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ON SUFFICIENT CONDITIONS FOR CARATHÉODORY FUNCTIONS WITH THE FIXED SECOND COEFFICIENT

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Abstract. In the present paper, we derive several sufficient conditions for Carathéodory functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ under the constraint that the second coefficient of the function is preassigned. And, we obtain some sufficient conditions for strongly starlike functions in $\mathbb{D}$.

1. Introduction

For $0 < r \leq 1$, let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ and let $\mathbb{D}_1 = \mathbb{D}$ be the open unit disk. Let $\mathcal{H}$ denote the class of analytic functions in $\mathbb{D}$. For a constant $a \in \mathbb{C}$ and a fixed positive integer n, let $\mathcal{H}[a,n]$ be its subclass consisting of function p of the form

$$p(z) = a + p_n z^n + p_{n+1} z^{n+1} + \cdots . \tag{1}$$

Let $\mathcal{A}_n$ be the class consisting of analytic functions f defined in $\mathbb{D}$ of the form

$$f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \cdots$$

and $\mathcal{A} := \mathcal{A}_1$. The class $\mathcal{SS}^*(\delta)$ of strongly starlike functions of order δ $(0 < \delta \leq 1)$ consists of functions $f \in \mathcal{A}$ satisfying the inequality

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2}\delta \quad (z \in \mathbb{D}).$$

More specially, $\mathcal{SS}^*(1)$ is the class of starlike functions which will be denoted by $\mathcal{S}^*$ throughout this paper. That is, the function f in $\mathcal{S}^*$

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satisfies the following inequality:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

Let $\mathcal{P}$ denote the class of functions $p \in \mathcal{H}$ which satisfy

$$\Re\{p(z)\} > 0 \quad (z \in \mathbb{D}).$$

It is well known that the function $p \in \mathcal{P}$ is called a Carathéodory function. These functions have been studied extensively by various authors (*e.g.* [4, 5, 6, 8, 9, 10, 11, 12, 13, 16, 19]).

In this paper, we derive several sufficient conditions for Carathéodory functions in $\mathbb{D}$ under the constraint that the second coefficient of a function is preassigned. It is remarkable that the second coefficient plays an important role in the univalent function theory. Accordingly, many researchers have found various properties of univalent functions with the fixed second coefficient. One can refer to [2, 3, 7, 14, 17] for the works on univalent functions in this direction. In the present paper, we find sufficient conditions for Carathéodory functions associated with the fixed second coefficient. Especially, we find some conditions, which are related to the following functionals, for Carathéodory functions $p \in \mathcal{H}_\beta[a,n]$:

(2)
$$p(z) + P(z)zp'(z), \quad p(z) + \frac{zp'(z)}{p(z)} \quad \text{and} \quad \frac{1}{p(z)}\left(p(z) + \frac{zp'(z)}{p(z)} - 1\right),$$

for a suitable function P defined in $\mathbb{D}$. Here, and throughout this paper, $\mathcal{H}_\beta[a,n]$ means the subclass of $\mathcal{H}[a,n]$ consisting of functions p of the form given by (1) with $p_n = \beta$. We remark that the functionals defined by (2) are dealt with by means of the first-order differential subordination in [4, 8, 18].

We say that f is subordinate to F in $\mathbb{D}$, written as $f \prec F$, if and only if, $f(z) = F(\omega(z))$ for some Schwarz function $\omega(z)$, $\omega(0) = 0$ and $|\omega(z)| < 1$, $z \in \mathbb{D}$. If $F(z)$ is univalent in $\mathbb{D}$, then the subordination $f \prec F$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{D}} \setminus E(q)$, where

$$E(q) = \left\{\zeta \in \partial\mathbb{D} : \lim_{z\to\zeta} q(z) = \infty\right\}$$

and are such that $q'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{D} \setminus E(q)$. Furthermore, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$.

To prove our main results, we need the following lemmas.

Lemma 1.1. [1, Lemma 2.2] *Let $q \in \mathcal{Q}(a)$ and let $p \in \mathcal{H}_\beta[a,n]$ with $p(z) \not\equiv a$. If there exists a point $z_0 \in \mathbb{D}$ such that $p(z_0) \in q(\partial\mathbb{D})$ and $p(\mathbb{D}_{|z_0|}) \subset q(\mathbb{D})$, then*

$$z_0 p'(z_0) = m\zeta_0 q'(\zeta_0)$$

and

$$\Re\left\{1 + \frac{z_0 p''(z_0)}{p'(z_0)}\right\} \geq m\Re\left\{1 + \frac{\zeta_0 q''(\zeta_0)}{q'(\zeta_0)}\right\},$$

where

$$q^{-1}(p(z_0)) = \zeta_0 = \mathrm{e}^{\mathrm{i}\theta_0}$$

and

$$m \geq n + \frac{|q'(0)| - |\beta||z_0|^n}{|q'(0)| + |\beta||z_0|^n} \geq n + \frac{|q'(0)| - |\beta|}{|q'(0)| + |\beta|}.$$

Lemma 1.2. [15] *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and univalent in $\mathbb{D}$ and suppose that q maps $\mathbb{D}$ onto a convex domain. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in $\mathbb{D}$ and satisfies the subordination $p \prec q$ in $\mathbb{D}$, then*

$$|A_n| \leq |B_1| \quad (n \in \mathbb{N}).$$

2. Main Results

With the aid of Lemma 1.1 and 1.2, we can obtain the following results.

Theorem 2.1. *Let α be a real number such that $0 \leq \alpha < \pi/2$ and let $P : \mathbb{D} \to \mathbb{C}$ with*

$$\Re\{P(z)\} > \Im\{P(z)\}\tan\alpha \geq 0 \quad (z \in \mathbb{D}).$$

Let p be an analytic function in $\mathbb{D}$ with the form given by

$$p(z) = 1 + \beta z^n + p_{n+1} z^{n+1} + \cdots \quad (n \in \mathbb{N}). \tag{3}$$

If p satisfies

$$\begin{aligned} &\Re\left\{p(z) + P(z) z p'(z)\right\} \\ &> \frac{1}{2\mu A}\left\{(2\mu A + \cos\alpha)\sin^2\alpha - \mu^2 A^2 \cos\alpha\right\} \quad (z \in \mathbb{D}), \end{aligned} \tag{4}$$

where

$$A = \Re\{P(z)\}\cos\alpha - \Im\{P(z)\}\sin\alpha \tag{5}$$

and

$$\mu = \mu(n, \alpha, \beta) := n + \frac{2\cos\alpha - |\beta|}{2\cos\alpha + |\beta|} \geq n, \tag{6}$$

then

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (z \in \mathbb{D}).$$

Proof. First, let us define functions q and $h_1 : \mathbb{D} \to \mathbb{C}$ by

$$q(z) = \mathrm{e}^{\mathrm{i}\alpha} p(z) \tag{7}$$

and

$$h_1(z) = \frac{\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{-\mathrm{i}\alpha} z}{1 - z}, \tag{8}$$

respectively. Then we see that q and h_1 are analytic in $\mathbb{D}$ with

$$q(0) = h_1(0) = \mathrm{e}^{\mathrm{i}\alpha} \in \mathbb{C}$$

and

$$h_1(\mathbb{D}) = \{w \in \mathbb{C} : \mathfrak{R}\{w\} > 0\}.$$

That is, it holds that $q \in \mathcal{H}_{\mathrm{e}^{\mathrm{i}\alpha}\beta}[\mathrm{e}^{\mathrm{i}\alpha}, n]$ and $h_1 \in \mathcal{Q}(\mathrm{e}^{\mathrm{i}\alpha})$. We also note that $h_1'(0) = 2\cos\alpha$.

Now we suppose that q is not subordinate to h_1. Then by Lemma 1.1, there exist points $z_1 \in \mathbb{D}$ and $\zeta_1 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$q(z_1) = h_1(\zeta_1) = \mathrm{i}\rho \quad (\rho \in \mathbb{R}) \tag{9}$$

and

$$z_1 q'(z_1) = m\zeta_1 h_1'(\zeta_1) \quad (m \geq \mu), \tag{10}$$

where μ is defined by (6), since $h_1'(0) = 2\cos\alpha$. Furthermore, we have

$$\zeta_1 h_1'(\zeta_1) = -\frac{\rho^2 - 2\rho\sin\alpha + 1}{2\cos\alpha} =: \sigma_1. \tag{11}$$

Using the all equations above and by letting

$$B_1 = \mathfrak{R}\{P(z_1)\}\cos\alpha + \mathfrak{I}\{P(z_1)\}\sin\alpha,$$

we obtain

$$\begin{aligned}
&\mathfrak{R}\left\{p(z_1) + P(z_1) z_1 p'(z_1)\right\} \\
&= \mathfrak{R}\left\{\mathrm{e}^{-\mathrm{i}\alpha} h(\zeta_1) + P(z_1)\mathrm{e}^{-\mathrm{i}\alpha} m\zeta_1 h'(\zeta_1)\right\} \\
&= \rho\sin\alpha + m\sigma_1 B_1 \\
&\leq \rho\sin\alpha + \mu\sigma_1 B_1 \\
&= \frac{1}{2\cos\alpha}\varphi(\rho),
\end{aligned} \tag{12}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the function defined by

$$\varphi(x) = -\mu B_1 x^2 + 2\sin\alpha(\cos\alpha + \mu B_1)x - \mu B_1.$$

Since $B_1 > 0$ and $\mu > 0$, φ takes its maximum at ρ^* on $\mathbb{R}$, where

$$\rho^* = \frac{\sin\alpha(\cos\alpha + \mu B_1)}{\mu B_1}.$$

Therefore, by (12), we get

$$\begin{aligned}
&\Re\left\{p(z_1) + P(z_1)z_1p'(z_1)\right\} \\
&\leq \frac{1}{2\cos\alpha}\varphi(\rho^*) \\
&= \frac{1}{2\mu B_1}\left\{(2\mu B_1 + \cos\alpha)\sin^2\alpha - \mu^2 B_1^2\cos\alpha\right\} \\
&\leq \frac{1}{2\mu A_1}\left\{(2\mu A_1 + \cos\alpha)\sin^2\alpha - \mu^2 A_1^2\cos\alpha\right\},
\end{aligned} \tag{13}$$

where A_1 is defined by

$$A_1 = \Re\{P(z_1)\}\cos\alpha - \Im\{P(z_1)\}\sin\alpha.$$

Hence, (13) is a contradiction to (4) and we obtained $q \prec h_1$ in $\mathbb{D}$. Thus, it holds that

$$\Re\left\{e^{i\alpha}p(z)\right\} > 0 \quad (z \in \mathbb{D}). \tag{14}$$

Next, we define the functions r and $h_2 : \mathbb{D} \to \mathbb{C}$ by

$$r(z) = e^{-i\alpha}p(z) \tag{15}$$

and

$$h_2(z) = \frac{e^{-i\alpha} + e^{i\alpha}z}{1 - z}, \tag{16}$$

respectively. Then we see that r and h_2 are analytic in $\mathbb{D}$ with

$$r(0) = h_2(0) = e^{-i\alpha} \in \mathbb{C}$$

and

$$h_2(\mathbb{D}) = \{w \in \mathbb{C} : \Re\{w\} > 0\} = h_1(\mathbb{D}).$$

We also see that $h_2'(0) = 2\cos\alpha$.

Suppose that r is not subordinate to h_2. Then, by Lemma 1.1, there exist points $z_2 \in \mathbb{D}$ and $\zeta_2 \in \partial\mathbb{D} \setminus \{1\}$ such that

$$r(z_2) = h_2(\zeta_2) = i\rho \quad (\rho \in \mathbb{R})$$

and

$$z_2 r'(z_2) = m\zeta_2 h_2'(\zeta_2) \quad (m \geq \mu),$$

where μ is given by (6). We also note that

$$\zeta_2 h_2'(\zeta_2) = -\frac{\rho^2 + 2\rho\sin\alpha + 1}{2\cos\alpha} =: \sigma_2.$$

Then applying the equations above, we get

$$\begin{aligned}
&\Re\left\{p(z_2)+P(z_2)z_2p'(z_2)\right\}\\
&=\Re\left\{\mathrm{e}^{\mathrm{i}\alpha}h(\zeta_2)+P(z_2)\mathrm{e}^{\mathrm{i}\alpha}m\zeta_2h'(\zeta_2)\right\}\\
&=-\rho\sin\alpha+m\sigma_2A_2\\
&\leq-\rho\sin\alpha+\mu\sigma_2A_2\\
&=-\frac{1}{2\cos\alpha}\psi(\rho),
\end{aligned}\tag{17}$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is the function defined by

$$\psi(x)=\mu A_2x^2+2\sin\alpha(\cos\alpha+\mu A_2)x+\mu A_2,$$

and where

$$A_2=\Re\{P(z_2)\}\cos\alpha-\Im\{P(z_2)\}\sin\alpha.$$

Moreover, we have

$$\psi(x)\geq\psi(\rho^{**})=-\frac{\sin^2\alpha(\cos\alpha+\mu A_2)^2}{\mu A_2}+\mu A_2,\tag{18}$$

where

$$\rho^{**}=-\frac{\sin\alpha(\cos\alpha+\mu A_2)}{\mu A_2}.$$

From (17) and (18), we get

$$\begin{aligned}
&\Re\left\{p(z_2)+P(z_2)z_2p'(z_2)\right\}\\
&\leq-\frac{1}{2\cos\alpha}\psi(x^{**})\\
&=\frac{1}{2\mu A_2}\left\{(2\mu A_2+\cos\alpha)\sin^2\alpha-\mu^2A_2^2\cos\alpha\right\},
\end{aligned}$$

which is a contradiction to (4). Therefore we obtain

$$\Re\left\{\mathrm{e}^{-\mathrm{i}\alpha}p(z)\right\}>0\quad(z\in\mathbb{D}).\tag{19}$$

Hence it follows from (14) and (19) that

$$|\arg\{p(z)\}|<\frac{\pi}{2}-\alpha\quad(z\in\mathbb{D}).$$

Finally, the inequality in (6) comes from Lemma 1.2, since $q^{(n)}(0)=\beta\cdot n!$ and $h_1'(0)=2\cos\alpha$. This completes the proof of Theorem 2.1. $\square$

By putting $P(z)\equiv\gamma>0$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *Let α and γ be real numbers such that $0 \le \alpha < \pi/2$ and $\gamma > 0$. Let p be an analytic function in $\mathbb{D}$ with the form given by* (3)*. If p satisfies*

$$\Re\left\{p(z) + \gamma z p'(z)\right\} > \frac{1}{2\mu\gamma}[(\mu\gamma+1)^2 \sin^2\alpha - \mu^2\gamma^2] \quad (z \in \mathbb{D}),$$

where μ is the quantity defined by (6)*, then*

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (z \in \mathbb{D}).$$

By putting $\alpha = 0$ in Corollary 2.2, we have the following special result.

Corollary 2.3. *Let $\gamma > 0$ and let p be an analytic function in $\mathbb{D}$ with the form given by* (3)*. If p satisfies*

$$\Re\left\{p(z) + \gamma z p'(z)\right\} > -\frac{\gamma(2(n+1) + (n-1)|\beta|)}{2(2+|\beta|)} \quad (z \in \mathbb{D}),$$

then $\Re\{p(z)\} > 0$ for all $z \in \mathbb{D}$.

Example 2.4. *Consider a function $\hat{p} : \mathbb{D} \to \mathbb{C}$ defined by*

$$\hat{p}(z) = \frac{2+z}{2-z} = 1 + z + \frac{1}{2}z^2 + \frac{1}{4}z^3 + \cdots . \tag{20}$$

We see that $\hat{p}$ belongs to the class $\mathcal{H}_1[1,1]$. Moreover, using Maximum principle for harmonic functions, we can check the following relations are true for all $z \in \mathbb{D}$:

$$\begin{aligned} &\Re\left\{\hat{p}(z) + z\hat{p}'(z)\right\} \\ &= \Re\left\{\frac{4+4z-z^2}{(2-z)^2}\right\} \\ &\ge \min\left\{\Re\left\{\frac{4+4e^{i\theta}-e^{i2\theta}}{(2-e^{i\theta})^2}\right\} : \theta \in [0,2\pi)\right\} \\ &= \min\left\{\frac{8\cos\theta - 1}{(5-2\cos\theta)^2} : \theta \in [0,2\pi)\right\} \\ &= \min\left\{\phi(\theta) : \theta \in [0,2\pi)\right\}, \end{aligned} \tag{21}$$

where ϕ is the function defined by

$$\phi(\theta) = \frac{8\cos\theta - 1}{(5-2\cos\theta)^2} \quad (\theta \in [0,2\pi)).$$

Furthermore, we can see that

$$\phi(\theta) \ge \phi(\pi) = -\frac{9}{49} > -\frac{2}{3}. \tag{22}$$

Combining (21) *and* (22) *lead us to get (see Figure 1 below)*

$$\Re\left\{\hat{p}(z)+z\hat{p}'(z)\right\} > -\frac{2}{3} \quad (z\in\mathbb{D}),$$

and it follows from Corollary 2.3 with $\gamma = 1$ *that* $\Re\{\hat{p}(z)\} > 0$ *for all* $z \in \mathbb{D}$.

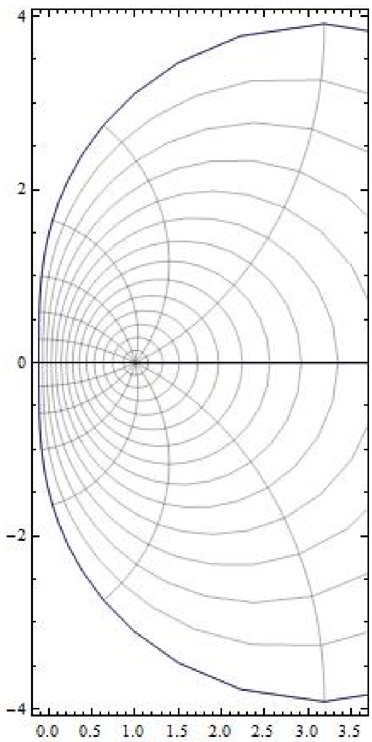

FIGURE 1. The image of $\hat{p}(z) + z\hat{p}'(z)$ on $\mathbb{D}$

Now, we find another sufficient conditions for Carathéodory functions.

Theorem 2.5. *Let* α *be a real number such that* $0 \le \alpha < \pi/2$ *and let* p *be a nonzero analytic function in* $\mathbb{D}$ *with the form given by* (3). *If*

$$\left|\Im\left\{p(z)+\frac{zp'(z)}{p(z)}\right\}\right| < \frac{1}{\cos\alpha}\left(\sqrt{(2\cos^2\alpha+\mu)\mu}-\mu\sin\alpha\right) \quad (z\in\mathbb{D}),$$

where μ is the quantity defined by (6), *then*

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (z \in \mathbb{D}). \tag{23}$$

Proof. We define the functions q and h_1 by (7) and (8), respectively. If q is not subordinate to h_1, then there exist points $z_1 \in \mathbb{D}$ and $\zeta_1 \in \partial\mathbb{D} \setminus \{1\}$ satisfying (9) and (10). By using the equations (9) and (10), we have

$$\begin{aligned} &\Im\left\{p(z_1) + \frac{z_1 p'(z_1)}{p(z_1)}\right\} \\ &= \Im\left\{\mathrm{e}^{-\mathrm{i}\alpha} h(\zeta_1) + \frac{m\zeta_1 h'(\zeta_1)}{h(\zeta_1)}\right\} \\ &= \rho\cos\alpha - \frac{m\sigma_1}{\rho}, \end{aligned} \tag{24}$$

where $m \geq \mu$, $\rho \in \mathbb{R}$ with $\rho \neq 0$ and $\sigma_1 < 0$ is given by (11).

For the case $\rho > 0$, since $\sigma_1 < 0$, using (11), we obtain

$$\begin{aligned} &\rho\cos\alpha - \frac{m\sigma_1}{\rho} \\ &\geq \rho\cos\alpha - \frac{\mu\sigma_1}{\rho} \\ &= \frac{1}{2\cos\alpha}\left[(2\cos^2\alpha + \mu)\rho - 2\mu\sin\alpha + \frac{\mu}{\rho}\right] \\ &\geq \frac{1}{\cos\alpha}\left[\sqrt{(2\cos^2\alpha + \mu)\mu} - \mu\sin\alpha\right]. \end{aligned} \tag{25}$$

By (24) and (25), we have

$$\Im\left\{p(z_1) + \frac{z_1 p'(z_1)}{p(z_1)}\right\} \geq \frac{1}{\cos\alpha}\left[\sqrt{(2\cos^2\alpha + \mu)\mu} - \mu\sin\alpha\right],$$

which is a contradiction to the assumption of Theorem 2.5.

For the case $\rho < 0$, using (11), we have

$$\begin{aligned} &\rho\cos\alpha - \frac{m\sigma_1}{\rho} \\ &\leq \rho\cos\alpha - \frac{\mu\sigma_1}{\rho} \\ &= -\frac{1}{\cos\alpha}\left[(2\cos^2\alpha + \mu)\tilde{\rho} + 2\mu\sin\alpha + \frac{\mu}{\tilde{\rho}}\right] \\ &\leq -\frac{1}{\cos\alpha}\left[\sqrt{(2\cos^2\alpha + \mu)\mu} + \mu\sin\alpha\right], \end{aligned}$$

where $\tilde{\rho} = -\rho > 0$. And this is also a contradiction to the assumption of Theorem 2.5. Hence, we have $q \prec h_1$ in $\mathbb{D}$ and

$$\Re\left\{e^{i\alpha}p(z)\right\} > 0 \quad (z \in \mathbb{D}). \tag{26}$$

We next consider the functions r and h_2 defined by (15) and (16), respectively. By using a similar method as the above, we obtain

$$\Re\left\{e^{-i\alpha}p(z)\right\} > 0 \quad (z \in \mathbb{D}). \tag{27}$$

Therefore making use of (26) and (27), we have (23) and this completes the proof of Theorem 2.5. □

Let $n \in \mathbb{N}$ and consider a function $f \in \mathcal{A}_n$ which has the expansion

$$f(z) = z + \beta z^{n+1} + a_{n+2}z^{n+2} + a_{n+3}z^{n+3} + \cdots . \tag{28}$$

If we define a function $p : \mathbb{D} \to \mathbb{C}$ by $p(z) = zf'(z)/f(z)$, then we have

$$p(z) = 1 + n\beta z^n + \cdots .$$

Therefore, we get $p \in \mathcal{H}_{n\beta}[1, n]$. Hence, for given $\delta \in (0, 1]$, by putting $p(z) = zf'(z)/f(z)$ and $\alpha = \pi(1 - \delta)/2$ in Theorem 2.5, we can obtain the following corollary.

Corollary 2.6. *Let $n \in \mathbb{N}$ and let δ be a real number such that $0 < \delta \leq 1$. Let $f \in \mathcal{A}_n$ be of the form given by* (28). *If*

$$\begin{aligned}&\left|\Im\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| \\ &< \csc\left(\frac{\pi}{2}\delta\right)\left[\sqrt{\left(2\sin^2\left(\frac{\pi}{2}\delta\right) + \tilde{\mu}\right)\tilde{\mu}} - \tilde{\mu}\cos\left(\frac{\pi}{2}\delta\right)\right] \quad (z \in \mathbb{D}),\end{aligned}$$

where

$$\tilde{\mu} = n + \frac{2\sin\left(\frac{\pi}{2}\delta\right) - n|\beta|}{2\sin\left(\frac{\pi}{2}\delta\right) + n|\beta|} \geq n, \tag{29}$$

then $f \in \mathcal{SS}^*(\delta)$.

More specially, by putting $\delta = 1$ in Corollary 2.6, we have a sufficient condition for starlike functions as follows:

Corollary 2.7. *Let $n \in \mathbb{N}$ and let $f \in \mathcal{A}_n$ be of the form given by* (28). *If f satisfies*

$$\left|\Im\left\{1 + \frac{zf''(z)}{f'(z)}\right\}\right| < \sqrt{\hat{\mu}(2 + \hat{\mu})} \quad (z \in \mathbb{D}),$$

where

$$\hat{\mu} = n + \frac{2 - n|\beta|}{2 + n|\beta|} \geq n,$$

then $f \in \mathcal{S}^*$.

Theorem 2.8. *Let* α *be a real number such that* $0 \leq \alpha < \pi/2$ *and let* p *be a nonzero analytic function in* $\mathbb{D}$ *with the form given by* (3). *If*

$$\left| p(z) + \frac{zp'(z)}{p(z)} - 1 \right| < (1 + \frac{1}{2}\mu)|p(z)| \cos\alpha \quad (z \in \mathbb{D}),$$

where μ *is the quantity defined by* (6), *then*

$$|\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (z \in \mathbb{D}).$$

Proof. Let $q(z) = \mathrm{e}^{\mathrm{i}\alpha}/p(z)$ and let h_1 be the function defined by (8). If q is not subordinate to h_1, then there exist points $z_1 \in \mathbb{D}$ and $\zeta_1 \in \partial\mathbb{D} \setminus \{1\}$ satisfying (9) and (10). By using the equations (9) and (10), we have

$$\left| \frac{p(z_1) + \frac{z_1 p'(z_1)}{p(z_1)} - 1}{p(z_1)} \right| = \left| \mathrm{e}^{-\mathrm{i}\alpha} h(\zeta_1) + \mathrm{e}^{-\mathrm{i}\alpha} m \zeta_1 h_1'(\zeta_1) - 1 \right|, \tag{30}$$

where $m \geq \mu$. And we have

$$\begin{aligned} &\left| \mathrm{e}^{-\mathrm{i}\alpha} h(\zeta_1) + \mathrm{e}^{-\mathrm{i}\alpha} m \zeta_1 h_1'(\zeta_1) - 1 \right|^2 \\ &= (\rho - \sin\alpha)^2 + (m\sigma_1 - \cos\alpha)^2 \\ &\geq (m\sigma_1 - \cos\alpha)^2. \end{aligned} \tag{31}$$

On the other hand, by (11), we have

$$m\sigma_1 - \cos\alpha = \frac{-m}{2\cos\alpha} \left(\rho^2 - 2\rho\sin\alpha + 1 + \frac{2}{m}\cos^2\alpha \right) \tag{32}$$

and by squaring (32), we get

$$\begin{aligned} &(m\sigma_1 - \cos\alpha)^2 \\ &= \frac{m^2}{4\cos^2\alpha} \left[(\rho - \sin\alpha)^2 + \left(1 + \frac{2}{m}\right)\cos^2\alpha \right]^2 \\ &\geq \left(1 + \frac{1}{2}m\right)^2 \cos^2\alpha. \end{aligned} \tag{33}$$

Therefore, by combining $m \geq \mu$, (30), (31) and (33), we have

$$\begin{aligned}\left|\frac{p(z_1)+\frac{z_1 p'(z_1)}{p(z_1)}-1}{p(z_1)}\right| &\geq \left(1+\frac{1}{2}m\right)\cos\alpha \\ &\geq \left(1+\frac{1}{2}\mu\right)\cos\alpha,\end{aligned}$$

which is a contradiction to the assumption of Theorem 2.8. Hence we have

$$\Re\left\{\frac{\mathrm{e}^{\mathrm{i}\alpha}}{p(z)}\right\} > 0 \quad (z \in \mathbb{D}). \tag{34}$$

Next, we consider the function r defined by $r(z) = \mathrm{e}^{-\mathrm{i}\alpha}/p(z)$ and the function h_2 defined by (16). Using a similar method as the above, we obtain

$$\Re\left\{\frac{\mathrm{e}^{-\mathrm{i}\alpha}}{p(z)}\right\} > 0 \quad (z \in \mathbb{D}). \tag{35}$$

Therefore by virtue of (34) and (35), we have the conclusion of Theorem 2.8. □

Corollary 2.9. *Let δ be a real number such that $0 < \delta \leq 1$ and let $f \in \mathcal{A}_n$ $(n \in \mathbb{N})$ be of the form given by* (28). *If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \left(1+\frac{1}{2}\tilde{\mu}\right)\left|\frac{zf'(z)}{f(z)}\right| \quad (z \in \mathbb{D}),$$

where $\tilde{\mu}$ is the quantity defined by (29), *then $f \in \mathcal{SS}^*(\delta)$.*

Corollary 2.10. *Let $f \in \mathcal{A}_n$ $(n \in \mathbb{N})$ be of the form given by* (28). *If*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \left(\frac{2n+6+(n^2+n)|\beta|}{2(2+n|\beta|)}\right)\left|\frac{zf'(z)}{f(z)}\right| \quad (z \in \mathbb{D}),$$

then $f \in \mathcal{S}^*$.

Example 2.11. *Consider a function $\tilde{f} : \mathbb{D} \to \mathbb{C}$ defined by*

$$\begin{aligned}\tilde{f}(z) &= z\exp\left(\int_0^z \frac{\hat{p}(\zeta)-1}{\zeta}\mathrm{d}\zeta\right) \\ &= z + z^2 + \frac{3}{4}z^3 + \frac{1}{2}z^4 + \cdots,\end{aligned}$$

where $\hat{p}$ is the function defined by (20). *We see that*

$$\frac{z\tilde{f}'(z)}{\tilde{f}(z)} = \hat{p}(z) = \frac{2+z}{2-z} \quad \text{and} \quad \frac{z\tilde{f}''(z)}{\tilde{f}'(z)} = \frac{2z(1+z)}{4-z^2}.$$

Therefore we have

$$\left(\frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right) / \left(\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\right) = \frac{2z(1+z)}{(2+z)^2} =: p_{\tilde{f}}(z). \tag{36}$$

Putting $z = re^{i\theta}$ *with* $r \in [0,1]$ *and* $\theta \in \mathbb{R}$, *we get*

$$\left|p_{\tilde{f}}(z)\right| = \frac{2r\sqrt{1+2r\cos\theta + r^2}}{4+4r\cos\theta + r^2}. \tag{37}$$

We now define a function $\Phi : [0,1] \times [-1,1] \to \mathbb{R}$ *by*

$$\Phi(r,x) = \frac{2r\sqrt{1+2rx+r^2}}{4+4rx+r^2}. \tag{38}$$

Then $(\partial\Phi/\partial x)(r,x) = 0$ *occurs when* $x = -3r/4 \in [-1,1]$ *and therefore we obtain the inequality*

(39)

$$\Phi(r,x) \leq \Phi\left(r, -\frac{3}{4}r\right) = \frac{r}{\sqrt{4-2r^2}} \leq \frac{\sqrt{2}}{2} \quad (r \in [0,1], x \in [-1,1]).$$

By combining (37), (38) *and* (39), *we have (see Figure 2 below)*

$$\left|p_{\tilde{f}}(z)\right| \leq \frac{\sqrt{2}}{2} \quad (z \in \mathbb{D}).$$

Hence, from (36), *we obtain*

$$\left|\frac{z\tilde{f}''(z)}{\tilde{f}'(z)}\right| \leq \frac{\sqrt{2}}{2}\left|\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\right| < \frac{5}{3}\left|\frac{z\tilde{f}'(z)}{\tilde{f}(z)}\right| \quad (z \in \mathbb{D}).$$

Thus, by Corollary 2.10 with $n = 1$ *and* $\beta = 1$, *it follows that* $\tilde{f} \in \mathcal{S}^*$.

3. Concluding Remark

In this section, we investigate a connection between the result already obtained by Kim and Cho [8] and the new one obtained in this paper by considering the fixed second coefficient. For this, we recall that the following result due to them [8, Theorem 1].

Theorem 3.1. *Let* $P : \mathbb{D} \to \mathbb{C}$ *with*

$$\Re\{P(z)\} > \Im\{P(z)\}\tan\alpha \geq 0 \quad (0 \leq \alpha < \pi/2).$$

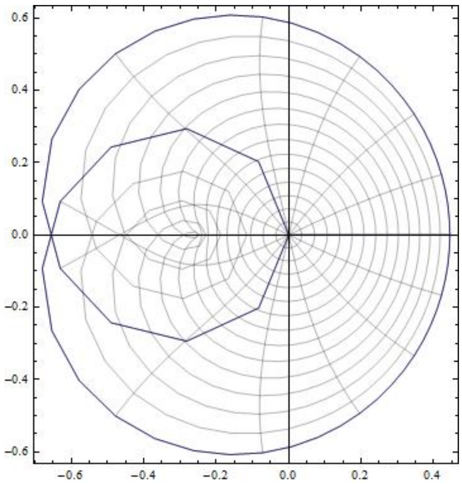

FIGURE 2. The image of $p_{\tilde{f}}$ on $\mathbb{D}$

If p is an analytic function in $\mathbb{D}$ with $p(0) = 1$ and

$$\begin{aligned}(40)\qquad &\Re\left\{p(z) + P(z)zp'(z)\right\} \\ &> \frac{1}{2A}\left\{(\cos\alpha + 2A)\sin^2\alpha - A^2\cos\alpha\right\} \quad (z \in \mathbb{D}),\end{aligned}$$

where A is defined by (5)*, then*

$$(41)\qquad |\arg\{p(z)\}| < \frac{\pi}{2} - \alpha \quad (z \in \mathbb{D}).$$

Keeping in mind that $\mu \geq 1$, we can check the following inequality holds:

$$\begin{aligned}(42)\qquad &\frac{1}{2\mu A}\left\{(2\mu A + \cos\alpha)\sin^2\alpha - \mu^2 A^2\cos\alpha\right\} \\ &\leq \frac{1}{2A}\left\{(\cos\alpha + 2A)\sin^2\alpha - A^2\cos\alpha\right\}.\end{aligned}$$

Hence, if $p \in \mathcal{H}$ with $p(0) = 1$ satisfies the inequality (40), then, $p \in \mathcal{H}_\beta[1, n]$ for some $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$. And, from the inequality (42), the function p satisfies the inequality (4). Therefore it follows from Theorem 2.1 that the inequality (41) holds. Conclusionally, the following relationship holds:

$$\text{Theorem 3.1} \quad \Longrightarrow \quad \text{Theorem 2.1.}$$

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