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RELATIVE (p,q)-φ ORDER AND RELATIVE (p,q)-φ TYPE ORIENTED GROWTH ANALYSIS OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. The main aim of this paper is to study some growth properties of composite entire functions on the basis of relative (p,q)-φ type and relative (p,q)-φ weak type where p and q are any two positive integers and $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function.

1. Introduction, Definitions and Notations

We denote by $\mathbb{C}$ the set of all finite complex numbers. Let f be an entire function defined on $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. Since $M_f(r)$ is strictly increasing and continuous, therefore there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0,\infty)$ with $\lim\limits_{s\to\infty} M_f^{-1}(s) = \infty$. The maximum term $\mu_f(r)$ of entire f can be defined as $\mu_f(r) = \max\limits_{n\geq 0}(|a_n|r^n)$. Obviously $\mu_f(r)$ is also a real and increasing function of r. For $x \in [0,\infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers. We also denote $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. Further we assume that throughout the present paper a, b, c, d, p, q, m, n, l, x and y always denote positive integers. Also throughout the paper occasionally $\varphi_1(r)$ will stand for r. Now considering this, let us recall that Juneja et al. [7] defined the (p,q)-th order and (p,q)-th lower order of an entire function, respectively, as follows:

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Definition 1.1. [7] *The (p,q)-th order and (p,q)-th lower order of an entire function f are defined as:*

$$\begin{matrix}\rho^{(p,q)}(f) \\ \lambda^{(p,q)}(f)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where $p \geq q$.

Extending the above notion, recently Shen et al. [9] introduced the new concept of (p,q)-φ order and (p,q)-φ lower order of entire function which are as follows:

Definition 1.2. [9] *Let $\varphi(r):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function, and $p \geq q$. Then the (p,q)-φ order and (p,q)-φ lower order of an entire function f are defined as:*

$$\begin{matrix}\rho^{(p,q)}(f,\varphi) \\ \lambda^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} M_f(r)}{\log^{[q]} \varphi(r)}.$$

The function f is said to be of regular (p,q)-φ growth when (p,q)-φ order and (p,q)-φ lower order of f are the same. Functions which are not of regular (p,q)-φ growth are said to be of irregular (p,q)-φ growth.

However the above definitions are very useful for measuring the growth of entire functions. If $\varphi(r) = r$, then Definition 1.1 is the special case of the above definition. Moreover if $p = 2$, $q = 1$ and $\varphi(r) = r$, then we respectively denote $\rho^{(2,1)}(f,r)$ and $\lambda^{(2,1)}(f,r)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire function f.

Further the definition of order (respectively lower order) does not seem to be feasible if an entire function f is of order zero (respectively lower order zero). To over come this situation and in order to study the growth of an entire function f of order zero (respectively lower order zero) precisely, Chern [6] introduced the concept of logarithmic order (respectively logarithmic lower order) by increasing $\log^+$ once in the denominator. Therefore the definition of logarithmic order $\rho_{\log}(f)$ (respectively logarithmic lower order $\lambda_{\log}(f)$) of an entire function f is define as:

Definition 1.3. [6] *The logarithmic order $\rho_{\log}(f)$ (respectively logarithmic lower order $\lambda_{\log}(f)$) of an entire function f is*

$$\rho_{\log}(f) = \limsup_{r\to+\infty} \frac{\log^{[2]} M_f(r)}{\log^{[2]} r} = \limsup_{r\to+\infty} \frac{\log^{[2]} M_f(r)}{\log^{[1+1]} r}$$

$$\left(\lambda_{\log}(f)=\liminf_{r\to+\infty}\frac{\log^{[2]}M_f(r)}{\log^{[2]}r}=\liminf_{r\to+\infty}\frac{\log^{[2]}M_f(r)}{\log^{[1+1]}r}\right).$$

Definition 1.3 is a special case of Definition 1.1 for $p=q=2$. Also if $p=q=2$ and $\varphi(r)=r$, then Definition 1.3 is also a special case of Definition 1.2.

Similarly, Definition 1.1 does not seem to be feasible if an entire function f is of (p,q)-th order zero. In this situation one may introduce the concept of l logarithmic (p,q)-th order and l logarithmic (p,q)-th lower order of an entire function in the following way:

Definition 1.4. *The l logarithmic (p,q)-th order $\rho^{(p,q)}_{\log^{[l]}}(f)$ and l logarithmic (p,q)-th lower order $\lambda^{(p,q)}_{\log^{[l]}}(f)$ of an entire function f are defined as:*

$$\begin{matrix}\rho^{(p,q)}_{\log^{[l]}}(f)\\ \lambda^{(p,q)}_{\log^{[l]}}(f)\end{matrix}=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}M_f(r)}{\log^{[q+l]}r},$$

where $p\geq q$. Now the two cases may arise: either $p\geq q+l$ or $p<q+l$. If $p=2$, $q=1$ and $l=1$, then Definition 1.3 is a special case of Definition 1.4.

In fact Definition 1.2 itself explain the above situation when $p\geq q$ and $l>p-q$ as follows:

$$\begin{aligned}\begin{matrix}\rho^{(p,q)}\left(f,\log^{[l]}r\right)\\ \lambda^{(p,q)}\left(f,\log^{[l]}r\right)\end{matrix}&=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}M_f(r)}{\log^{[q]}\left(\log^{[l]}r\right)}\\ &=\lim_{r\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p]}M_f(r)}{\log^{[q+l]}r}=\begin{matrix}\rho^{(p,q+l)}(f)\\ \lambda^{(p,q+l)}(f)\end{matrix},\end{aligned}$$

and in this case obviously $p<q+l$.

Combining Definition 1.1 and Definition 1.4, recently Biswas (see, e.g., [3]) introduce a new definitions of the (p,q)-th order and (p,q)-th lower order of an entire function avoiding the restriction $p\geq q$ of the original definitions introduced by Juneja et al. [7]. Subsequently Biswas (see, e.g., [3]) also rewrite Definition 1.2 avoiding the restriction $p\geq q$ of the original definitions introduced by Shen et al. [9].

However extending the notion of index-pair (p,q) introduced by Juneja et al. [7], one may also introduce the definition of index-pair (p,q)-φ in the following manner:

Definition 1.5. *An entire function f is said to have index-pair (p,q)-φ if $b < \rho^{(p,q)}(f,\varphi) < \infty$ and $\rho^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f,\varphi) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \quad . \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f,\varphi) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \quad . \end{cases}$$

If $\varphi(r) = r$, then Definition 1.5 reduces to the the definition of index-pair (p,q) of an entire function. However, mainly the growth investigation of entire functions has usually been done through their maximum moduli in comparison with those of exponential function. But if one is paying attention to evaluate the growths of any entire function with respect to a new entire function, the notion of relative growth indicator (see e.g. [1, 2]) will come. Extending this notion, Sánchez Ruiz et al. [8] gave the definitions of relative (p,q)-th order and relative (p,q)-th lower order of an entire function with respect to another entire function in the light of index-pair. Further revisiting the ideas developed by Shen et al. [9], recently Biswas [3] introduce the definitions of relative (p,q)-φ order and relative (p,q)-φ lower order of an entire function with respect to another entire function in the following way:

Definition 1.6. [3] *Let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Also let f and g be any two entire functions with index-pair (m,q)-φ and (m,p) respectively. The relative (p,q)-φ order and the relative (p,q)-φ lower order of f with respect to g are defined as*

$$\begin{matrix} \rho_g^{(p,q)}(f,\varphi) \\ \lambda_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} \varphi(r)}.$$

Further if relative (p,q)-φ order and the relative (p,q)-φ lower order of f with respect to g are the same, then f is called a function of regular relative (p,q)-φ growth with respect to g. Otherwise, f is said to be irregular relative (p,q)-φ growth.with respect to g. Also for any non-decreasing unbounded function $\varphi(r) : [0,+\infty) \to (0,+\infty)$, if $\varphi(r)$ satisfies the condition $\lim\limits_{r\to+\infty} \frac{\log^{[q]} r}{\log^{[q]} \varphi(r)} = \alpha$ where $\alpha > 0$, then for any

entire function f, one can easily verify that $\rho_g^{(p,q)}(f,\varphi) = \alpha\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f,\varphi) = \alpha\lambda_g^{(p,q)}(f)$.

In this connection we also introduce the following definition which will be needed in the sequel:

Definition 1.7. *An entire function f is said to have relative index-pair (p,q)-φ with respect to an entire function g if $b < \rho_g^{(p,q)}(f,\varphi) < \infty$ and $\rho_g^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho_g^{(p,q)}(f,\varphi) < \infty$, then*

$$\begin{cases} \rho_g^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \quad . \end{cases}$$

Similarly for $0 < \lambda_g^{(p,q)}(f,\varphi) < \infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f,\varphi) = \infty & \text{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f,\varphi) = 0 & \text{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f,\varphi) = 1 & \text{for} \quad n = 1,2,\cdots \quad . \end{cases}$$

Throughout the paper, whenever we deal with any entire function f having relative index-pair (p,q)-φ with respect to an entire function g, we mean that f has positive relative (p,q)-φ lower order and finite relative (p,q)-φ order with respect to g.

Now in order to refine the above growth scale, one may introduce the definitions of other growth indicators, such as relative (p,q)-φ type and relative (p,q)-φ lower type of entire functions with respect to another entire function which are as follows:

Definition 1.8. [3] *Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ type and the relative (p,q)-φ lower type of an entire function f with respect to another entire function g having non-zero finite relative (p,q)-φ order $\rho_g^{(p,q)}(f,\varphi)$ are defined as :*

$$\begin{matrix} \sigma_g^{(p,q)}(f,\varphi) \\ \overline{\sigma}_g^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\rho_g^{(p,q)}(f,\varphi)}}.$$

Analogously, to determine the relative growth of f having same non zero finite relative (p,q)-φ lower order with respect to g, one can introduce the definition of relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f)$ and the growth

indicator $\tau_g^{(p,q)}(f)$ of f with respect to g of finite positive relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f)$ in the following way:

Definition 1.9. [3] *Let $\varphi : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. The relative (p,q)-φ weak type $\overline{\tau}_g^{(p,q)}(f,\varphi)$ and the growth indicator $\tau_g^{(p,q)}(f,\varphi)$ of an entire function f with respect to another entire function g having non-zero finite relative (p,q)-φ lower order $\lambda_g^{(p,q)}(f,\varphi)$ are defined as :*

$$\begin{matrix}\tau_g^{(p,q)}(f,\varphi) \\ \overline{\tau}_g^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\left[\log^{[q-1]} \varphi(r)\right]^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

If we consider $\varphi(r) = r$, then $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and $\overline{\tau}_g^{(p,q)}(f,r)$ are respectively known as relative (p,q)-th order (relative (p,q)-th lower order), relative (p,q)-th type (relative (p,q)-th lower type) and relative (p,q)-th weak type of f with respect to g. Further for $\varphi(r) = r$, we simplify to denote $\rho_g^{(p,q)}(f,r)$ $(\lambda_g^{(p,q)}(f,r))$, $\sigma_g^{(p,q)}(f,r)$ $(\overline{\sigma}_g^{(p,q)}(f,r))$ and $\tau_g^{(p,q)}(f,r)$ $(\overline{\tau}_g^{(p,q)}(f,r))$ by $\rho_g^{(p,q)}(f)$ $(\lambda_g^{(p,q)}(f))$, $\sigma_g^{(p,q)}(f)$ $(\overline{\sigma}_g^{(p,q)}(f))$ and $\tau_g^{(p,q)}(f)$ $(\overline{\tau}_g^{(p,q)}(f))$ respectively.

In terms of maximum terms of entire functions, Definition 1.6 can be reformulated as:

Definition 1.10. *The growth indicators $\rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi)$ of an entire function f with respect to another entire function g are defined as:*

$$\begin{matrix}\rho_g^{(p,q)}(f,\varphi) \\ \lambda_g^{(p,q)}(f,\varphi)\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p]} \mu_g^{-1}(\mu_f(r))}{\log^{[q]} \varphi(r)}.$$

In fact, for any nondecreasing unbounded function $\varphi(r) : [0,+\infty) \to (0,+\infty)$, if we assume $\lim\limits_{r\to+\infty} \frac{\log^{[q]} \varphi(ar)}{\log^{[q]} \varphi(r)} = 1$ for all $\alpha > 0$, then in view of Lemma 6 of [4], one can easily verify the equivalence of Definition 1.6 and Definition 1.10.

The purpose of this paper is to deal with some growth properties of composite entire functions in the light of their relative (p,q)-φ order, relative (p,q)-φ type and relative (p,q)-φ weak type after improving some results of Xu et.al [10]. We use the standard notations and definitions of the theory of entire functions which are available in [11] and [12], and therefore we do not explain those in details.

2. Lemma

In this section we present a lemma which will be needed in the sequel.

Lemma 2.1. [5] *Let f and g are any two entire functions with $g(0) = 0$. Also let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)) \ .$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right)\right) \ .$$

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f, g, h be any three entire functions such that the relative index pair of f with respect to h and the index pair of g are (p,q)-φ_1 and (m,n)-φ respectively. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim\limits_{r\to+\infty} \frac{\log^{[n]} \varphi(ar)}{\log^{[n]} \varphi(r)} = 1$ for all $\alpha > 0$. Then*

(i) *the relative index-pair of $f \circ g$ is (p,n)-φ when $q = m$ and either $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ or $\lambda^{(m,n)}(g,\varphi) > 0$. Also*

(a) $\lambda_h^{(p,q)}(f,\varphi_1)\, \rho^{(m,n)}(g,\varphi) \leq \rho_h^{(p,n)}(f\circ g,\varphi) \leqslant \rho_h^{(p,q)}(f,\varphi_1)\, \rho^{(m,n)}(g,\varphi)$ *if* $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ *and*

(b) $\rho_h^{(p,q)}(f,\varphi_1)\, \lambda^{(m,n)}(g,\varphi) \leq \rho_h^{(p,n)}(f\circ g,\varphi) \leqslant \rho_h^{(p,q)}(f,\varphi_1)\, \rho^{(m,n)}(g,\varphi)$ *if* $\lambda^{(m,n)}(g,\varphi) > 0$;

(ii) *the relative index-pair of $f \circ g$ is $(p, q+n-m)$-φ when $q > m$ and either $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ or $\lambda^{(m,n)}(g,\varphi) > 0$. Also*

(a) $\lambda_h^{(p,q)}(f,\varphi_1) \leq \rho_h^{(p,q+n-m)}(f\circ g,\varphi) \leqslant \rho_h^{(p,q)}(f,\varphi_1)$ *if* $\lambda_h^{(p,q)}(f,\varphi_1) > 0$ *and*

(b) $\rho_h^{(p,q+n-m)}(f\circ g,\varphi) = \rho_h^{(p,q)}(f,\varphi_1)$ *if* $\lambda^{(m,n)}(g,\varphi) > 0$;

(iii) *the relative index-pair of $f\circ g$ is $(p+m-q,n)$-φ when $q<m$ and either $\lambda_h^{(p,q)}(f,\varphi_1)>0$ or $\lambda^{(m,n)}(g,\varphi)>0$. Also*

(a) $\rho_h^{(p+m-q,n)}(f\circ g,\varphi)=\rho^{(m,n)}(g,\varphi)$ *if* $\lambda_h^{(p,q)}(f,\varphi_1)>0$ *and*

(b) $\lambda^{(m,n)}(g,\varphi)\leq\rho_h^{(p+m-q,n)}(f\circ g,\varphi)\leqslant\rho^{(m,n)}(g,\varphi)$ *if* $\lambda^{(m,n)}(g,\varphi)>0$.

Proof. In view of Lemma 2.1, it follows for all sufficiently large positive numbers of r that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(r))\geq\left(\lambda_h^{(p,q)}(f,\varphi_1)-\varepsilon\right)\log^{[q]}M_g\left(\frac{r}{2}\right)+O(1) \tag{1}$$

and also for a sequence of positive numbers of r tending to infinity we get that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(r))\geq\left(\rho_h^{(p,q)}(f,\varphi_1)-\varepsilon\right)\log^{[q]}M_g\left(\frac{r}{2}\right)+O(1)\ . \tag{2}$$

Similarly, we have for all sufficiently large positive numbers of r that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(r))\leqslant\left(\rho_h^{(p,q)}(f,\varphi_1)+\varepsilon\right)\log^{[q]}M_g(r)\ . \tag{3}$$

Now the following three cases may arise:

Case I. Let $q=m$. In this case we have from (3) for all sufficiently large positive numbers of r that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(r))\leqslant\left(\rho_h^{(p,q)}(f,\varphi_1)+\varepsilon\right)\left(\rho^{(m,n)}(g,\varphi)+\varepsilon\right)\log^{[n]}\varphi(r)$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[n]}\varphi(r)}\leqslant\rho_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\ . \tag{4}$$

Also from (1) and in view of the condition $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$, we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]}M_h^{-1}(M_{f\circ g}(r))\geq\left(\lambda_h^{(p,q)}(f,\varphi_1)-\varepsilon\right)\left(\rho^{(m,n)}(g,\varphi)-\varepsilon\right)\log^{[n]}\varphi(r)+O(1)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[n]}\varphi(r)}\geq\lambda_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\ . \tag{5}$$

Moreover in view of the condition $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$, we have from (2) for a sequence of positive numbers of r tending to infinity

that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geq \left(\rho_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\left(\lambda^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)} \geq \rho_h^{(p,q)}\left(f,\varphi_1\right)\lambda^{(m,n)}\left(g,\varphi\right)\ . \tag{6}$$

Therefore from (4) and (5), we get for $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ that

$$\lambda_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right)\leq\rho_h^{(p,n)}\left(f\circ g,\varphi\right)\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{7}$$

Likewise, from (4) and (6) we obtain for $\lambda^{(m,n)}\left(g,\varphi\right)>0$ that

$$\rho_h^{(p,q)}\left(f,\varphi_1\right)\lambda^{(m,n)}\left(g,\varphi\right)\leq\rho_h^{(p,n)}\left(f\circ g,\varphi\right)\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\rho^{(m,n)}\left(g,\varphi\right). \tag{8}$$

Also from (7) and (8) one can easily verify that $\rho_h^{(p-1,n)}\left(f\circ g,\varphi\right)=\infty$, $\rho_h^{(p,n-1)}\left(f\circ g,\varphi\right)=0$ and $\rho_h^{(p+1,n+1)}\left(f\circ g,\varphi\right)=1$ and therefore we obtain that the relative index-pair of $f\circ g$ is (p,n)-φ when $q=m$ and either $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ or $\lambda^{(m,n)}\left(g,\varphi\right)>0$ and thus the first part of the theorem is established.

Case II. Let $q>m$. Now we obtain from (3) for all sufficiently large positive numbers of r that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\leqslant\left(\rho_h^{(p,q)}\left(f,\varphi_1\right)+\varepsilon\right)\log^{[q-m]}\log^{[m]} M_g\left(r\right)$$

$$i.e.,\ \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\leqslant \left(\rho_h^{(p,q)}\left(f,\varphi_1\right)+\varepsilon\right)\log^{[q-m]}\left[\left(\rho^{(m,n)}\left(g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)\right]$$

$$i.e.,\ \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\leqslant\left(\rho_h^{(p,q)}\left(f,\varphi_1\right)+\varepsilon\right)\log^{[q+n-m]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)}\leqslant\rho_h^{(p,q)}\left(f,\varphi_1\right)\ . \tag{9}$$

Also from (1) and in view of the condition $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$, we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\geq \left(\lambda_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q-m]}\left[\left(\rho^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)\right]+O(1)$$

$$i.e.,\ \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geq \left(\lambda_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q-m+n]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)} \geq \lambda_h^{(p,q)}\left(f,\varphi_1\right)\ . \tag{10}$$

Further in view of the condition $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$, we get from (2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geq$$

$$\left(\rho_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q-m]}\left[\left(\lambda^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)\right]+O(1)$$

$$i.e.,\ \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geq \left(\rho_h^{(p,q)}\left(f,\varphi_1\right)-\varepsilon\right)\log^{[q+n-m]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[q+n-m]}\varphi\left(r\right)} \geq \rho_h^{(p,q)}\left(f,\varphi_1\right)\ . \tag{11}$$

Therefore from (9) and (10), we get for $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ that

$$\lambda_h^{(p,q)}\left(f,\varphi_1\right) \leq \rho_h^{(p,q+n-m)}\left(f\circ g,\varphi\right) \leqslant \rho_h^{(p,q)}\left(f,\varphi_1\right). \tag{12}$$

Likewise, from (9) and (11) we get for $\lambda^{(m,n)}\left(g,\varphi\right)>0$ that

$$\rho_h^{(p,q+n-m)}\left(f\circ g,\varphi\right)=\rho_h^{(p,q)}\left(f,\varphi_1\right). \tag{13}$$

Hence from (12) and (13) one can easily verify that $\rho_h^{(p-1,q+n-m)}\left(f\circ g,\varphi\right)=\infty$, $\rho_h^{(p,q+n-m-1)}\left(f\circ g,\varphi\right)=0$ and $\rho_h^{(p+1,q+n-m+1)}\left(f\circ g,\varphi\right)=1$ and therefore we get that the relative index-pair of $f\circ g$ is $(p,q+n-m)$-φ when $q>m$ and either $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ or $\lambda^{(m,n)}\left(g,\varphi\right)>0$ and thus the second part of the theorem follows.

Case III. Let $q<m$. Then we obtain from (3) for all sufficiently large positive numbers of r that

$$\log^{[p+m-q]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \leqslant \log^{[m]} M_g\left(r\right)+O(1)$$

$$i.e.,\ \log^{[p+m-q]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \leqslant \left(\rho^{(m,n)}\left(g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)+O(1)$$

$$i.e.,\ \lim_{r\to+\infty}\frac{\log^{[p+m-q]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)} \leqslant \rho^{(m,n)}\left(g,\varphi\right)\ . \tag{14}$$

Also from (1) and in view of the condition $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$, we have for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\ \geq\ \log^{[m]}M_g\left(\frac{r}{2}\right)+O(1)$$

$$i.e.,\ \log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\ \geq\ \left(\rho^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)+O(1)$$

$$\limsup_{r\to+\infty}\frac{\log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)}\geq\rho^{(m,n)}\left(g,\varphi\right)\ . \tag{15}$$

Further, we get from (2) for a sequence of positive numbers of r tending to infinity that

$$\log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\ \geq\ \log^{[m]}M_g\left(\frac{r}{2}\right)+O(1)$$

$$i.e.,\ \log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)\ \geq\ \left(\lambda^{(m,n)}\left(g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)+O(1)$$

$$\limsup_{r\to+\infty}\frac{\log^{[p+m-q]}M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[n]}\varphi\left(r\right)}\geq\lambda^{(m,n)}\left(g,\varphi\right)\ . \tag{16}$$

Therefore from (14) and (15), we obtain for $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ that

$$\rho_h^{(p+m-q,n)}\left(f\circ g,\varphi\right)=\rho^{(m,n)}\left(g,\varphi\right). \tag{17}$$

Similarly, from (14) and (16) we get for $\lambda^{(m,n)}\left(g,\varphi\right)>0$ that

$$\lambda^{(m,n)}\left(g,\varphi\right)\leq\rho_h^{(p+m-q,n)}\left(f\circ g,\varphi\right)\leqslant\rho^{(m,n)}\left(g,\varphi\right). \tag{18}$$

So from (17) and (18) one can easily verify that $\rho_h^{(p+m-q-1,n)}\left(f\circ g,\varphi\right)=\infty$,
$\rho_h^{(p+m-q,n-1)}\left(f\circ g,\varphi\right)=0$ and $\rho_h^{(p+m-q+1,n+1)}\left(f\circ g,\varphi\right)=1$ and therefore we obtain that the relative index-pair of $f\circ g$ is $(p+m-q,n)$-φ when $q<m$ and either $\lambda_h^{(p,q)}\left(f,\varphi_1\right)>0$ or $\lambda^{(m,n)}\left(g,\varphi\right)>0$ and thus the third part of the theorem is established. □

In the line of Theorem 3.1 one can easily deduce the conclusion of the following theorem and so its proof is omitted.

Theorem 3.2. *Let f, g, h be any three entire functions such that the relative index pair of f with respect to h and the index pair of g are (p,q)-φ_1 and (m,n)-φ respectively. Also let $\varphi\left(r\right):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=$*

1 *for all* $\alpha > 0$. *Then*

$$(i)\ \ \lambda_h^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi) \leq \lambda_h^{(p,n)}(f\circ g,\varphi)$$
$$\leqslant \min\left\{\rho_h^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi),\lambda_h^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\right\}$$
$$\text{if } q=m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{ and } \lambda^{(m,n)}(g,\varphi)>0;$$

$$(ii)\ \ \lambda_h^{(p,q+n-m)}(f\circ g,\varphi)=\lambda_h^{(p,q)}(f,\varphi_1)$$
$$\text{if } q>m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{ and } \lambda^{(m,n)}(g,\varphi)>0$$

and

$$(iii)\ \ \lambda_h^{(p+m-q,n)}(f\circ g,\varphi)=\lambda^{(m,n)}(g,\varphi)$$
$$\text{if } q<m,\ \lambda_h^{(p,q)}(f,\varphi_1)>0 \text{and } \lambda^{(m,n)}(g,\varphi)>0\ .$$

Corollary 3.3. *Let* f, g, h *be any three entire functions such that the relative index pair of* f *with respect to* h *and the index pair of* g *are* $(p-l,m-l)$-φ_1 *and* (m,n)-φ *respectively such that* $p-l>0$ *and* $m-l>0$. *Also let* $\varphi(r):[0,+\infty)\to(0,+\infty)$ *is a nondecreasing unbounded function and satisfies* $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ *for all* $\alpha>0$. *Then*

$$\rho_h^{(p,n)}(f\circ g,\varphi)=\rho^{(m,n)}(g,\varphi) \text{ and } \lambda_h^{(p,n)}(f\circ g,\varphi)=\lambda^{(m,n)}(g,\varphi)\ .$$

Proof. In view of Definition 1.7 $\rho_h^{(p,m)}(f,\varphi_1)=\lambda_h^{(p,m)}(f,\varphi_1)=1$. Therefore the conclusion of above corollary immediately follows from the first part of Theorem 3.1 and Theorem 3.2. □

Corollary 3.4. *Let* f *and* g *be any two entire functions with index pairs* (p,q)-φ_1 *and* (m,n)-φ *respectively. Also let* $\varphi(r):[0,+\infty)\to(0,+\infty)$ *is a nondecreasing unbounded function and satisfies*

$$\lim_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$$

for all $\alpha>0$. *Then*
(i) *the index-pair of* $f\circ g$ *is* (p,n)-φ *when* $q=m$ *and either* $\lambda^{(p,q)}(f,\varphi_1)>0$ *or* $\lambda^{(m,n)}(g,\varphi)>0$. *Also*

$$(a)\ \ \lambda^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\leq\rho^{(p,n)}(f\circ g,\varphi)\leqslant$$
$$\rho^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi) \text{ if } \lambda^{(p,q)}(f,\varphi_1)>0 \text{ and}$$

(b) $\rho^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi) \leq \rho^{(p,n)}(f\circ g,\varphi) \leqslant \rho^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)$ *if* $\lambda^{(m,n)}(g,\varphi)>0$;

(ii) *the index-pair of* $f\circ g$ *is* $(p,q+n-m)$-φ *when* $q>m$ *and either* $\lambda^{(p,q)}(f,\varphi_1)>0$ *or* $\lambda^{(m,n)}(g,\varphi)>0$. *Also*

(a) $\lambda^{(p,q)}(f,\varphi_1) \leq \rho^{(p,q+n-m)}(f\circ g,\varphi) \leqslant \rho^{(p,q)}(f,\varphi_1)$ *if* $\lambda^{(p,q)}(f,\varphi_1)>0$

and

(b) $\rho^{(p,q+n-m)}(f\circ g,\varphi)=\rho^{(p,q)}(f,\varphi_1)$ *if* $\lambda^{(m,n)}(g,\varphi)>0$;

(iii) *the index-pair of* $f\circ g$ *is* $(p+m-q,n)$-φ *when* $q<m$ *and either* $\lambda^{(p,q)}(f,\varphi_1)>0$ *or* $\lambda^{(m,n)}(g,\varphi)>0$. *Also*

(a) $\rho^{(p+m-q,n)}(f\circ g,\varphi)=\rho^{(m,n)}(g,\varphi)$ *if* $\lambda^{(p,q)}(f,\varphi_1)>0$ *and*

(b) $\lambda^{(m,n)}(g,\varphi) \leq \rho^{(p+m-q,n)}(f\circ g,\varphi) \leqslant \rho^{(m,n)}(g,\varphi)$ *if* $\lambda^{(m,n)}(g,\varphi)>0$.

Corollary 3.5. *Let* f *and* g *be any two entire functions with index pairs* (p,q)-φ_1 *and* (m,n)-φ *respectively. Also let* $\varphi(r):[0,+\infty)\to(0,+\infty)$ *is a nondecreasing unbounded function and satisfies*

$$\lim_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$$

for all $\alpha>0$. *Then*

(i) $\lambda^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi) \leq \lambda^{(p,n)}(f\circ g,\varphi)$
$\leqslant \min\left\{\rho^{(p,q)}(f,\varphi_1)\,\lambda^{(m,n)}(g,\varphi),\lambda^{(p,q)}(f,\varphi_1)\,\rho^{(m,n)}(g,\varphi)\right\}$
if $q=m$, $\lambda^{(p,q)}(f,\varphi_1)>0$ *and* $\lambda^{(m,n)}(g,\varphi)>0$;

(ii) $\lambda^{(p,q+n-m)}(f\circ g,\varphi)=\lambda^{(p,q)}(f,\varphi_1)$
if $q>m$, $\lambda^{(p,q)}(f,\varphi_1)>0$ *and* $\lambda^{(m,n)}(g,\varphi)>0$

and

(iii) $\lambda^{(p+m-q,n)}(f\circ g,\varphi)=\lambda^{(m,n)}(g,\varphi)$
if $q<m$, $\lambda^{(p,q)}(f,\varphi_1)>0$ *and* $\lambda^{(m,n)}(g,\varphi)>0$.

Reasoning similarly as in the proofs of the Theorem 3.1 and Theorem 3.2, one can easily deduce the conclusions of the above two corollaries, and so their proofs are omitted.

Corollary 3.6. *Let f, g, h be any three entire functions such that the relative index pair of f with respect to h and the index pair of g are $(p-1,m-1)$-φ_1 and (m,n)-φ respectively such that $p>1$ and $m>1$. Also let $\varphi(r):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim\limits_{r\to+\infty}\frac{\log^{[n-1]}\varphi(ar)}{\log^{[n-1]}\varphi(r)}=1$ for all $\alpha>0$. Then*

$$\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)\leq\sigma_h^{(p,n)}(f\circ g,\varphi)\leq\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)$$

and

$$\lambda^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi)\leq\overline{\sigma}^{(p,n)}(f\circ g,\varphi)\leq\rho^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi).$$

Proof. In view of Lemma 2.1 and Corollary 3.3, we get that

$$\sigma_h^{(p,n)}(f\circ g,\varphi)=\limsup_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_{f\circ g}(r))}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho_h^{(p,n)}(f\circ g,\varphi)}}$$

i.e., $\sigma_h^{(p,n)}(f\circ g,\varphi)\leq$

$$\limsup_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_f(M_g(r)))}{\log^{[m-1]}M_g(r)}\cdot\limsup_{r\to+\infty}\frac{\log^{[m-1]}M_g(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$\text{(19)}\qquad i.e.,\ \sigma_h^{(p,n)}(f\circ g,\varphi)\leq\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)\ .$$

Similarly

$$\sigma_h^{(p,n)}(f\circ g,\varphi)=\limsup_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_{f\circ g}(r))}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho_h^{(p,n)}(f\circ g,\varphi)}}$$

i.e., $\sigma_h^{(p,n)}(f\circ g,\varphi)\geq$

$$\liminf_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}\left(M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right)\right)}{\log^{[m-1]}M_g\left(\frac{r}{2}\right)+O(1)}\cdot\limsup_{r\to+\infty}\frac{\log^{[m-1]}M_g\left(\frac{r}{2}\right)+O(1)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$\text{(20)}\qquad i.e.,\ \sigma_h^{(p,n)}(f\circ g,\varphi)\geq\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\sigma^{(m,n)}(g,\varphi)\ .$$

Hence the first part of corollary follows from (19) and (20) .

$$\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)=\liminf_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_{f\circ g}(r))}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho_h^{(p,n)}(f\circ g,\varphi)}}$$

i.e., $\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\geq$

$$\liminf_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}\left(M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right)\right)}{\log^{[m-1][m-1]}M_g\left(\frac{r}{2}\right)+O(1)}\cdot\liminf_{r\to+\infty}\frac{\log^{[m-1]}M_g\left(\frac{r}{2}\right)+O(1)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

Again

$$\text{(21)}\qquad i.e.,\ \overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\geq\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi)\ .$$

Also

$$\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)=\liminf_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_{f\circ g}(r))}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho_h^{(p,n)}(f\circ g,\varphi)}}$$

i.e., $\overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\leq$

$$\limsup_{r\to+\infty}\frac{\log^{[p-1]}M_h^{-1}(M_f(M_g(r)))}{\log^{[m-1]}M_g(r)}\cdot\liminf_{r\to+\infty}\frac{\log^{[m-1]}M_g(r)}{\left(\log^{[n-1]}\varphi(r)\right)^{\rho^{(m,n)}(g,\varphi)}}$$

$$\text{(22)}\qquad i.e.,\ \overline{\sigma}_h^{(p,n)}(f\circ g,\varphi)\leq\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\sigma}^{(m,n)}(g,\varphi)\ .$$

Therefore the second part of corollary follows from (21) and (22) .

Thus the corollary follows. □

Corollary 3.7. *Let f, g, h be any three entire functions such that the relative index pair of f with respect to h and the index pair of g are $(p-1,m-1)$-φ_1 and (m,n)-φ respectively such that $p>1$ and $m>1$. Also let $\varphi(r):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim_{r\to+\infty}\frac{\log^{[n-1]}\varphi(ar)}{\log^{[n-1]}\varphi(r)}=1$ for all $\alpha>0$. Then*

$$\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\tau^{(m,n)}(g,\varphi)\leq\tau_h^{(p,n)}(f\circ g,\varphi)\leq\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\tau^{(m,n)}(g,\varphi).$$

and

$$\lambda_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\tau}^{(m,n)}(g,\varphi)\leq\overline{\tau}_h^{(p,n)}(f\circ g,\varphi)\leq\rho_h^{(p-1,q-1)}(f,\varphi_1)\,\overline{\tau}^{(m,n)}(g,\varphi)\ .$$

Reasoning similarly as in the proof of the Corollary 3.6 one can easily deduce the conclusion of Corollary 3.7, and so its proof is omitted.

Theorem 3.8. *Let f, g, h and k be any four entire functions with index pairs (p,q)-φ_1, (m,n)-φ, (a,b)-φ_1 and (c,d)-φ_1 respectively. Also let $\varphi(r):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies the condition (i) $\lim\limits_{r\to+\infty}\frac{\log^{[q]} r}{\log^{[q]}\varphi(r)}=\alpha$ where $\alpha>0$, (ii) $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$.*

(i) *If either $(q=m,\ a=c=p,\ q\geq n)$ or $(q<m,\ c=p,\ a=p+m-q,\ q\geq n)$ holds and $\lambda^{(p,q)}(f,\varphi_1)>0$, $0<\lambda_h^{(b,n)}(f\circ g,\varphi)\leq\rho_h^{(b,n)}(f\circ g,\varphi)<\infty$ then*

$$\begin{aligned}\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}&\leq\liminf_{r\to+\infty}\frac{\log^{[b]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]}M_k^{-1}\left(M_f\left(\exp^{[q-n]}r\right)\right)}\\&\leq\min\left\{\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)},\frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}\right\}\leq\\&\max\left\{\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)},\frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}\right\}\leq\\&\limsup_{r\to+\infty}\frac{\log^{[b]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]}M_k^{-1}\left(M_f\left(\exp^{[q-n]}r\right)\right)}\leq\frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)}\ .\end{aligned}$$

and

(ii) *If $q>m$, $a=c=p$, $\lambda^{(p,q)}(f,\varphi_1)>0$ and $0<\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\rho_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, then*

$$\begin{aligned}\frac{\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}&\leq\liminf_{r\to+\infty}\frac{\log^{[b]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]}M_k^{-1}\left(M_f\left(\exp^{[m-n]}r\right)\right)}\leq\\&\min\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)},\frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}\right\}\leq\\&\max\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)},\frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,q)}(f,\varphi)}\right\}\leq\\&\limsup_{r\to+\infty}\frac{\log^{[b]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]}M_k^{-1}\left(M_f\left(\exp^{[m-n]}r\right)\right)}\leq\frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\lambda_k^{(d,q)}(f,\varphi)}.\end{aligned}$$

Proof. Let either $(q = m,\ a = c = p,\ q \geq n)$ or $(q < m,\ c = p,\ a = p+m-q,\ q \geq n)$ hold and $\lambda^{(p,q)}(f, \varphi_1) > 0$. Then in view of Theorem 3.4, the index-pair of $f \circ g$ is (p, n)-φ or $(p + m - q, n)$-φ respectively and therefore by Definition 1.6, $\rho_h^{(b,n)}(f \circ g, \varphi)$ $\left(\text{respectively } \lambda_h^{(b,n)}(f \circ g, \varphi)\right)$ exist. Since the index pairs of f and k are (p, q)-φ_1 and (c, d)-φ_1 respectively, then in view of $c = p$ and $\lim\limits_{r\to+\infty} \frac{\log^{[q]} r}{\log^{[q]} \varphi(r)} = \alpha$ where $\alpha > 0$, we get that $0 < \rho_k^{(d,q)}(f, \varphi) < \infty$. Therefore the relative index-pair of f with respect to k is (d, q)-φ.

Now from the definition of $\rho_k^{(d,q)}(f, \varphi)$ and $\lambda_h^{(b,n)}(f \circ g, \varphi)$, we have for arbitrary positive ε and for all sufficiently large positive numbers of r that

$$\log^{[b]} M_h^{-1}(M_{f\circ g}(r)) \geqslant \left(\lambda_h^{(b,n)}(f \circ g, \varphi) - \varepsilon\right) \log^{[n]} \varphi(r) \tag{23}$$

and

$$\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right) \leq \left(\rho_k^{(d,q)}(f, \varphi) + \varepsilon\right) \log^{[n]} \varphi(r)\ . \tag{24}$$

Now from (23) and (24), it follows for all sufficiently large positive numbers of r that

$$\frac{\log^{[b]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geqslant \frac{\left(\lambda_h^{(b,n)}(f \circ g, \varphi) - \varepsilon\right) \log^{[n]} \varphi(r)}{\left(\rho_k^{(d,q)}(f, \varphi) + \varepsilon\right) \log^{[n]} \varphi(r)}\ .$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty} \frac{\log^{[b]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geqslant \frac{\lambda_h^{(b,n)}(f \circ g, \varphi)}{\rho_k^{(d,q)}(f, \varphi)}\ . \tag{25}$$

Again we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[b]} M_h^{-1}(M_{f\circ g}(r)) \leq \left(\lambda_h^{(b,n)}(f \circ g, \varphi) + \varepsilon\right) \log^{[n]} \varphi(r) \tag{26}$$

and for all sufficiently large positive numbers of r that

$$\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right) \geqslant \left(\lambda_k^{(d,q)}(f, \varphi) - \varepsilon\right) \log^{[n]} \varphi(r)\ . \tag{27}$$

Combining (26) and (27), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\left(\lambda_h^{(b,n)}(f \circ g, \varphi) + \varepsilon\right) \log^{[n]} \varphi(r)}{\left(\lambda_k^{(d,q)}(f, \varphi) - \varepsilon\right) \log^{[n]} \varphi(r)}\ .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\lambda_h^{(b,n)}\left(f\circ g,\varphi\right)}{\lambda_k^{(d,q)}\left(f,\varphi\right)} \ . \tag{28}$$

Also for a sequence of positive numbers of r tending to infinity that

$$\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right) \leq \left(\lambda_k^{(d,q)}\left(f,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right) \ . \tag{29}$$

Now from (23) and (29), we obtain for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geq \frac{\left(\lambda_h^{(b,n)}\left(f\circ g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)}{\left(\lambda_k^{(d,q)}\left(f,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)} \ .$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geq \frac{\lambda_h^{(b,n)}\left(f\circ g,\varphi\right)}{\lambda_k^{(d,q)}\left(f,\varphi\right)} \ . \tag{30}$$

Also we obtain for all sufficiently large positive numbers of r that

$$\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \leq \left(\rho_h^{(b,n)}\left(f\circ g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right) \ . \tag{31}$$

Now it follows from (27) and (31) for all sufficiently large positive numbers of r that

$$\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\left(\rho_h^{(b,n)}\left(f\circ g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)}{\left(\lambda_k^{(d,q)}\left(f,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)} \ .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\rho_h^{(b,n)}\left(f\circ g,\varphi\right)}{\lambda_k^{(d,q)}\left(f,\varphi\right)} \ . \tag{32}$$

Further from the definition of $\rho_k^{(d,q)}\left(f\right)$, we get for a sequence of positive numbers of r tending to infinity that

$$\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right) \geqslant \left(\rho_k^{(d,q)}\left(f,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right) \ . \tag{33}$$

Now from (31) and (33), it follows for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\left(\rho_h^{(b,n)}\left(f\circ g,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)}{\left(\rho_k^{(d,q)}\left(f,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \leq \frac{\rho_h^{(b,n)}\left(f\circ g,\varphi\right)}{\rho_k^{(d,q)}\left(f,\varphi\right)} . \tag{34}$$

Again we obtain for a sequence of positive numbers of r tending to infinity that

$$\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geqslant \left(\rho_h^{(b,n)}\left(f\circ g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right) . \tag{35}$$

So combining (24) and (35), we get for a sequence of positive numbers of r tending to infinity that

$$\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geqslant \frac{\left(\rho_h^{(b,n)}\left(f\circ g,\varphi\right)-\varepsilon\right)\log^{[n]}\varphi\left(r\right)}{\left(\rho_k^{(d,q)}\left(f,\varphi\right)+\varepsilon\right)\log^{[n]}\varphi\left(r\right)} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[d]} M_k^{-1}\left(M_f\left(\exp^{[q-n]} r\right)\right)} \geqslant \frac{\rho_h^{(b,n)}\left(f\circ g,\varphi\right)}{\rho_k^{(d,q)}\left(f,\varphi\right)} . \tag{36}$$

Thus the first part of the theorem follows from (25), (28), (30), (32), (34) and (36).
Analogously, the second part of the theorem can be derived in a like manner. □

The following theorem can be proved in the line of Theorem 3.8 and so its proof is omitted.

Theorem 3.9. *Let f, g, h and k be any four entire functions with index pairs (p,q)-φ_1, (m,n)-φ, (a,b)-φ_1 and (c,d)-φ_1 respectively. Also let $\varphi\left(r\right):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies the condition (i) $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ for all $\alpha>0$.*
(i) *If either $(q=m=c,\ a=p)$ or $(q<m=c,\ a=p+m-q)$ holds,*

$\lambda^{(m,n)}(g,\varphi) > 0$, $0 < \lambda_h^{(b,n)}(f\circ g,\varphi) \leq \rho_h^{(b,n)}(f\circ g,\varphi) < \infty$, *then*

$$\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)} \leq \liminf_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]} M_k^{-1}(M_g(r))} \leq$$

$$\min\left\{\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,n)}(g,\varphi)}, \frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)}\right\} \leq$$

$$\max\left\{\frac{\lambda_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,n)}(g,\varphi)}, \frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)}\right\} \leq$$

$$\limsup_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}(M_{f\circ g}(r))}{\log^{[d]} M_k^{-1}(M_g(r))} \leq \frac{\rho_h^{(b,n)}(f\circ g,\varphi)}{\lambda_k^{(d,n)}(g,\varphi)}\ .$$

and

(ii) *If* $q > m = c$, $a = p$, $\lambda^{(m,n)}(g,\varphi) > 0$ *and* $0 < \lambda_h^{(b,q+n-m)}(f\circ g,\varphi) \leq \rho_h^{(b,q+n-m)}(f\circ g,\varphi) < \infty$, *then*

$$\frac{\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)} \leq \liminf_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(\exp^{[q-m]} r\right)\right)}{\log^{[d]} M_k^{-1}(M_g(r))} \leq$$

$$\min\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)}{\lambda_k^{(d,n)}(g,\varphi)}, \frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)}\right\} \leq$$

$$\max\left\{\frac{\lambda_h^{(b,q+n-m)}(f\circ g)}{\lambda_k^{(d,n)}(g,\varphi)}, \frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\rho_k^{(d,n)}(g,\varphi)}\right\} \leq$$

$$\limsup_{r\to+\infty}\frac{\log^{[b]} M_h^{-1}\left(M_{f\circ g}\left(\exp^{[q-m]} r\right)\right)}{\log^{[d]} M_k^{-1}(M_g(r))} \leq \frac{\rho_h^{(b,q+n-m)}(f\circ g,\varphi)}{\lambda_k^{(d,n)}(g,\varphi)}\ .$$

The proof of the following theorem can be carried out as of the Theorem 3.8, therefore we omit the details.

Remark 3.10. *The same results Theorem 3.8 and Theorem 3.9 in terms of maximum terms of entire functions can also be deduced with the help of Definition 1.10.*

Theorem 3.11. *Let f, g, h and k be any four entire functions with index pairs (p,q)-φ_1, (m,n)-φ, (a,b)-φ_1 and (c,d)-φ_1 respectively. Also let $\varphi(r) : [0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies the condition (i) $\lim\limits_{r\to+\infty}\frac{\log^{[q]} r}{\log^{[q]}\varphi(r)} = \alpha$ where $\alpha > 0$, (ii)*

$\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ *for all* $\alpha>0$.

(i) *If either* $(q=m,\ a=c=p,\ q\geq n)$ *or* $(q<m,\ c=p,\ a=p+m-q,\ q\geq n)$ *holds,* $0<\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)\leq\sigma_h^{(b,n)}(f\circ g,\varphi)<\infty$, $0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ *and* $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$, *then*

$$\frac{\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\leq\liminf_{r\to+\infty}\frac{\log^{[b-1]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d-1]}M_k^{-1}\left(M_f\left(\exp^{[q-n]}r\right)\right)}\leq$$

$$\min\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)},\frac{\sigma_h^{(b,n)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\right\}\leq$$

$$\max\left\{\frac{\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)},\frac{\sigma_h^{(b,n)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\right\}\leq$$

$$\limsup_{r\to+\infty}\frac{\log^{[b-1]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d-1]}M_k^{-1}\left(M_f\left(\exp^{[q-n]}r\right)\right)}\leq\frac{\sigma_h^{(b,n)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)}\ .$$

and

(ii) *If* $q>m$, $a=c=p$, $0<\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, $0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ *and* $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$, *then*

$$\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\leq\liminf_{r\to+\infty}\frac{\log^{[b-1]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d-1]}M_k^{-1}\left(M_f\left(\exp^{[m-n]}r\right)\right)}\leq$$

$$\min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)},\frac{\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\right\}\leq$$

$$\max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)},\frac{\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)}{\sigma_k^{(d,q)}(f,\varphi)}\right\}\leq$$

$$\limsup_{r\to+\infty}\frac{\log^{[b-1]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[d-1]}M_k^{-1}\left(M_f\left(\exp^{[m-n]}r\right)\right)}\leq\frac{\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)}{\overline{\sigma}_k^{(d,q)}(f,\varphi)}.$$

Remark 3.12. *In Theorem 3.11, if we will replace the conditions* "$0<\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)\leq\sigma_h^{(b,n)}(f\circ g,\varphi)<\infty$, $0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ *and* $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *and* "$0<\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, $0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ *and* $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *by* "$0<\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)\leq\tau_h^{(b,n)}(f\circ g,\varphi)$

$<\infty$, $0<\overline{\tau}_k^{(d,q)}(f,\varphi)\leq\tau_k^{(d,q)}(f,\varphi)<\infty$ and $\lambda_h^{(b,n)}(f\circ g,\varphi)=\lambda_k^{(d,q)}(f,\varphi)$" and "$0<\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\tau_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, $0<\overline{\tau}_k^{(d,q)}(f,\varphi)\leq\tau_k^{(d,q)}(f,\varphi)<\infty$ and $\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)=\lambda_k^{(d,q)}(f,\varphi)$" *respectively, then the conclusion of Theorem 3.11 remains valid with* $\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)$, $\tau_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\tau}_k^{(d,q)}(f,\varphi)$, $\tau_k^{(d,q)}(f,\varphi)$, $\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\tau_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\overline{\tau}_k^{(d,q)}(f,\varphi)$ *and* $\tau_k^{(d,q)}(f,\varphi)$ *replaced by* $\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)$, $\sigma_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\sigma}_k^{(d,q)}(f,\varphi)$, $\sigma_k^{(d,q)}(f,\varphi)$, $\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\overline{\sigma}_k^{(d,q)}(f,\varphi)$ *and* $\sigma_k^{(d,q)}(f,\varphi)$ *respectively.*

Remark 3.13. *In Theorem 3.11, if we will replace the conditions* "$0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ and $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *and* "$0<\overline{\sigma}_k^{(d,q)}(f,\varphi)\leq\sigma_k^{(d,q)}(f,\varphi)<\infty$ and $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *by* "$0<\overline{\tau}_k^{(d,q)}(f,\varphi)\leq\tau_k^{(d,q)}(f,\varphi)<\infty$ and $\rho_h^{(b,n)}(f\circ g,\varphi)=\lambda_k^{(d,q)}(f,\varphi)$" *and* "$0<\overline{\tau}_k^{(d,q)}(f,\varphi)\leq\tau_k^{(d,q)}(f,\varphi)<\infty$ and $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\lambda_k^{(d,q)}(f,\varphi)$" *respectively, then the conclusion of Theorem 3.11 remains valid with* $\overline{\tau}_k^{(d,q)}(f,\varphi)$, $\tau_k^{(d,q)}(f,\varphi)$, $\overline{\tau}_k^{(d,q)}(f,\varphi)$ *and* $\tau_k^{(d,q)}(f,\varphi)$ *replaced by* $\overline{\sigma}_k^{(d,q)}(f,\varphi)$, $\sigma_k^{(d,q)}(f,\varphi)$, $\overline{\sigma}_k^{(d,q)}(f,\varphi)$ *and* $\sigma_k^{(d,q)}(f,\varphi)$ *respectively.*

Remark 3.14. *In Theorem 3.11, if we will replace the conditions* "$0<\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)\leq\sigma_h^{(b,n)}(f\circ g,\varphi)<\infty$ and $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *and* "$0<\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$ and $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *by* "$0<\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)\leq\tau_h^{(b,n)}(f\circ g,\varphi)<\infty$ and $\lambda_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *and* "$0<\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\tau_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$ and $\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,q)}(f,\varphi)$" *respectively, then the conclusion of Theorem 3.11 remains valid with* $\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)$, $\tau_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)$ *and* $\tau_h^{(b,q+n-m)}(f\circ g,\varphi)$ *replaced by* $\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)$, $\sigma_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)$ *and* $\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)$ *respectively.*

Analogously one may formulate the following theorem without its proof.

Theorem 3.15. *Let f, g, h and k be any four entire functions with index pairs (p,q)-φ_1, (m,n)-φ, (a,b)-φ_1 and (c,d)-φ_1 respectively. Also let $\varphi(r):[0,+\infty)\to(0,+\infty)$ is a nondecreasing unbounded function and satisfies the condition* (i) $\lim\limits_{r\to+\infty}\frac{\log^{[n]}\varphi(ar)}{\log^{[n]}\varphi(r)}=1$ *for all* $\alpha>0$.

(i) *If either* $(q = m = c,\ a = p)$ *or* $(q < m = c,\ a = p + m - q)$ *holds,* $0 < \overline{\sigma}_h^{(b,n)}(f \circ g, \varphi) \leq \sigma_h^{(b,n)}(f \circ g, \varphi) < \infty$, $0 < \overline{\sigma}_k^{(d,n)}(g, \varphi) \leq \sigma_k^{(d,n)}(g, \varphi) < \infty$ *and* $\rho_h^{(b,n)}(f \circ g, \varphi) = \rho_k^{(d,n)}(g, \varphi)$ *then*

$$\frac{\overline{\sigma}_h^{(b,n)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)} \leq \liminf_{r \to +\infty} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r))}{\log^{[d-1]} M_k^{-1}(M_g(r))} \leq$$

$$\min\left\{\frac{\overline{\sigma}_h^{(b,n)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}, \frac{\sigma_h^{(b,n)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)}\right\} \leq$$

$$\max\left\{\frac{\overline{\sigma}_h^{(b,n)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}, \frac{\sigma_h^{(b,n)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)}\right\} \leq$$

$$\limsup_{r \to +\infty} \frac{\log^{[b-1]} M_h^{-1}(M_{f \circ g}(r))}{\log^{[d-1]} M_k^{-1}(M_g(r))} \leq \frac{\sigma_h^{(b,n)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}\ .$$

and

(ii) *If* $q > m = c$, $a = p$, $0 < \overline{\sigma}_h^{(b,q+n-m)}(f \circ g, \varphi) \leq \sigma_h^{(b,q+n-m)}(f \circ g, \varphi) < \infty$, $0 < \overline{\sigma}_k^{(d,n)}(g, \varphi) \leq \sigma_k^{(d,n)}(g, \varphi) < \infty$ *and* $\rho_h^{(b,q+n-m)}(f \circ g, \varphi) = \rho_k^{(d,n)}(g, \varphi)$ *then*

$$\frac{\overline{\sigma}_h^{(b,q+n-m)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)} \leq \liminf_{r \to +\infty} \frac{\log^{[b-1]} M_h^{-1}\left(M_{f \circ g}\left(\exp^{[q-m]} r\right)\right)}{\log^{[d-1]} M_k^{-1}(M_g(r))} \leq$$

$$\min\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}, \frac{\sigma_h^{(b,q+n-m)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)}\right\} \leq$$

$$\max\left\{\frac{\overline{\sigma}_h^{(b,q+n-m)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}, \frac{\sigma_h^{(b,q+n-m)}(f \circ g, \varphi)}{\sigma_k^{(d,n)}(g, \varphi)}\right\} \leq$$

$$\limsup_{r \to +\infty} \frac{\log^{[b-1]} M_h^{-1}\left(M_{f \circ g}\left(\exp^{[q-m]} r\right)\right)}{\log^{[d-1]} M_k^{-1}(M_g(r))} \leq \frac{\sigma_h^{(b,q+n-m)}(f \circ g, \varphi)}{\overline{\sigma}_k^{(d,n)}(g, \varphi)}\ .$$

Remark 3.16. *In Theorem 3.15, if we will replace the conditions "*$0 < \overline{\sigma}_h^{(b,n)}(f \circ g, \varphi) \leq \sigma_h^{(b,n)}(f \circ g, \varphi) < \infty$, $0 < \overline{\sigma}_k^{(d,n)}(g, \varphi) \leq \sigma_k^{(d,n)}(g, \varphi) < \infty$ *and* $\rho_h^{(b,n)}(f \circ g, \varphi) = \rho_k^{(d,n)}(g, \varphi)$*" and "*$0 < \overline{\sigma}_h^{(b,q+n-m)}(f \circ g, \varphi) \leq \sigma_h^{(b,q+n-m)}(f \circ g, \varphi) < \infty$, $0 < \overline{\sigma}_k^{(d,n)}(g, \varphi) \leq \sigma_k^{(d,n)}(g, \varphi) < \infty$ *and* $\rho_h^{(b,q+n-m)}(f \circ g, \varphi) = \rho_k^{(d,n)}(g, \varphi)$*" by "*$0 < \overline{\tau}_h^{(b,n)}(f \circ g, \varphi) \leq \tau_h^{(b,n)}(f \circ g, \varphi) < \infty$, $0 < \overline{\tau}_k^{(d,n)}(g, \varphi) \leq \tau_k^{(d,n)}(g, \varphi) < \infty$ *and* $\lambda_h^{(b,n)}(f \circ g, \varphi) = \lambda_k^{(d,n)}(g, \varphi)$*"*

and "$0<\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\tau_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty, 0<\overline{\tau}_k^{(d,n)}(g,\varphi)\leq\tau_k^{(d,n)}(g,\varphi)<\infty$ *and* $\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)=\lambda_k^{(d,n)}(g,\varphi)$*" respectively, then the conclusion of Theorem 3.15 remains valid with* $\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)$, $\tau_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\tau}_k^{(d,n)}(g,\varphi)$, $\tau_k^{(d,n)}(g,\varphi)$, $\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\tau_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\overline{\tau}_k^{(d,n)}(g,\varphi)$ *and* $\tau_k^{(d,n)}(g,\varphi)$ *replaced by* $\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)$, $\sigma_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\sigma}_k^{(d,n)}(g,\varphi)$, $\sigma_k^{(d,n)}(g,\varphi)$, $\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)$, $\overline{\sigma}_k^{(d,n)}(g,\varphi)$ *and* $\sigma_k^{(d,n)}(g,\varphi)$ *respectively.*

Remark 3.17. *In Theorem 3.15, if we will replace the conditions "*$0<\overline{\sigma}_k^{(d,n)}(g,\varphi)\leq\sigma_k^{(d,n)}(g,\varphi)<\infty$ *and* $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" and "*$0<\overline{\sigma}_k^{(d,n)}(g,\varphi)\leq\sigma_k^{(d,n)}(g,\varphi)<\infty$ *and* $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" by "*$0<\overline{\tau}_k^{(d,n)}(g,\varphi)\leq\tau_k^{(d,n)}(g,\varphi)<\infty$ *and* $\rho_h^{(b,n)}(f\circ g,\varphi)=\lambda_k^{(d,n)}(g,\varphi)$*" and "*$0<\overline{\tau}_k^{(d,n)}(g,\varphi)\leq\tau_k^{(d,n)}(g,\varphi)<\infty$ *and* $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\lambda_k^{(d,n)}(g,\varphi)$*" respectively, then the conclusion of Theorem 3.15 remains valid with* $\overline{\tau}_k^{(d,n)}(g,\varphi)$, $\tau_k^{(d,n)}(g,\varphi)$, $\overline{\tau}_k^{(d,n)}(g,\varphi)$ *and* $\tau_k^{(d,n)}(g,\varphi)$ *replaced by* $\overline{\sigma}_k^{(d,n)}(g,\varphi)$, $\sigma_k^{(d,n)}(g,\varphi)$, $\overline{\sigma}_k^{(d,n)}(g,\varphi)$ *and* $\sigma_k^{(d,n)}(g,\varphi)$ *respectively.*

Remark 3.18. *In Theorem 3.15, if we will replace the conditions "*$0<\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)\leq\sigma_h^{(b,n)}(f\circ g,\varphi)<\infty$, $\rho_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" and "*$0<\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, $\rho_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" by "*$0<\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)\leq\tau_h^{(b,n)}(f\circ g,\varphi)<\infty$, $\lambda_h^{(b,n)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" and "*$0<\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)\leq\tau_h^{(b,q+n-m)}(f\circ g,\varphi)<\infty$, $\lambda_h^{(b,q+n-m)}(f\circ g,\varphi)=\rho_k^{(d,n)}(g,\varphi)$*" respectively, then the conclusion of Theorem 3.15 remains valid with* $\overline{\tau}_h^{(b,n)}(f\circ g,\varphi)$, $\tau_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\tau}_h^{(b,q+n-m)}(f\circ g,\varphi)$ *and* $\tau_h^{(b,q+n-m)}(f\circ g,\varphi)$ *replaced by* $\overline{\sigma}_h^{(b,n)}(f\circ g,\varphi)$, $\sigma_h^{(b,n)}(f\circ g,\varphi)$, $\overline{\sigma}_h^{(b,q+n-m)}(f\circ g,\varphi)$ *and* $\sigma_h^{(b,q+n-m)}(f\circ g,\varphi)$ *respectively.*

Theorem 3.19. *Let* f, g *and* h *be any three entire functions such that* $\rho_k^{(p,q)}(g,\varphi)<\infty$ *and* $\lambda_h^{(p,q)}(f\circ g,\varphi)=\infty$. *Then*

$$\lim_{r\to\infty}\frac{\log^{[p]}M_h^{-1}(M_{f\circ g}(r))}{\log^{[p]}M_k^{-1}(M_g(r))}=\infty\ .$$

Proof. If possible, let there exists a constant β such that for a sequence of values of r tending to infinity we have

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \leq \beta \cdot \log^{[p]} M_k^{-1}\left(M_g\left(r\right)\right) \ . \tag{37}$$

Again from the definition of $\rho_k^{(p,q)}\left(g,\varphi\right)$, it follows for all sufficiently large values of r that

$$\log^{[p]} M_k^{-1}\left(M_g\left(r\right)\right) \leq \left(\rho_k^{(p,q)}\left(g,\varphi\right)+\varepsilon\right)\log^{[q]}\varphi\left(r\right) \ . \tag{38}$$

Now combining (37) and (38) we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) &\leq \beta\cdot\left(\rho_k^{(p,q)}\left(g,\varphi\right)+\varepsilon\right)\log^{[q]}\varphi\left(r\right) \\ i.e.,\ \lambda_h^{(p,q)}\left(f\circ g,\varphi\right) &\leq \beta\cdot\left(\rho_k^{(p,q)}\left(g,\varphi\right)+\varepsilon\right), \end{aligned}$$

which contradicts the condition $\lambda_h^{(p,q)}\left(f\circ g,\varphi\right)=\infty$. So for all sufficiently large values of r we get that

$$\log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) \geq \beta\cdot\log^{[p]} M_k^{-1}\left(M_g\left(r\right)\right),$$

from which the theorem follows. □

Remark 3.20. *Theorem 3.19 is also valid with "limit superior" instead of "limit" if $\lambda_h^{(p,q)}\left(f\circ g,\varphi\right)=\infty$ is replaced by $\rho_h^{(p,q)}\left(f\circ g,\varphi\right)=\infty$ and the other conditions remain the same.*

Corollary 3.21. *Under the assumptions of Theorem 3.19 and Remark 3.20,*

$$\lim_{r\to\infty}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[m-1]} M_k^{-1}\left(M_g\left(r\right)\right)}=\infty \text{ and } \overline{\lim_{r\to\infty}}\frac{\log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right)}{\log^{[m-1]} M_k^{-1}\left(M_g\left(r\right)\right)}=\infty$$

respectively.

Proof. By Theorem 3.19 we obtain for all sufficiently large values of r and for $K>1$,

$$\begin{aligned} \log^{[p]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) &> K\cdot\log^{[m]} M_k^{-1}\left(M_g\left(r\right)\right) \\ i.e.,\ \log^{[p-1]} M_h^{-1}\left(M_{f\circ g}\left(r\right)\right) &> \left\{\log^{[m-1]} M_k^{-1}\left(M_g\left(r\right)\right)\right\}^K, \end{aligned}$$

from which the first part of the corollary follows.

Similarly using Remark 3.20, we obtain the second part of the corollary. □

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