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ON THE HYERS-ULAM-RASSIAS STABILITY OF A GENERAL QUARTIC FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate Hyers-Ulam-Rassias stability of the general quartic functional equation

$f(5x+y)-5f(4x+y)+10f(3x+y)-10f(2x+y)+5f(x+y)-f(y)=0.$

1. Introduction

In 1940, Ulam [8] raised the question about the stability of group homomorphisms, and the following year Hyers [3] partially solved Ulam's question about the stability of Cauchy functional equation. In 1978, Rassias [7] generalized Hyers' results (Refer to Găvruta's paper [2] for a more generalized result). The concept of stability used by Rassias is called *Hyers-Ulam-Rassias stability.*

Let V and W be real vector spaces. A mapping $f : V \to W$ is called a general quartic mapping if f satisfies the functional equation

$$f(5x+y)-5f(4x+y)+10f(3x+y)-10f(2x+y)+5f(x+y)-f(y)=0 \tag{1}$$

which is called a *general quartic functional equation.* For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sum_{i=0}^{4} a_i x^i$, $a_i \in \mathbb{R}$, satisfies the above functional equation. In other words, the general quartic mapping is called *generalized polynomial mapping of degree 4* in Baker's paper [1]. A more detailed concept of general polynomial function equation of degree 4 can be found in Baker's paper [1].

Lee [6, 5] and Kim etc. [4] has previously studied the stability of a general quadratic functional equation and a general cubic functional equation.

In this paper, we will investigate Hyers-Ulam-Rassias stability of the functional equation (1).

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2. Main theorems

Throughout this paper, let X be a real normed space and Y a real Banach space. For a given mapping $f : X \to Y$, we use the following abbreviations:

$$\begin{aligned} f_o(x) &:= \frac{f(x)-f(-x)}{2}, \quad f_e(x) := \frac{f(x)+f(-x)}{2}, \\ Df(x,y) &:= f(5x+y) - 5f(4x+y) + 10f(3x+y) \\ &\quad - 10f(2x+y) + 5f(x+y) - f(y), \\ \Gamma f(x) &:= Df(x,-x) + 5Df(x,-2x) \end{aligned}$$

for all $x, y \in X$.

Let $p \notin \{1,2,3,4\}$ be a real number. For a given mapping $f : X \to Y$ with $f(0) = 0$, let $J_n f : X \to Y$ be the mappings defined by

$$J_n f(x) := \begin{cases} -\frac{4^n}{3}\left(f_e\left(\frac{x}{2^n}\right) - 16 f_e\left(\frac{x}{2^{n+1}}\right)\right) + \frac{16^{n+1}}{12}\left(f_e\left(\frac{x}{2^n}\right) - 4 f_e\left(\frac{x}{2^{n+1}}\right)\right) \\ \quad + \frac{4\cdot 8^n - 2^n}{3} f_o\left(\frac{x}{2^n}\right) - \frac{8^{n+1}-2^{n+3}}{3} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 4 < p, \\ \frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 16^n} \\ \quad + \frac{4\cdot 8^n - 2^n}{3} f_o\left(\frac{x}{2^n}\right) - \frac{8^{n+1}-2^{n+3}}{3} f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 3 < p < 4, \\ \frac{4^n}{12}\left(16 f_e(2^{-n}x) - f_e(2^{-n+1}x)\right) - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 16^n} \\ \quad - \frac{2^{n-1}}{3}\left(f_o\left(\frac{x}{2^{n-1}}\right) - 8 f_o\left(\frac{x}{2^n}\right)\right) + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } 2 < p < 3, \\ \frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 16^n} \\ \quad - \frac{2^{n-1}}{3}\left(f_o\left(\frac{x}{2^{n-1}}\right) - 8 f_o\left(\frac{x}{2^n}\right)\right) + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } 1 < p < 2, \\ \frac{16 f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 4^n} - \frac{4f_e(2^n x) - f_e(2^{n+1}x)}{12\cdot 16^n} \\ \quad + \frac{8 f_o(2^n x) - f_o(2^{n+1}x)}{6\cdot 2^n} + \frac{f_o(2^{n+1}x) - 2f_o(2^n x)}{6\cdot 8^n} & \text{if } p < 1 \end{cases}$$

for all $x \in X$ and all nonnegative integers n. Then, by the definition of $J_n f$ and Γf, we can calculate that

$J_n f(x) - J_{n+1} f(x) =$

(2)

$$\begin{cases} -\frac{4^n}{3}\Gamma f_e\left(\frac{x}{2^{n+2}}\right) + \frac{4^{2n+1}}{3}\Gamma f_e\left(\frac{x}{2^{n+2}}\right) + \frac{4\cdot 8^n}{3}\Gamma f_o\left(\frac{x}{2^{n+2}}\right) - \frac{2^n}{3}\Gamma f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 4 < p, \\ -\frac{4^n}{12}\Gamma f_e\left(\frac{x}{2^{n+1}}\right) - \frac{\Gamma f_e(2^n x)}{192\cdot 16^n} + \frac{4\cdot 8^n}{3}\Gamma f_o\left(\frac{x}{2^{n+2}}\right) - \frac{2^n}{3}\Gamma f_o\left(\frac{x}{2^{n+2}}\right) & \text{if } 3 < p < 4, \\ -\frac{4^n}{12}\Gamma f_e\left(\frac{x}{2^{n+1}}\right) - \frac{\Gamma f_e(2^n x)}{192\cdot 16^n} - \frac{1}{48\cdot 8^n}\Gamma f_o(2^n x) - \frac{2^{n-1}}{3}\Gamma f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 2 < p < 3, \\ \frac{\Gamma f_e(2^n x)}{48\cdot 4^n} - \frac{\Gamma f_e(2^n x)}{192\cdot 16^n} - \frac{1}{48\cdot 8^n}\Gamma f_o(2^n x) - \frac{2^{n-1}}{3}\Gamma f_o\left(\frac{x}{2^{n+1}}\right) & \text{if } 1 < p < 2, \\ \frac{\Gamma f_e(2^n x)}{48\cdot 4^n} - \frac{\Gamma f_e(2^n x)}{192\cdot 16^n} + \frac{1}{12\cdot 2^n}\Gamma f_o(2^n x) - \frac{1}{48\cdot 8^n}\Gamma f_o(2^n x) & \text{if } p < 1 \end{cases}$$

holds for all $x \in X$ and all nonnegative integers n. Therefore, together with the equality $f(x) - J_n f(x) = \sum_{i=0}^{n-1}(J_i f(x) - J_{i+1} f(x))$, we obtain the following lemma.

Lemma 2.1. *If $f : X \to Y$ is a mapping such that*

$$Df(x,y) = 0$$

for all $x, y \in X$*, then*

$$J_n f(x) = f(x)$$

for all $x \in X$ *and all positive integers* n*.*

From Lemma 2.1, we can prove the following stability theorem.

Theorem 2.2. *Let* $p \neq 1, 2, 3, 4$ *be a real number. Suppose that* $f : X \to Y$ *is a mapping such that*

$$\|Df(x, y)\| \leq \theta(\|x\|^p + \|y\|^p) \tag{3}$$

for all $x, y \in X \backslash \{0\}$*. Then there exists a unique solution mapping* F *of the functional equation* $DF(x, y) = 0$ *such that*

$$\|f(x) - f(0) - F(x)\| \leq \left(\frac{7 + 5 \cdot 2^p}{|2^p - 16||2^p - 4|} + \frac{7 + 5 \cdot 2^p}{|2^p - 8||2^p - 2|}\right)\theta\|x\|^p \tag{4}$$

for all $x \in X \backslash \{0\}$ *and* $F(0) = 0$*.*

Proof. If $\tilde{f}$ is the mapping defined by $\tilde{f}(x) = f(x) - f(0)$, then $D\tilde{f}(x, y) = Df(x, y)$ and $\tilde{f}(0) = 0$. By the definition of Γf and (3), we have

$$\begin{aligned} \|\Gamma \tilde{f}_e(x)\| &= \left\|Df_e(x, x) + 5Df_e(x, -2x)\right\| \leq \theta(7 + 5 \cdot 2^p)\|x\|^p, \\ \|\Gamma \tilde{f}_o(x)\| &= \left\|Df_o(x, x) + 5Df_o(x, -2x)\right\| \leq \theta(7 + 5 \cdot 2^p)\|x\|^p \end{aligned}$$

for all $x \in X$. It follows from (2) and (3) that

$$\frac{\|J_n\tilde{f}(x) - J_{n+1}\tilde{f}(x)\|}{7+5\cdot 2^p} \leq$$

$$\begin{cases} \left(\frac{4^n(4^{n+1}-1)}{3\cdot 2^{(n+2)p}} + \frac{(4\cdot 8^n - 2^n)}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 4 < p, \\ \left(\frac{2^{np}}{12\cdot 16^{n+1}} + \frac{4^{n-1}}{3\cdot 2^{(n+1)p}} + \frac{(4\cdot 8^n - 2^n)}{3\cdot 2^{(n+2)p}}\right)\theta\|x\|^p & \text{if } 3 < p < 4, \\ \left(\frac{2^{np}}{12\cdot 16^{n+1}} + \frac{4^{n-1}}{3\cdot 2^{(n+1)p}} + \frac{2^{np}}{6\cdot 8^{n+1}} + \frac{2^n}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 2 < p < 3, \\ \left(\frac{(4^{n+1}-1)2^{np}}{3\cdot 4^{2n+3}} + \frac{2^{np}}{6\cdot 8^{n+1}} + \frac{2^n}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 1 < p < 2, \\ \left(\frac{(4^{n+1}-1)2^{np}}{3\cdot 4^{2n+3}} + \frac{(4^{n+1}-1)2^{np}}{6\cdot 8^{n+1}}\right)\theta\|x\|^p & \text{if } 0 < p < 1 \end{cases}$$

for all $x \in X \backslash \{0\}$. Together with the equality $J_n\tilde{f}(x) - J_{n+m}\tilde{f}(x) = \sum_{i=n}^{n+m-1}(J_i\tilde{f}(x) - J_{i+1}\tilde{f}(x))$ for all $x \in X$, we get

$$\frac{\|J_n\tilde{f}(x)-J_{n+m}\tilde{f}(x)\|}{7+5\cdot 2^p} \leq$$

$$(5)\quad \begin{cases} \sum_{i=n}^{n+m-1}\left(\frac{4^i(4^{i+1}-1)}{3\cdot 2^{(i+2)p}}+\frac{(4\cdot 8^i-2^i)}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 4<p, \\ \sum_{i=n}^{n+m-1}\left(\frac{2^{ip}}{12\cdot 16^{i+1}}+\frac{4^{i-1}}{3\cdot 2^{(i+1)p}}+\frac{(4\cdot 8^i-2^i)}{3\cdot 2^{(i+2)p}}\right)\theta\|x\|^p & \text{if } 3<p<4, \\ \sum_{i=n}^{n+m-1}\left(\frac{2^{ip}}{12\cdot 16^{i+1}}+\frac{4^{i-1}}{3\cdot 2^{(i+1)p}}+\frac{2^{ip}}{6\cdot 8^{i+1}}+\frac{2^i}{6\cdot 2^{(i+1)p}}\right)\theta\|x\|^p & \\ & \text{if } 2<p<3, \\ \sum_{i=n}^{n+m-1}\left(\frac{(4^{i+1}-1)2^{ip}}{3\cdot 4^{2i+3}}+\frac{2^{ip}}{6\cdot 8^{i+1}}+\frac{2^i}{6\cdot 2^{(n+1)p}}\right)\theta\|x\|^p & \text{if } 1<p<2, \\ \sum_{i=n}^{n+m-1}\left(\frac{(4^{i+1}-1)2^{ip}}{3\cdot 4^{2i+3}}+\frac{(4^{i+1}-1)2^{ip}}{6\cdot 8^{i+1}}\right)\theta\|x\|^p & \text{if } p<1 \end{cases}$$

for all $x \in X\backslash\{0\}$ and $n, m \in \mathbb{N}\cup\{0\}$. It follows from (5) that the sequence $\{J_n\tilde{f}(x)\}$ is a Cauchy sequence for all $x \in X\backslash\{0\}$. Since Y is complete and $\tilde{f}(0)=0$, the sequence $\{J_n\tilde{f}(x)\}$ converges for all $x \in X$. Hence we can define a mapping $F : X \to Y$ by

$$F(x) := \lim_{n\to\infty} J_n\tilde{f}(x)$$

for all $x \in X$. Moreover, letting $n=0$ and passing the limit $n \to \infty$ in (5) we get the inequality (4). For the case $2 < p < 3$, from the definition of F, we easily get

$$\begin{aligned} \|DF(x,y)\| = \lim_{n\to\infty}\Big\| & \frac{4^n}{12}\left(-Df_e\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+16Df_e\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right) \\ & +\frac{Df_e\left(2^{n+1}x,2^{n+1}y\right)-4Df_e\left(2^nx,2^ny\right)}{12\cdot 16^n} \\ & +\frac{2^n}{6}\left(-Df_o\left(\frac{2x}{2^n},\frac{2y}{2^n}\right)+8Df_o\left(\frac{x}{2^n},\frac{y}{2^n}\right)\right) \\ & +\frac{Df_o\left(2^{n+1}x,2^{n+1}y\right)-2Df_o\left(2^nx,2^ny\right)}{6\cdot 8^n}\Big\| \\ \leq \lim_{n\to\infty} & \left(\frac{4^n(2^p+16)}{12\cdot 2^{np}}+\frac{2^{np}(2^p+4)}{12\cdot 16^n}+\frac{2^n(2^p+8)}{6\cdot 2^{np}}+\frac{2^{np}(2^p+2)}{6\cdot 8^n}\right)\times \\ & \theta(\|x\|^p+\|y\|^p) \\ =&0 \end{aligned}$$

for all $x, y \in X\backslash\{0\}$. Since $DF(-x,0) = -DF(x,-5x)$ and $DF(0,y) = 0$ for all $x, y \in X\backslash\{0\}$, if $F : X \to Y$ satisfies the equality $DF(x,y) = 0$ for all $x, y \in X\backslash\{0\}$ with $F(0) = 0$, then $DF(x,y) = 0$ for all $x, y \in X$. Also we easily show that $DF(x,y) = 0$ by the similar method for the other cases, either $p < 1$ or $1 < p < 2$ or $3 < p < 4$ or $4 < p$. To prove the uniqueness of F, let $F' : X \to Y$ be another solution mapping satisfying (4). By Lemma 2.1, the equality $F'(x) = J_n F'(x)$ holds for all $n \in \mathbb{N}$. For the case $2 < p < 3$, we have

$$\begin{aligned}
&\|J_n\tilde{f}(x) - F'(x)\| \\
&=\|J_n\tilde{f}(x) - J_nF'(x)\| \\
&\leq\Big\| -\frac{2^n}{6}\Big(\tilde{f}_o\Big(\frac{2x}{2^n}\Big) - 8\tilde{f}_o\Big(\frac{x}{2^n}\Big)\Big) - \frac{4^{n-1}}{3}\Big(\tilde{f}_e\Big(\frac{2x}{2^n}\Big) - 16\tilde{f}_e\Big(\frac{x}{2^n}\Big)\Big) \\
&\quad + \frac{\tilde{f}_o(2^{n+1}x) - 2\tilde{f}_o(2^nx)}{6\cdot 8^n} + \frac{\tilde{f}_e(2^{n+1}x) - 4\tilde{f}_e(2^nx)}{12\cdot 16^n} \\
&\quad + \frac{2^n}{6}\Big(F_o'\Big(\frac{2x}{2^n}\Big) - 8F_o'\Big(\frac{x}{2^n}\Big)\Big) + \frac{4^{n-1}}{3}\Big(F_e'\Big(\frac{2x}{2^n}\Big) - 16F_e'\Big(\frac{x}{2^n}\Big)\Big) \\
&\quad - \frac{F_o'(2^{n+1}x) - 2F_o'(2^nx)}{6\cdot 8^n} - \frac{F_e'(2^{n+1}x) - 4F_e'(2^nx)}{12\cdot 16^n}\Big\| \\
&\leq\frac{2^n}{6}\Big\|(\tilde{f}_o - F_o')\Big(\frac{2x}{2^n}\Big)\Big\| + \frac{2^{n+3}}{6}\Big\|(\tilde{f}_o - F_o')\Big(\frac{x}{2^n}\Big)\Big\| + \frac{4^n}{12}\Big\|(\tilde{f}_e - F_e')\Big(\frac{2x}{2^n}\Big)\Big\| \\
&\quad + \frac{4^{n+1}}{3}\Big\|(\tilde{f}_e - F_e')\Big(\frac{x}{2^n}\Big)\Big\| + \frac{\|\tilde{(}f_o - F_o')(2^{n+1}x)\|}{6\cdot 8^n} \\
&\quad + \frac{2\|(\tilde{f}_o - F_o')(2^nx)\|}{6\cdot 8^n} + \frac{\|(\tilde{f}_e - F_e')(2^nx)\|}{3\cdot 16^n} + \frac{\|(\tilde{f}_e - F_e')(2^{n+1}x)\|}{12\cdot 16^n} \\
&\leq\Big(\frac{2^{n-1+p} + 2^{n+2} + 4^{n-1+p} + 4^{n+1}}{3\cdot 2^{np}} + \frac{2^{(n+1)p} + 2^{np+1}}{3\cdot 2^{3n+1}} + \frac{2^{np+2} + 2^{(n+1)p}}{3\cdot 4^{2n+1}}\Big) \\
&\quad \times \Big[\frac{1}{|2^p - 16||2^p - 4|} + \frac{1}{|2^p - 8||2^p - 2|}\Big](7 + 5\cdot 2^p)\theta\|x\|^p
\end{aligned}$$

for all $x \in X\backslash\{0\}$ and all positive integers n. Taking the limit in the above inequality as $n \to \infty$, we can conclude that $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in X\backslash\{0\}$. For the other cases, either $0 < p < 1$ or $1 < p < 2$ or $3 < p < 4$ or $4 < p$, we also easily show that $F'(x) = \lim_{n\to\infty} J_n\tilde{f}(x)$ for all $x \in X\backslash\{0\}$ by the similar method. With the property $F'(0) = 0$, we know that $F(x) = F'(x)$ for all $x \in X$. □

Theorem 2.3. *Let $p < 0$ be a real number. If a mapping $f : X \to Y$ satisfies the inequality* (3) *for all $x, y \in X\backslash\{0\}$, then $f : X \to Y$ satisfies the equality $Df(x, y) = 0$ for all $x, y \in X$.*

Proof. According to Theorem 2.2, there is a unique solution mapping F of the functional equation $DF(x, y) = 0$ such that

$$\|\tilde{f}(x) - F(x)\| \leq \Big(\frac{7 + 5\cdot 2^p}{|2^p - 16||2^p - 4|} + \frac{7 + 5\cdot 2^p}{|2^p - 8||2^p - 2|}\Big)\theta\|x\|^p$$

for all $x \in X\backslash\{0\}$, where $\tilde{f}(x) = f(x) - f(0)$. From the equality

$$\begin{aligned} D\tilde{f}(nx, -(n-1)x) =& D\tilde{f}(nx, -(n-1)x) - DF(nx, -(n-1)x) \\ =& (\tilde{f} - F)((4n+1)x) - 5(\tilde{f} - F)((3n+1)x) \\ &+ 10(\tilde{f} - F)((2n+1)x) - 10(\tilde{f} - F)((n+1)x) \\ &+ 5(\tilde{f} - F)(x) - (\tilde{f} - F)((-n+1)x) \end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}$, we have the inequality

$$\begin{aligned} &\|5(\tilde{f} - F)(x)\| \\ &= \Big\| (\tilde{f} - F)((4n+1)x) - 5(\tilde{f} - F)((3n+1)x) \\ &\quad + 10(\tilde{f} - F)((2n+1)x) - 10(\tilde{f} - F)((n+1)x) - (f - F)(-(n-1)x) \\ &\quad - Df(nx, -(n-1)x) \Big\| \\ &\leq \Bigg[\left(\frac{7 + 5 \cdot 2^p}{|2^p - 16||2^p - 4|} + \frac{7 + 5 \cdot 2^p}{|2^p - 8||2^p - 2|} \right) ((4n+1)^p + 5(3n+1)^p \\ &\quad + 10(2n+1)^p + 10(n+1)^p + (n-1)^p) + n^p + (n-1)^p \Bigg] \theta \|x\|^p \end{aligned}$$

for all $x \in X\backslash\{0\}$ and $n \in \mathbb{N}$. Since $(4n+1)^p$, $(3n+1)^p$, $(2n+1)^p$, $(n+1)^p$, n^p, $(n-1)^p$ tend to 0 as $n \to \infty$ and $\tilde{f}(0) = F(0)$, we get $\tilde{f}(x) = F(x)$ for all $x \in X$. Therefore $Df(x, y) = D\tilde{f}(x, y) = DF(x, y) = 0$ for all $x, y \in X$. □

References

[1] J. Baker, *A general functional equation and its stability*, Proc. Natl. Acad. Sci. **133(6)** (2005), 1657–1664.

[2] P. Găvruta, *A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[3] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. USA **27** (1941), 222–224.

[4] K.-W. Jun and H.-M. Kim, *On the Hyers-Ulam-Rassias stability of a general cubic functional equation*, Math. Inequal. Appl. **6** (2003), 289–302.

[5] Y.-H. Lee, *On the generalized Hyers-Ulam stability of the generalized polynomial function of degree 3*, Tamsui Oxford Journal of Mathematical Sciences **24(4)** (2008), 429–444.

[6] Y.-H. Lee, *On the Hyers-Ulam-Rassias stability of the generalized polynomial function of degree 2*, J. Chungcheong Math. Soc. **22(2)** (2009), 201–209.

[7] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[8] S.M. Ulam, *A Collection of Mathematical Problems*, Interscience, New York, 1960.

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