\eta(Y)hX \\ &\quad +\mu\eta(X)hY) + (\alpha - \nu)(\eta(Y)\phi hX - \eta(X)\phi hY), \end{aligned} \tag{24}$$

$$\begin{aligned} (\nabla_X h)Y - (\nabla_Y h)X &= (\kappa + \alpha^2)(\eta(Y)\phi X - \eta(X)\phi Y \\ &\quad +2g(\phi X, Y)\xi) + \mu(\eta(Y)\phi hX - \eta(X)\phi hY) \\ &\quad +(\alpha - \nu)(\eta(Y)hX - \eta(X)hY), \end{aligned} \tag{25}$$

for all vector fields X, Y on M^{2n+1}.

Proof. From (2) we get

$$lX = R(X,\xi)\xi = \kappa(X - \eta(X)\xi) + \mu hX + \nu\phi hX. \tag{26}$$

Replacing X by ϕX in (26), it gives

$$l\phi X = \kappa\phi X + \mu\phi hX + \nu\phi^2 hX.$$

Thus we have

$$l\phi X - \phi lX = \mu(h\phi X - \phi hX) - 2\nu\phi^2 hX,$$

so it completes the proof of (16). By using (26) we deduce

$$\phi l\phi X = \phi\kappa\phi X + \phi\mu\phi hX + \phi\nu\phi^2 hX. \tag{27}$$

Taking into account (26) and (27) we get

$$lX - \phi l\phi X = -2\kappa\phi^2 X.$$

Again using (26) we have

$$-2\kappa\phi^2 X = 2\alpha^2\phi^2 X - 2h^2 X,$$

which gives the proof of (17). Moreover, differentiating (17) along ξ we get

$$(\nabla_\xi h)X = -\kappa\phi X - \mu\phi hX + \nu hX - \alpha^2\phi X - 2\alpha hX + (\kappa + \alpha^2)\phi X.$$

Alternately, using (17), we obtain

$$\nabla_\xi h^2 = 2(\nu - 2\alpha)h^2 X.$$

The proof of (19) is obvious from (18). Then differentiating (19) along ξ we find

$$2(\nu - 2\alpha)(\kappa + \alpha^2)\phi^2 X = [\xi(\kappa)]\,\phi^2 X.$$

Since $g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X)$, we have

$$\begin{aligned} g(R(Y,Z)\xi,X) =& \kappa(\eta(Z)g(Y,X) - \eta(Y)g(Z,X)) + \mu(\eta(Z)g(hY,X) \\ & -\eta(Y)g(hZ,X)) + \nu(\eta(Z)g(\phi hY,X) - \eta(Y)g(\phi hZ,X)). \end{aligned}$$

The last equation completes the proof of (21). Contracting (21) with respect to X, Y and using the definition of Ricci tensor, we obtain

$$S(\xi, Z) = \sum_{i=1}^{2n+1} g(R(\xi, E_i)E_i, Z) = 2n\kappa\eta(Z),$$

for any vector field Z. Thus (22) is clear. In addition, (22) implies that

$$\begin{aligned} g(R_{\xi X}Y, Z) =& \kappa\,[g(X,Y)\eta(Z) - \eta(Y)g(X,Z)] + \mu g(hX,Y)\eta(Z) \\ & - \mu\eta(Y)g(hX,Z) + \nu\,[g(\phi hY,X)\eta(Z) - \eta(Y)g(\phi hX,Z)]\,. \end{aligned}$$

Summing the left side of (14) with the help of the above equation for $\xi(\alpha) = 0$, then we deduce

$$-2\kappa\,[\eta(Y)g(X,Z) - \eta(Z)g(X,Y)]\,.$$

Thus (14) reduces to

$$\begin{aligned} & - 2\kappa\,[\eta(Y)g(X,Z) - \eta(Z)g(X,Y)] \\ =& 2(\nabla_{hX}\Phi)(Y,Z) + 2\alpha^2\eta(Y)g(X,Z) - 2\alpha^2\eta(Z)g(X,Y) \\ & - 2\alpha\eta(Y)g(\phi hX,Z) + 2\alpha\eta(Z)g(\phi hX,Y) + 2\alpha\eta(Z)g(\phi hX,Y). \end{aligned}$$

Also, we have

$$\begin{aligned} -(\nabla_{hX}\Phi)(Y,Z) \quad = \quad & (\kappa + \alpha^2)\,[\eta(Y)g(X,Z) - \eta(Z)g(X,Y)] \\ & -\alpha\,[\eta(Y)g(\phi hX,Z) - \eta(Z)g(\phi hX,Y)]\,. \end{aligned} \tag{28}$$

Using (28) then we obtain (23). Then in view of (8), we also have

$$\begin{aligned} & (\nabla_X \phi h)Y - (\nabla_Y \phi h)X = -R(X,Y)\xi \\ & (\alpha^2 + \xi(\alpha))\,[\eta(X)Y - \eta(Y)X] - \alpha\,[\eta(X)\phi hY - \eta(Y)\phi hX]\,. \end{aligned} \tag{29}$$

The proof of (24) is obvious from (2) and (25) is an immediate consequence of (29). □

Remark 3.2. *(23) shows that an almost α-cosymplectic (κ, μ, ν)-space satisfies the Kaehlerian condition.*

Theorem 3.3. *The following differential equation is satisfied on almost α-cosymplectic (κ, μ, ν)-space for*

$$\begin{aligned} 0 = &\xi(\kappa)(\eta(Y)X - \eta(X)Y) + \xi(\mu)(\eta(Y)hX - \eta(X)hY) \\ &+\xi(\nu)(\eta(Y)\phi hX - \eta(X)\phi hY) - X(\kappa)\phi^2 Y + X(\mu)hY \\ &+X(\nu)\phi hY - Y(\mu)hX - Y(\nu)\phi hX + Y(\kappa)\phi^2 X \\ &+2(\kappa + \alpha^2)\mu g(\phi X, Y)\xi + 2\mu g(hX, \phi hY)\xi. \end{aligned} \tag{30}$$

for all vector fields X, Y.

Proof. Differentiating (2) along a vector field Z and using (3) we have

$$\begin{aligned} (\nabla_Z R)(X,Y)\xi = &Z(\kappa)\left[\eta(Y)X - \eta(X)Y\right] + Z(\mu)\left[\eta(Y)hX - \eta(X)hY\right] \\ &+ Z(\nu)\left[\eta(Y)\phi hX - \eta(X)\phi hY\right] + \kappa\left[\alpha g(Z,X)Y\right] \\ &+ \kappa\left[-\alpha g(X,Z)Y + g(X,\phi hZ)Y - g(Y,\phi hZ)X\right] \\ &+ \mu\left[-g(Y,\phi hZ)hX + \eta(Y)(\nabla_Z h)X + \alpha g(Y,Z)hX\right] \\ &+ \mu\left[-\alpha g(X,Z)hY + g(X,\phi hZ)hY - \eta(X)(\nabla_Z h)Y\right] \\ &+ \nu\left[\alpha g(Y,Z)\phi hX - g(Y,\phi hZ)\phi hX + \eta(Y)(\nabla_Z \phi h)X\right] \\ &+ \nu\left[-\alpha g(X,Z)\phi hY + g(X,\phi hZ)\phi hY - \eta(X)(\nabla_Z \phi h)Y\right] \\ &- \alpha R(X,Y)Z + R(X,Y)\phi hZ. \end{aligned}$$

Next, using the last equation and the second Bianchi identity, we obtain

$$\begin{aligned} 0 = &Z(\kappa)\left[\eta(Y)X - \eta(X)Y\right] + Z(\mu)\left[\eta(Y)hX - \eta(X)hY\right] \\ &+ Z(\nu)\left[\eta(Y)\phi hX - \eta(X)\phi hY\right] + X(\kappa)\left[\eta(Z)Y - \eta(Y)Z\right] \\ &+ X(\mu)\left[\eta(Z)hY - \eta(Y)hZ\right] + X(\nu)\left[\eta(Z)\phi hY - \eta(Y)\phi hZ\right] \\ &+ Y(\kappa)\left[\eta(X)Z - \eta(Z)X\right] + Y(\mu)\left[\eta(X)hZ - \eta(Z)hX\right] \\ &+ Y(\nu)\left[\eta(X)\phi hZ - \eta(Z)\phi hX\right] + \mu\left[\eta(Y)\left((\nabla_Z h)X - (\nabla_X h)Z\right)\right] \\ &+ \mu\left[\eta(Z)\left((\nabla_X h)Y - (\nabla_Y h)X\right) + \eta(X)\left((\nabla_Y h)Z - (\nabla_Z h)Y\right)\right] \\ &+ \nu\left[\eta(Y)\left((\nabla_Z \phi h)X - (\nabla_X \phi h)Z\right) + \eta(Z)\left((\nabla_X \phi h)Y - (\nabla_Y \phi h)X\right)\right] \\ &+ \nu\left[\eta(X)\left((\nabla_Y \phi h)Z - (\nabla_Z \phi h)Y\right)\right] + R(X,Y)\phi hZ + R(Y,Z)\phi hX \\ &- \alpha\left[R(X,Y)Z + R(Y,Z)X + R(Z,X)Y\right] + R(Z,X)\phi hY \end{aligned}$$

for all vector fields X, Y, Z. Putting ξ instead of Z in the above equation, we obtain

$$\begin{aligned} 0 &= \xi(\kappa)\left[\eta(Y)X - \eta(X)Y\right] + \xi(\mu)\left[\eta(Y)hX - \eta(X)hY\right] \\ &\quad + \xi(\nu)\left[\eta(Y)\phi hX - \eta(X)\phi hY\right] - X(\kappa)\phi^2 Y + X(\mu)hY \\ &\quad + X(\nu)\phi hY + Y(\kappa)\phi^2 X - Y(\mu)hX - Y(\nu)\phi hX \\ &\quad + \mu\eta(Y)\left[-(\kappa+\alpha^2)\phi X - \mu\phi hX - (\alpha-\nu)hX\right] \\ &\quad + \mu(\kappa+\alpha^2)\left[\eta(Y)\phi X - \eta(X)\phi Y + 2g(\phi X, Y)\xi\right] \\ &\quad + \mu^2\left[\eta(Y)\phi hX - \eta(X)\phi hY\right] + \mu(\alpha-\nu)\left[\eta(Y)hX - \eta(X)hY\right] \\ &\quad + \mu\eta(X)\left[(\kappa+\alpha^2)\phi Y + \mu h\phi Y + (\alpha-\nu)hY\right] \\ &\quad + \nu\eta(Y)\left[-(\kappa+\alpha^2)\phi^2 X + \mu hX - (\alpha-\nu)\phi hX\right] \\ &\quad - \nu(\kappa+\alpha^2)\left[\eta(Y)X - \eta(X)Y\right] - \nu\mu\left[\eta(Y)hX - \eta(X)hY\right] \\ &\quad + \nu(\alpha-\nu)\left[\eta(Y)\phi hX - \eta(X)\phi hY\right] + \nu\eta(X)(\kappa+\alpha^2)\phi^2 Y \\ &\quad + \nu\eta(X)\left[-\mu hY + (\alpha-\nu)\phi hY\right] - R(\xi, Y)\phi hX + R(\xi, X)\phi hY. \end{aligned}$$

Finally, substituting (21), (24) and (25) in the last equation, we deduce (30). □

Lemma 3.4. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-cosymplectic (κ, μ, ν)-space. For every $p \in N$, there exists neighborhood W of p and orthonormal local vector fields $X_i, \phi X_i$ and ξ for $i = 1, \ldots, n$, defined on W, such that*

$$hX_i = \lambda X_i, \quad h\phi X_i = -\lambda X_i, \quad h\xi = 0, \tag{31}$$

for $i = 1, \ldots, n$, where $\lambda = \sqrt{-(\kappa+\alpha^2)}$.

Proof. According to Koufogiorgos ([13], Lemma 4.2), the proof can be easily carried out for almost α-cosymplectic (κ, μ, ν)-space. □

Theorem 3.5. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost α-cosymplectic (κ, μ, ν)-space for $n > 1$. Then the functions κ, μ and ν are non-constant functions on M^{2n+1} such that $df \wedge \eta = 0$.*

Proof. By means of Lemma 1, the existence of a local orthonormal basis $\{X_i, \phi X_i, \xi\}$ such that

$$he_i = \lambda e_i, \; h\phi e_i = -\lambda\phi e_i, \; h\xi = 0, \; \lambda = \sqrt{-(\kappa+\alpha^2)},$$

on W. Substituting $X = e_i$ and $Y = \phi e_i$ in (30), we obtain

$$\left[e_i(\kappa) - \lambda e_i(\mu) - \lambda\phi e_i(\nu)\right]\phi e_i + \left[\lambda e_i(\nu) - \lambda\phi e_i(\mu) - \phi e_i(\kappa)\right] = 0.$$

Since $\{e_i, eX_i\}$ is linearly independent, we have

$$\begin{aligned} e_i(\kappa) - \lambda e_i(\mu) - \lambda\phi e_i(\nu) = 0, \\ \lambda e_i(\nu) - \lambda\phi e_i(\mu) - \phi e_i(\kappa) = 0. \end{aligned} \tag{32}$$

In addition, replacing X and Y by e_i and e_j, respectively, for $i \neq j$, (30) provides that

$$\begin{aligned} e_i(\kappa) + \lambda e_i(\mu) = 0, \\ e_i(\nu) = 0. \end{aligned} \tag{33}$$

Besides, substituting $X = \phi e_i$ and $Y = \phi e_j$ in (30) for $i \neq j$, we get

$$\phi e_i(\kappa) - \lambda\phi e_i(\mu) = 0,\ \phi e_i(\nu) = 0 \tag{34}$$

In view of (32), (34) and (33) we deduce

$$e_i(\kappa) = e_i(\mu) = e_i(\nu) = \phi e_i(\kappa) = \phi e_i(\mu) = \phi e_i(\nu) = 0.$$

For an arbitrary function κ, we obtain $d\kappa = \xi(\kappa)\eta$ in the last equation system. In this way, we have

$$0 = d^2\kappa = d(d\kappa) = d\xi(\kappa) \wedge \eta + \xi(\kappa)d\eta. \tag{35}$$

Since $d\eta = 0$, it follows that $d\xi(\kappa) \wedge \eta = 0$. Thus the proof is completed. □

Corollary 3.6. *The functions κ, μ and ν are constants iff these functions are constants along the characteristic vector field ξ for almost α-cosymplectic (κ, μ, ν)-space with $n > 1$.*

4. On Three Dimensional Case

In this section, we investigate the existence of almost α-cosymplectic (κ, μ, ν)-space in 3-dimensional case.

Let U be the open subset of M^3 where the tensor field $h \neq 0$ and let U' be the open subset of points $p \in M^3$ such that $h = 0$ in a neighborhood of p. Thus the association set of $U \cup U'$ is an open and dense subset of M^3. For every $p \in U$ there exists an open neighborhood of p such that $he = \lambda e$ and $h\phi e = -\lambda\phi e$, where λ is a positive non-vanishing smooth function. So every properties satisfying on $U \cup U'$ is valid on M^3. Therefore, there exists a local orthonormal basis $\{e, \phi e, \xi\}$ of smooth eigenfunctions of h in a neighborhood of p for every point $p \in U \cup U'$. This basis is called ϕ-basis. So we state the following Lemma.

Lemma 4.1. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. Then we have the following relations for the covariant derivatives on U*

$$\begin{array}{llllll} \nabla_\xi e & = & -a\phi e, & \nabla_\xi \phi e & = & ae, \\ \nabla_e \xi & = & \alpha e - \lambda\phi e, & \nabla_{\phi e}\xi & = & -\lambda e + \alpha\phi e, \\ \nabla_e e & = & b\phi e - \alpha\xi, & \nabla_{\phi e}\phi e & = & ce - \alpha\xi, \\ \nabla_e \phi e & = & -be + \lambda\xi, & \nabla_{\phi e} e & = & -c\phi e + \lambda\xi, \end{array}$$

where a is a smooth function, $b = g(\nabla_e e, \phi e)$ and $c = g(\nabla_{\phi e}\phi e, e)$ defined by

$$b = \frac{1}{2\lambda}\left[(\phi e)(\lambda) + \sigma(e)\right], \quad \sigma(e) = S(\xi, e) = g(Q\xi, e),$$

and

$$c = \frac{1}{2\lambda}\left[e(\lambda) + \sigma(\phi e)\right], \quad \sigma(\phi e) = S(\xi, \phi e) = g(Q\xi, \phi e),$$

respectively.

Proof. Replacing X by e and ϕe in (3), we get

$$\nabla_e \xi = \alpha e - \lambda\phi e, \quad \nabla_{\phi e}\xi = \alpha\phi e - \lambda e,$$

for any vector field X. Furthermore, we have

$$\nabla_\xi e = -g(e, \nabla_\xi \phi e)\phi e$$

where a is defined by $a = g(e, \nabla_\xi \phi e)$. Following this procedure, the other covariant derivative equalities can easily find. We recall that the curvature tensor R is given by

$$\begin{array}{l} R(X,Y)Z = -S(X,Z)Y + S(Y,Z)X - g(X,Z)QY \\ +g(Y,Z)QX + \frac{r}{2}[g(X,Z)Y - g(Y,Z)X], \end{array} \tag{36}$$

in dimension 3 for any vector fields X, Y, Z. Putting $X = e$, $Y = \phi e$ and $Z = \xi$ in the last equation, we obtain

$$R(e, \phi e)\xi = -g(Qe, \xi)\phi e + g(Q\phi e, \xi)e.$$

Since $\sigma(X) = g(Q\xi, X)$, we have

$$R(e, \phi e)\xi = -\sigma(e)\phi e + \sigma(\phi e)e, \tag{37}$$

for any vector field X. By using the curvature properties of the Riemannian tensor, we also have

$$R(e, \phi e)\xi = (2\lambda c - e(\lambda))e + (-2\lambda b + (\phi e)(\lambda))\phi e. \tag{38}$$

Thus combining (38) and (37), we deduce

$$\sigma(e) = 2\lambda b - (\phi e)(\lambda), \ \sigma(\phi e) = 2\lambda c - e(\lambda). \tag{39}$$

Hence, the functions b and c are obvious from (39). $\square$

Proposition 4.2. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. On U, we have*

$$\nabla_\xi h = 2ah\phi + \xi(\lambda)s, \tag{40}$$

where s is the tensor field of type $(1,1)$ defined by $s\xi = 0$, $se = e$ and $s\phi e = -\phi e$.

Proof. First, differentiating of the tensor field h along ξ we have

$$(\nabla_\xi h)e = -2\lambda a\phi e + \xi(\lambda)e,\ (\nabla_\xi h)\phi e = -2\lambda ae - \xi(\lambda)\phi e.$$

In addition, we also have $(\nabla_\xi h)\xi = 0$. With the help of the last equation, we obtain (40). It is notice that $tr(s) = 0$. □

Proposition 4.3. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. Then we have*

$$h^2 - \alpha^2\phi^2 = \tfrac{tr(l)}{2}\phi^2. \tag{41}$$

Proof. Using (13), we get $tr(l) = -2\left[\alpha^2 + \lambda^2\right]$ for all vector fields on M^3. Besides, we have

$$h^2 e - \alpha^2\phi^2 e = \tfrac{tr(l)}{2}\phi^2 e,\ h^2\phi e - \alpha^2\phi^3 e = \tfrac{tr(l)}{2}\phi^2\phi e.$$

It follows that $h^2\xi - \alpha^2\phi^2\xi = \frac{tr(l)}{2}\phi^2\xi = 0$. Thus it completes the proof. □

Lemma 4.4. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. Then the Ricci operator Q satisfies the following relation*

$$\begin{aligned} Q &= \tilde{a}I + \tilde{b}\eta\otimes\xi + 2\alpha\phi h + \phi(\nabla_\xi h) - \sigma(\phi^2)\otimes\xi \\ &\quad +\sigma(e)\eta\otimes e + \sigma(\phi e)\eta\otimes\phi e, \end{aligned} \tag{42}$$

where the smooth functions $\tilde{a}$ and $\tilde{b}$ are defined by $\tilde{a} = \frac{1}{2}r + \alpha^2 + \lambda^2$ and $\tilde{b} = -\frac{1}{2}r - 3\alpha^2 - 3\lambda^2$ respectively.

Proof. For 3-dimensional case, we deduce

$$lX = tr(l)X - S(X,\xi)\xi + QX - \eta(X)Q\xi - \tfrac{r}{2}\left(X - \eta(X)\xi\right),$$

for any vector field X. It follows that

$$\begin{aligned} &QX = \alpha^2\phi^2 X + 2\alpha\phi hX - h^2X + \phi(\nabla_\xi h)X - tr(l)X \\ &-S(X,\xi)\xi + \eta(X)Q\xi + \tfrac{r}{2}\left(X - \eta(X)\xi\right). \end{aligned}$$

Moreover, since $S(X,\xi) = -S(\phi^2X,\xi) + \eta(X)tr(l)$, we have

$$\begin{aligned} QX = -\tfrac{tr(l)}{2}\phi^2X + 2\alpha\phi hX + \phi(\nabla_\xi h)X - tr(l)X \\ -S(\phi^2X,\xi)\xi + \eta(X)tr(l)\xi + \eta(X)Q\xi - \tfrac{r}{2}\phi^2X, \end{aligned} \tag{43}$$

and

$$Q\xi = \sigma(e)e + \sigma(\phi e)\phi e + tr(l)\xi. \tag{44}$$

Next, using (43) and (44) we obtain

$$\begin{aligned} QX &= \left[\tfrac{1}{2}r + \alpha^2 + \lambda^2\right] X + \left[-\tfrac{1}{2}r - 3\alpha^2 - 3\lambda^2\right] \eta(X)\xi \\ &+2\alpha\phi hX + \phi(\nabla_\xi h)X - S(\phi^2 X, \xi)\xi + \eta(X)\sigma(e)e + \eta(X)\sigma(\phi e)\phi e. \end{aligned}$$

Thus (42) is obvious for any vector field X. □

Theorem 4.5. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. If $\sigma \equiv 0$, then the (κ, μ, ν)-structure exists on every open and dense subset of M^3.*

Proof. Substituting $\sigma \equiv 0$ and $s = \frac{1}{\lambda}h$ in (42) we have

$$Q = \tilde{a}I + \tilde{b}\eta \otimes \xi + 2ah + (2\alpha + \frac{\xi(\lambda)}{\lambda})\phi h, \tag{45}$$

which yields

$$Q\xi = tr(l)\xi, \tag{46}$$

for any vector fields on M^3. Since $C \equiv 0$, taking ξ instead of Z in (36) we obtain

$$\begin{aligned} R(X,Y)\xi &= -S(X,\xi)Y + S(Y,\xi)X + \eta(Y)QX \\ &-\eta(X)QY - \tfrac{r}{2}[\eta(Y)X - \eta(X)Y], \end{aligned} \tag{47}$$

and replacing X by ξ , then we get $Q\xi = tr(l)$. Hence, it follows that

$$S(Y,\xi) = tr(l)\eta(Y), \tag{48}$$

for any vector field Y. Thus by virtue of (45), (46) and (48), we have

$$\begin{aligned} R(X,Y)\xi &= -\left(\alpha^2 + \lambda^2\right)(\eta(Y)X - \eta(X)Y) \\ &+2a(\eta(Y)hX - \eta(X)hY) + (2\alpha + \tfrac{\xi(\lambda)}{\lambda})(\eta(Y)\phi hX - \eta(X)\phi hY). \end{aligned}$$

Therefore, we obtain κ, μ and ν defined $\kappa = \frac{tr(l)}{2}$, $\mu = 2a$ and $\nu = 2\alpha + \frac{\xi(\lambda)}{\lambda}$, respectively. □

Theorem 4.6. *Let $(M^3, \phi, \xi, \eta, g)$ be an almost α-cosymplectic manifold. If the following relation is held*

$$Q\phi - \phi Q = f_1 h\phi + f_2 h, \tag{49}$$

then the manifold is an almost α-cosymplectic (κ, μ, ν)-space, where the functions f_1, $f_2 \in C^\infty$.

Proof. By the hypothesis, we have

$$
\begin{aligned}
&\alpha^2\phi^2X + 2\alpha\phi hX - h^2X + \phi(\nabla_\xi h)X \\
&= QX - 2tr(l)\eta(X)\xi + tr(l)X - \tfrac{r}{2}\left(X - \eta(X)\xi\right).
\end{aligned} \tag{50}
$$

Applying ϕ both two sides of (50), we get

$$
-\alpha^2\phi X - \phi h^2X - 2\alpha hX - (\nabla_\xi h)X = \phi QX + tr(l)\phi X - \tfrac{r}{2}\phi X. \tag{51}
$$

Also, replacing X by ϕX in (51), we find

$$
\begin{aligned}
&-\alpha^2\phi X + 2\alpha hX - h^2\phi X + (\nabla_\xi h)X \\
&= Q\phi X + tr(l)\phi X - \tfrac{r}{2}\phi X.
\end{aligned} \tag{52}
$$

Then combining (51) and (52) we deduce

$$
Q\phi X + \phi QX = -2\left[\alpha^2\phi + \phi h^2\right]X - 2tr(l)\phi X + r\phi X.
$$

Next, substituting (41) in the last equation and using (49), we obtain

$$
Q\phi X + \phi QX = -tr(l)\phi X + r\phi X.
$$

By virtue of (49), (51) and (52) we also obtain

$$
(\nabla_\xi h)X = \tfrac{1}{2}f_1 h\phi X + \tfrac{1}{2}(f_2 - 4\alpha)hX. \tag{53}
$$

Using (53) in (42), we have

$$
QX = \tilde{a}X + \tilde{b}\eta(X)\xi + 2\alpha\phi hX + \tfrac{1}{2}f_1 hX + \tfrac{1}{2}(f_2 - 4\alpha)\phi hX, \tag{54}
$$

for $\sigma \equiv 0$. Finally, substituting (54) in (47), we deduce

$$
\begin{aligned}
&R(X,Y)\xi = (tr(l) + \tilde{a} - \tfrac{r}{2})\left[\eta(Y)X - \eta(X)Y\right] \\
&+\tfrac{1}{2}f_1\left[\eta(Y)hX - \eta(X)hY\right] + \tfrac{1}{2}f_2\left[\eta(Y)\phi hX - \eta(X)\phi hY\right].
\end{aligned} \tag{55}
$$

Follows from (55), there exists a (κ,μ,ν)-space where $\tilde{a} = \frac{1}{2}r + \alpha^2 + \lambda^2$. □

Example 4.7. *Let (M^3,ϕ,ξ,η,g) be an almost α-cosymplectic manifold. Then there exists a (κ,μ,ν)-structure such that*

$$
\begin{aligned}
&R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + 2a(\eta(Y)hX - \eta(X)hY) \\
&+(2\alpha + \tfrac{\xi(\lambda)}{\lambda})(\eta(Y)\phi hX - \eta(X)\phi hY),
\end{aligned}
$$

where the functions $\kappa,\mu,\nu \in \mathcal{R}^3_\eta(M)$ defined by

$$
d\kappa = \xi(\kappa)\eta,\ d\mu = \xi(\mu)\eta,\ d\nu = \xi(\nu)\eta.
$$

Now, let us consider ϕ-basis on M^3 such that $he = \lambda e$, $h\phi e = -\lambda\phi e$ and $h\xi = 0$. With respect to ϕ-basis, we have

$$\begin{aligned} e(\kappa) &= (d\kappa)e = \xi(\kappa)\eta(e) = 0, \\ e(\mu) &= (d\mu)e = \xi(\mu)\eta(e) = 0, \\ e(\nu) &= (d\nu)e = \xi(\nu)\eta(e) = 0, \end{aligned}$$

and similarly, we have

$$(\phi e)(\kappa) = 0,\ (\phi e)(\mu) = 0,\ (\phi e)(\nu) = 0.$$

Moreover, it follows that

$$\begin{gathered} \sigma(e) = 0, \sigma(\phi e) = 0,\ \lambda = \sqrt{-(\kappa + \alpha^2)}, \\ b = \tfrac{1}{2\lambda}(\phi e)(\lambda) = 0,\ c = \tfrac{1}{2\lambda}e(\lambda) = 0. \end{gathered}$$

Consider the three dimensional manifold

$$M^3 = \left\{(x, y, z) \in \mathbb{R}^3, \quad z \neq 0\right\},$$

where (x, y, z) are the cartesian coordinates in $\mathbb{R}^3$. We define three vector fields on M^3 as

$$\begin{aligned} & e = \frac{\partial}{\partial x},\ \phi e = \frac{\partial}{\partial y}, \\ & \xi = \left[\alpha x - y(e^{-2\alpha z} + z)\right]\frac{\partial}{\partial x} \\ & + \left[x(z - e^{-2\alpha z}) + \alpha y\right]\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \end{aligned}$$

Then we set

$$\begin{aligned} [e, \phi e] &= 0, \\ [e, \xi] &= \alpha e + (z - e^{-2\alpha z})\phi e, \\ [\phi e, \xi] &= -(e^{-2\alpha z} + z)e + \alpha\phi e. \end{aligned}$$

Moreover, the matrice form of the metric tensor g, the tensor fields φ and h are given by

$$g = \begin{pmatrix} 1 & 0 & -d \\ 0 & 1 & -k \\ -d & -k & 1 + d^2 + k^2 \end{pmatrix},$$

and

$$\phi = \begin{pmatrix} 0 & -d & k \\ 1 & 0 & -d \\ 0 & 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} e^{-2z} & 0 & -de^{-2z} \\ 0 & -e^{-2z} & ke^{-2z} \\ 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} d &= \alpha x - y(e^{-2\alpha z} + z), \\ k &= x(z - e^{-2\alpha z}) + \alpha y. \end{aligned}$$

Let η *be the* 1*-form defined by* $\eta = k_1 dx + k_2 dy + k_3 dz$ *for all vector fields on* M^3*. Since* $\eta(X) = g(X, \xi)$*, we obtain that* $\eta(e) = 0$*,* $\eta(\phi e) = 0$ *and* $\eta(\xi) = 1$*. Then we get* $\eta = dz$ *for all vector fields. Since* $d\eta = d(dz) = d^2 z$*, we have* $d\eta = 0$*. Using Koszul's formula, we have seen that* $d\Phi = 2\alpha\eta \wedge \Phi$*. Hence,* M^3 *is an almost* α*-cosymplectic manifold. Thus we obtain*

$$R(X,Y)\xi = -(e^{-4\alpha z} + \alpha^2)\left[\eta(Y)X - \eta(X)Y\right] + 2z\left[\eta(Y)hX - \eta(X)hY\right],$$

where $\kappa = -(e^{-4\alpha z} + \alpha^2)$ *and* $\mu = 2z$*.*

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Hakan Öztürk
Afyon Vocational School, Afyon Kocatepe University,
Campus of ANS, Afyonkarahisar/TURKEY.
E-mail: hozturk@aku.edu.tr