East Asian Math. J.
Vol. 35 (2019), No. 3, pp. 331–340
http://dx.doi.org/10.7858/eamj.2019.029

ON THE STABILITY OF A GENERAL QUADRATIC-CUBIC FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN NORMED SPACES

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Abstract. In this paper, we investigate the stability for the functional equation
$f(x+ky)-kf(x+y)+kf(x-y)-f(x-ky)-(k^3-k)f(y)+(k^3-k)f(-y)=0$
in the sense of M. S. Moslehian and Th. M. Rassias.

1. Introduction

In 1940, Ulam [18] proposed the problem concerning the stability of group homomorphisms. In 1941, Hyers [6] gave an affirmative answer to this problem for additive mappings between Banach spaces. Subsequently many mathematicians came to deal with this problem [5, 15].

Recently M. S. Moslehian and Th. M. Rassias [13] discussed the Hyers-Ulam stability of the Cauchy functional equation

$$f(x+y)=f(x)+f(y)$$

and the quadratic functional equation

$$f(x+y)+f(x-y)-2f(x)-2f(y)=0$$

in non-Archimedean normed spaces. The following definitions and terminologies were introduced by M. S. Moslehian and Th. M. Rassias [13].

Definition 1. Let $\mathbb{K}$ be a field. A function $|\cdot|:\mathbb{K}\to[0,\infty)$ is a non-Archimedean valuation if the following conditions hold:
(i) $|r|=0$ if and only if $r=0$,
(ii) $|rs|=|r||s|$, and
(iii) $|r+s|\le\max\{|r|,|s|\}$ for all $r,s\in\mathbb{K}$.
A scalar field $\mathbb{K}$ with a non-Archimedean valuation $|\cdot|$ is called a non-Archimedean field.

Received December 19, 2018; Revised April 4, 2019; Accepted May 8, 2019.
2010 *Mathematics Subject Classification.* 39B22,39B82,39B52.
Key words and phrases. stability; general quadratic-cubic functional equation; general quadratic-cubic mapping; direct method.

(pISSN 1226-6973, eISSN 2287-2833)

Clearly $|1| = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{N}$.

Definition 2. Let X be a vector space over a scalar field $\mathbb{K}$ with a non-Archimedean non- trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to \mathbb{R}$ is *a non-Archimedean norm (valuation)* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\|$ $(r \in \mathbb{K}, x \in X)$;
(iii) the strong triangle inequality (ultrametric); namely,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \ (x, y \in X).$$

Then $(X, \|\cdot\|)$ is called *a non-Archimedean space.*

Due to the fact that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n-1\}(n > m)$$

a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero in a non-Archimedean space. By *a complete non-Archimedean space* we mean one in which every Cauchy sequence is convergent.

A solution of the quadratic functional equation is called a quadratic mapping and a solution of the functional equation

$$f(x + 2y) - 3f(x + y) + 3f(x) - f(x - y) - 6f(y) = 0.$$

is called a cubic mapping [7, 12, 14]. A mapping f is called a general quadratic-cubic mapping if f is represented by sum of a quadratic mapping, a cubic mapping and a constant mapping. A functional equation is called a general quadratic-cubic functional equation provided that each solution of that equation is a general quadratic-cubic mapping and every general quadratic-cubic mapping is a solution of that equation [2, 3, 10, 11, 12, 17, 19]. Now, consider the following functional equation

$$\begin{aligned} f(x + ky) - kf(x + y) + kf(x - y) - f(x - ky) \\ - (k^3 - k)f(y) + (k^3 - k)f(-y) = 0. \end{aligned} \tag{1}$$

It is easy to see that the mapping $f(x) = ax^3 + bx^2 + c$ is a solution of the functional equation (1), where $f : \mathbb{R} \to \mathbb{R}$ and a, b, c are real constants.

When k is a fixed rational number such that $k \neq 0, \pm 1$, the functional equation (1) is a general quadratic-cubic functional equation.

In this paper, we investigate the general stability of that functional equation in non-Archimedean normed spaces.

2. Stability of the quadratic-cubic functional equation

Throughout this section, assume that V and W are real vector spaces, X and Y are non-Archimedean normed spaces over $\mathbb{K}$ with $|k| < 1$, and k is a real number such that $k \neq 0, \pm 1$. For a given mapping $f : V \to W$, we use the following abbreviations:

$$\begin{aligned} f_o(x) :=& \frac{f(x) - f(-x)}{2}, \\ f'_e(x) :=& \frac{f(x) + f(-x)}{2} - f(0), \\ Qf(x,y) :=& f(x+y) + f(x-y) - 2f(x) - 2f(y), \\ Cf(x,y) :=& f(x+2y) - 3f(x+y) + 3f(x) - f(x-y) - 6f(y), \\ D_k f(x,y) :=& f(x+ky) - kf(x+y) + kf(x-y) - f(x-ky) \\ & - (k^3 - k)f(y) + (k^3 - k)f(-y) \end{aligned}$$

for all $x, y \in V$.

We need the following particular case of Baker's theorem [1] to prove main theorem.

Theorem 2.1. (Theorem 1 in [1]) *Suppose that V and W are vector spaces over $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and $\alpha_0, \beta_0, \ldots, \alpha_m, \beta_m$ are scalar such that $\alpha_j \beta_l - \alpha_l \beta_j \neq 0$ whenever $0 \leq j < l \leq m$. If $f_l : V \to W$ for $0 \leq l \leq m$ and*

$$\sum_{l=0}^{m} f_l(\alpha_l x + \beta_l y) = 0$$

for all $x, y \in V$, then each f_l is a generalized polynomial mapping of degree at most $m - 1$.

We easily obtain the following theorem from Theorem 2.1.

Theorem 2.2. *Let k be a rational number such that $k \neq 0, \pm 1$. If a mapping $f : V \to W$ satisfies the functional equation $D_k f(x,y) = 0$ for all $x, y \in V$, then f is a generalized polynomial mapping of degree at most 3.*

Suppose that $f, g : V \to W$ are generalized polynomial mapping of degree at most 3. It is well known that if the equalities $f(kx) = k^2 f(x)$ and $g(kx) = k^3 g(x)$ hold for all $x \in V$ and any rational number k such that $k \neq 0, \pm 1$, then f and g are a quadratic mapping and a cubic mapping, respectively.

In the next theorem we will show that the functional equation $D_k f \equiv 0$ is a general quadratic-cubic functional equation when k be a fixed rational number such that $k \neq 0, \pm 1$.

Theorem 2.3. *Let k be a fixed rational number such that $k \neq 0, \pm 1$. A mapping f satisfies the functional equation $D_k f(x,y) = 0$ for all $x, y \in V$ if and only if f'_e is quadratic and f_o is cubic.*

Proof. If a mapping f satisfies the functional equation $D_k f(x,y) = 0$ for all $x, y \in V$, then the equalities $f_o(kx) = k^3 f_o(x)$ and $f'_e(kx) = k^2 f'_e(x)$ are obtained from the equalities

$$\begin{aligned} f_o(kx) - k^3 f_o(x) =& \frac{D_k f(0,x)}{2}, \\ f'_e(kx) - k^2 f'_e(x) =& \frac{1}{2}\left(D_k f\left(\frac{kx}{2}, \frac{x}{2}\right) + D_k f\left(-\frac{kx}{2}, -\frac{x}{2}\right)\right. \\ & \left. + kD_k f\left(\frac{x}{2}, \frac{x}{2}\right) + kD_k f\left(-\frac{x}{2}, -\frac{x}{2}\right)\right) \end{aligned}$$

for all $x \in V$. Since f_o and f'_e are generalized polynomial mappings of degree at most 3, f_o is a cubic mapping and f'_e is a quadratic mapping.

Conversely, assume that f_o is a cubic mapping and f'_e is a quadratic mapping, i.e., f is a general quadratic-cubic mapping. Notice that f_o satisfies the equality $f_o(kx) = k^3 f_o(x)$ and $f_o(x) = -f_o(-x)$, f_e satisfies $f'_e(kx) = k^2 f'_e(x)$ and $f'_e(x) = f'_e(-x)$ for all $x \in V$ and all $k \in \mathbb{Q}$, and $f(x) = f_o(x) + f'_e(x) + f(0)$.

The equalities $D_2 f(x,y) = 0$ and $D_3 f(x,y) = 0$ follow from the equalities

$$\begin{aligned} D_2 f_o(x,y) &= Cf_o(x,y) + Cf_o(x-y,y), \\ D_2 f'_e(x,y) &= Qf'_e(x+y,y) - Qf'_e(x-y,y), \\ D_3 f_o(x,y) &= Cf_o(x-y,2y), \\ D_3 f'_e(x,y) &= Qf'_e(x+y,2y) - Qf'_e(x-y,2y), \\ D_2 f(0,0) &= 0, \\ D_3 f(0,0) &= 0 \end{aligned}$$

for all $x, y \in V$. If the equality $D_j f(x,y) = 0$ holds for all $j \in \mathbb{N}$ when $2 \le j \le n-1$, then the equality $D_n f(x,y) = 0$ follows from the equality

$$D_n f(x,y) = D_{n-1} f(x+y,y) + D_{n-1} f(x-y,y) - D_{n-2} f(x,y) + (n-1) D_2 f(x,y)$$

for all $x, y \in V$. Using mathematical induction, we obtain

$$D_n f(x,y) = 0$$

for all $x, y \in V$ and any $n \in \mathbb{N}$. If $k \in \mathbb{Q}$ is represented by either $k = \frac{n}{m}$ or $k = \frac{-n}{m}$ for some $n, m \in \mathbb{N}$ with $m \neq n$, then the desired equality $D_k f(x,y) = 0$ follows from the equalities

$$\begin{aligned} D_{\frac{n}{m}} f(x,y) =& D_n f\left(x, \frac{y}{m}\right) - \frac{n}{m} D_m f\left(x, \frac{y}{m}\right), \\ D_{\frac{-n}{m}} f(x,y) =& - D_{\frac{n}{m}} f(x,y) \end{aligned}$$

for all $x, y \in X$ and $n, m \in \mathbb{N}$. □

Theorem 2.4. *Let $\varphi : V^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} \frac{\varphi(k^n x, k^n y)}{|k|^{3n}} = 0 \tag{2}$$

for all $x, y \in V$ and let $\tilde{\varphi}(x)$ be the limit

$$\lim_{n\to\infty} \max\Big\{\varphi_j(x), \varphi_j(-x) : 0 \le j < n\Big\} \tag{3}$$

for each $x \in V$, where

$$\varphi_j(x) := \max\Big\{\frac{\varphi(0, k^j x)}{|2||k|^{3j+3}}, \frac{\varphi(\frac{k^{j+1}x}{2}, \frac{k^j x}{2})}{|2||k|^{2j+2}}, \frac{\varphi(-\frac{k^{j+1}x}{2}, -\frac{k^j x}{2})}{|2||k|^{2j+2}}, \frac{\varphi(\frac{k^j x}{2}, \frac{k^j x}{2})}{|2||k|^{2j+1}}, \frac{\varphi(-\frac{k^j x}{2}, -\frac{k^j x}{2})}{|2||k|^{2j+1}}\Big\}.$$

Suppose that $f : V \to Y$ is a mapping satisfying $f(0) = 0$ and

$$\|D_k f(x, y)\| \le \varphi(x, y) \tag{4}$$

for all $x, y \in V$. Then there exists a unique general quadratic-cubic mapping $T : V \to Y$ such that

$$\|f(x) - T(x)\| \le \tilde{\varphi}(x) \tag{5}$$

for all $x \in V$. In particular, T is given by

$$T(x) = \lim_{n\to\infty} \frac{f'(k^n x) + f'(-k^n x)}{2 \cdot k^{2n}} + \frac{f(k^n x) - f(-k^n x)}{2 \cdot k^{3n}} + f(0)$$

for all $x \in V$, where $f' : V \to Y$ is a mapping defined by $f'(x) := f(x) - f(0)$.

Proof. Let $J_n f : V \to Y$ be a mapping defined by

$$J_n f(x) = \frac{f'(k^n x) + f'(-k^n x)}{2 \cdot k^{2n}} + \frac{f(k^n x) - f(-k^n x)}{2 \cdot k^{3n}} + f(0)$$

for all $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned}
&\|J_j f(x) - J_{j+1} f(x)\| \\
&= \Bigg\| \frac{-D_k f(0, k^j x)}{2 \cdot k^{3j+3}} - \frac{D_k f\big(\frac{k^j x}{2}, \frac{k^j x}{2}\big) + D_k f\big(-\frac{k^j x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+1}} \\
&\quad - \frac{D_k f\big(\frac{k^{j+1} x}{2}, \frac{k^j x}{2}\big) + D_k f\big(-\frac{k^{j+1} x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+2}} \Bigg\| \\
&\le \max\Bigg\{ \left\|\frac{D_k f(0, k^j x)}{2 \cdot k^{3j+3}}\right\|, \left\|\frac{D_k f\big(\frac{k^{j+1} x}{2}, \frac{k^j x}{2}\big)}{2 \cdot k^{2j+2}}\right\|, \left\|\frac{D_k f\big(-\frac{k^{j+1} x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+2}}\right\|, \\
&\qquad \left\|\frac{D_k f\big(\frac{k^j x}{2}, \frac{k^j x}{2}\big)}{2 \cdot k^{2j+1}}\right\|, \left\|\frac{D_k f\big(-\frac{k^j x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+1}}\right\| \Bigg\} \\
&\le \varphi_j(x) \qquad (6)
\end{aligned}$$

for all $x \in V$ and $j \in \mathbb{N} \cup \{0\}$. It follows from (2) and (6) that the sequence $\{J_n f(x)\}$ is Cauchy. Since Y is complete, $\{J_n f(x)\}$ is convergent. Set

$$T(x) := \lim_{n\to\infty} J_n f(x).$$

Using induction one can show that

$$\|J_n f(x) - f(x)\| \le \max\{\varphi_j(x) : 0 \le j < n\} \tag{7}$$

for all $n \in \mathbb{N}$ and all $x \in V$. By taking n to approach infinity in (7) and using (3) one obtains (5). Replacing x and y by $k^n x$ and $k^n y$, respectively, in (4) we get

$$\begin{aligned}\|DJ_n f(x,y)\| =& \Bigg\| \frac{Df(k^n x, k^n y) - Df(-k^n x, -k^n y)}{2 \cdot k^{3n}} \\ &+ \frac{Df(k^n x, k^n y) + Df(-k^n x, -k^n y)}{2 \cdot k^{2n}} \Bigg\| \\ \le& \max\Big\{ \frac{\varphi(k^n x, k^n y)}{|2| \cdot |k|^{3n}}, \frac{\varphi(-k^n x, -k^n y)}{|2| \cdot |k|^{3n}}, \\ & \frac{\varphi(k^n x, k^n y)}{|2| \cdot |k|^{2n}}, \frac{\varphi(-k^n x, -k^n y)}{|2| \cdot |k|^{2n}} \Big\}\end{aligned}$$

for all $x, y \in V$ and $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \to \infty$ and using (2) we get $DT(x,y) = 0$ for all $x, y \in V$. If T_1 is another general quadratic-cubic mapping satisfying (5), then the equality

$$T_1(x) - J_n T_1(x)$$

follows from the equality

$$\begin{aligned}T_1(x) - J_n T_1(x) =& \sum_{j=0}^{n-1} \Bigg(\frac{-D_k T_1(0, k^j x)}{2 \cdot k^{3j+3}} - \frac{D_k T_1\big(\frac{k^j x}{2}, \frac{k^j x}{2}\big) + D_k T_1\big(-\frac{k^j x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+1}} \\ & - \frac{D_k T_1\big(\frac{k^{j+1} x}{2}, \frac{k^j x}{2}\big) + D_k T_1\big(-\frac{k^{j+1} x}{2}, -\frac{k^j x}{2}\big)}{2 \cdot k^{2j+2}} \Bigg)\end{aligned}$$

for any $n \in \mathbb{N}$ and so we have

$$\begin{aligned}\|T(x) - T_1(x)\| =& \lim_{n\to\infty} \|J_n T(x) - J_n T_1(x)\| \\ \le& \lim_{n\to\infty} \max\{\|J_n T(x) - J_n f(x)\|, \|J_n f(x) - J_n T_1(x)\|\} \\ \le& \lim_{n\to\infty} \max\Big\{ \frac{\|T(k^n x) - f(k^n x)\|}{|2||k|^{3n}}, \frac{\|T(-k^n x) - f(-k^n x)\|}{|2||k|^{3n}}, \\ & \frac{\|f(k^n x) - T_1(k^n x)\|}{|2||k|^{3n}}, \frac{\|f(-k^n x) - T_1(-k^n x)\|}{|2||k|^{3n}} \Big\} \\ \le& \lim_{n\to\infty} \lim_{m\to\infty} \max\Big\{ \frac{\varphi_j(k^n x)}{|2||k|^{3n}}, \frac{\varphi_j(-k^n x)}{|2||k|^{3n}} : 0 \le j < m \Big\} \\ =& 0\end{aligned}$$

all $x \in V$. Therefore $T = T_1$. This completes the proof of the uniqueness of T. □

Corollary 2.5. *Let X and Y be non-Archimedean normed spaces over $\mathbb{K}$ with $|k| < 1$. If Y is complete and for some $3 < r$, $f : X \to Y$ satisfies the condition*

$$\|Df(x, y)\| \leq \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique general quadratic-cubic mapping $T : X \to Y$ such that

(8)
$$\|f(x) - T(x)\| \leq \theta \cdot \max\{|2|^{-1}|k|^{-3}, |2|^{-1-r}(1 + |k|^r)|k|^{-2}, 2|2|^{-1-r}|k|^{-1}\}\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$. Since $|k| < 1$ and $r - 3 > 0$,

$$\lim_{n\to\infty} |k|^{-3n}\varphi(k^n x, k^n y) = \lim_{n\to\infty} |k|^{n(r-3)}\varphi(x, y) = 0$$

for all $x, y \in X$. Therefore the conditions of Theorem 2.4 are fulfilled. It is easy to see that $\tilde{\varphi}(x) = \theta \cdot \max\{|2|^{-1}|k|^{-3}, |2|^{-1-r}(1+|k|^r)|k|^{-2}, 2|2|^{-1-r}|k|^{-1}\}\|x\|^r$. By Theorem 2.4 there is a unique general quadratic-cubic mapping $T : X \to Y$ satisfying (8). □

Theorem 2.6. *Let $\varphi : V^2 \to [0, \infty)$ be a function such that*

$$\lim_{n\to\infty} |k|^{2n}\varphi(k^{-n}x, k^{-n}y) = 0 \tag{9}$$

for all $x, y \in V$ and let for each $x \in V$ the limit

$$\lim_{n\to\infty} \max\{\varphi_{-j-1}(x), \varphi_{-j-1}(-x) : 0 \leq j < n\} \tag{10}$$

denoted by $\tilde{\varphi}(x)$, exists, where φ_j is defined as in Theorem 2.1. Suppose that $f : V \to Y$ is a mapping satisfying $f(0) = 0$ and

$$\|D_k f(x, y)\| \leq \varphi(x, y) \tag{11}$$

for all $x, y \in V$. Then there exists a unique general quadratic-cubic mapping $T : V \to Y$ such that

$$\|f'(x) - T(x)\| \leq \tilde{\varphi}(x) \tag{12}$$

for all $x \in V$. In particular, T is given by

$$T(x) = \lim_{n\to\infty} \left(\frac{k^{2n}}{2}\left(f'\left(\frac{x}{k^n}\right) + f'\left(\frac{-x}{k^n}\right)\right) + \frac{k^{3n}}{2}\left(f\left(\frac{x}{k^n}\right) - f\left(\frac{-x}{k^n}\right)\right)\right) + f(0)$$

for all $x \in V$.

Proof. Let $J_n f : V \to Y$ be a mapping defined by

$$J_n f(x) = \frac{k^{2n}}{2}\left(f'\left(\frac{x}{k^n}\right) + f'\left(\frac{-x}{k^n}\right)\right) + \frac{k^{3n}}{2}\left(f\left(\frac{x}{k^n}\right) - f\left(\frac{-x}{k^n}\right)\right) + f(0)$$

for all $x \in V$ and $n \in \mathbb{N} \cup \{0\}$. Notice that $J_0 f(x) = f(x)$ and

$$\begin{aligned}
&\|J_j f(x) - J_{j+1} f(x)\| \\
&= \Big\| \frac{k^{3j} D_k f(0, \frac{x}{k^{j+1}})}{2} + \frac{k^{2j}}{2} \Big(D_k f\Big(\frac{x}{2 \cdot k^j}, \frac{x}{2 \cdot k^{j+1}}\Big) + D_k f\Big(\frac{-x}{2 \cdot k^j}, \frac{-x}{2 \cdot k^{j+1}}\Big) \\
&\quad + k D_k f\Big(\frac{x}{2 \cdot k^{j+1}}, \frac{x}{2 \cdot k^{j+1}}\Big) + k D_k f\Big(\frac{-x}{2 \cdot k^{j+1}}, \frac{-x}{2 \cdot k^{j+1}}\Big)\Big)\Big\| \\
&\le \max\Big\{ \frac{|k|^{3j}}{|2|} \Big\| D_k f\Big(0, \frac{x}{k^{j+1}}\Big)\Big\|, \frac{|k|^{2j}}{|2|} \Big\| D_k f\Big(\frac{x}{2 \cdot k^j}, \frac{x}{2 \cdot k^{j+1}}\Big)\Big\|, \\
&\quad \frac{|k|^{2j}}{|2|} \Big\| D_k f\Big(\frac{-x}{2 \cdot k^j}, \frac{-x}{2 \cdot k^{j+1}}\Big)\Big\|, \frac{|k|^{2j+1}}{|2|} \Big\| D_k f\Big(\frac{x}{2 \cdot k^{j+1}}, \frac{x}{2 \cdot k^{j+1}}\Big)\Big\|, \\
&\quad \frac{|k|^{2j+1}}{|2|} \Big\| D_k f\Big(\frac{-x}{2 \cdot k^{j+1}}, \frac{-x}{2 \cdot k^{j+1}}\Big)\Big\| \Big\} \\
&= \varphi_{-j-1}(x) \qquad (13)
\end{aligned}$$

for all $x \in V$ and $j \in \mathbb{N} \cup \{0\}$. It follows from (13) and (9) that the sequence $\{J_n f(x)\}$ is Cauchy. Since Y is complete, $\{J_n f(x)\}$ is convergent. Set

$$T(x) := \lim_{n \to \infty} J_n f(x).$$

Using induction one can show that

$$\|J_n f(x) - f(x)\| \le \max\{\varphi_{-j-1}(x) : 0 \le j < n\} \qquad (14)$$

for all $n \in N$ and all $x \in V$. By taking n to approach infinity in (14) and using (10) one obtains (12). Replacing x and y by $k^{-n}x$ and $k^{-n}y$, respectively, in (11) we get

$$\begin{aligned}
\|DJ_n f(x,y)\| &= \Big\| \frac{k^{3n}}{2} Df\left(\frac{x}{k^n}, \frac{y}{k^n}\right) - \frac{k^{3n}}{2} Df\left(\frac{-x}{k^n}, \frac{-y}{k^n}\right) \\
&\quad + \frac{k^{2n}}{2} Df\left(\frac{x}{k^n}, \frac{y}{k^n}\right) + \frac{k^{2n}}{2} Df\left(\frac{-x}{k^n}, \frac{-y}{k^n}\right) \Big\| \\
&\le \max\left\{ |2|^{-1}|k|^{2n} \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right), |2|^{-1}|k|^{2n} \varphi\left(\frac{-x}{2^n}, \frac{-y}{2^n}\right) \right\}
\end{aligned}$$

for all $n \in \mathbb{N}$ and all $x, y \in V$. Taking the limit as $n \to \infty$ and using (9) we get $DT(x,y) = 0$. If T_1 is another general quadratic-cubic mapping satisfying (12), then the equality $T_1(x) = J_n T_1(x)$ follows from the equality

$$\begin{aligned}
&T_1(x) - J_n T_1(x) \\
&= \sum_{j=0}^{n-1} \Big(\frac{k^{3j} D_k T_1(0, \frac{x}{k^{j+1}})}{2} + \frac{k^{2j}}{2} \Big(D_k T_1\Big(\frac{x}{2 \cdot k^j}, \frac{x}{2 \cdot k^{j+1}}\Big) + D_k T_1\Big(\frac{-x}{2 \cdot k^j}, \frac{-x}{2 \cdot k^{j+1}}\Big) \\
&\quad + k D_k T_1\Big(\frac{x}{2 \cdot k^{j+1}}, \frac{x}{2 \cdot k^{j+1}}\Big) + k D_k T_1\Big(\frac{-x}{2 \cdot k^{j+1}}, \frac{-x}{2 \cdot k^{j+1}}\Big)\Big)\Big)
\end{aligned}$$

for any $k \in \mathbb{N}$ and so

$$\begin{aligned}\|T(x)-T_1(x)\| &= \lim_{n\to\infty} \|J_nT(x)-J_nT_1(x)\| \\ &\le \lim_{n\to\infty} \max\{\|J_nT(x)-J_nf(x)\|, \|J_nf(x)-J_nT_1(x)\|\} \\ &\le \lim_{n\to\infty} \frac{|k|^{2n}}{|2|} \max\left\{ \left\|T\left(\frac{x}{k^n}\right)-f\left(\frac{x}{k^n}\right)\right\|, \left\|T\left(\frac{-x}{k^n}\right)-f\left(\frac{-x}{k^n}\right)\right\|,\right. \\ &\qquad \left.\left\|f\left(\frac{x}{k^n}\right)-T_1\left(\frac{x}{k^n}\right)\right\|, \left\|f\left(-\frac{x}{k^n}\right)-T_1\left(-\frac{x}{k^n}\right)\right\| \right\} \\ &\le \lim_{n\to\infty} \frac{|k|^{2n}}{|2|} \lim_{m\to\infty} \max\left\{\varphi_{-j-1}(k^{-n}x), \varphi_{-j-1}(-k^{-n}x) : 0 \le j < m\right\} \\ &=0\end{aligned}$$

for all $x \in V$. Therefore $T = T_1$. This completes the proof of the uniqueness of T. □

Corollary 2.7. *Let X and Y be non-Archimedean normed spaces over $\mathbb{K}$ with $|k| < 1$. If Y is complete and for some $r < 2$, $f : X \to Y$ satisfies the condition*

$$\|D_kf(x,y)\| \le \theta(\|x\|^r + \|y\|^r)$$

for all $x, y \in X$. Then there exists a unique general quadratic-cubic mapping $T : X \to Y$ such that

(15)
$$\|f(x)-T(x)\| \le \theta \cdot \max\{|2|^{-1}|k|^{-r}, |2|^{-1-r}(1+|k|^{-r}), 2|2|^{-1-r}|k|^{1-r}\}\|x\|^r$$

for all $x \in X$.

Proof. Let $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Since $|k| < 1$ and $2-r > 0$,

$$\lim_{n\to\infty} |k|^{2n}\varphi(k^{-n}x, k^{-n}y) = \lim_{n\to\infty} |k|^{n(2-r)}\varphi(x,y) = 0$$

for all $x, y \in X$. Therefore the conditions of Theorem 2.6 are fulfilled. It is easy to see that

$$\tilde{\varphi}(x) = \theta \cdot \max\{|2|^{-1}|k|^{-r}, |2|^{-1-r}(1+|k|^{-r}), 2|2|^{-1-r}|k|^{1-r}\}\|x\|^r$$

for all $x \in X$. By Theorem 2.6 there is a unique general quadratic-cubic mapping $T : X \to Y$ satisfying (15). □

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