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TWO WEIGHT ESTIMATE FOR THE PARAPRODUCT IN THE SPACE OF HOMOGENEOUS TYPE

Daewon Chung

Abstract. In this paper, we provide sufficient conditions of a pair of weights (u, v) and a function b so that the dyadic paraproduct is bounded from $L^2_u(X)$ into $L^2_v(X)$, where X is a space of homegeneous type. In order to prove the main result we use the honest dyadic system introduced in [10].

1. Introduction and statement of Main results

The aim of this article is to establish quantitative two weight estimates for the dyadic paraproduct in the space of homogeneous type. Spaces of homogeneous type (or SHTs) were first introduced in [1]. Spaces of homogeneous type are characterized by quasi-metrics and doubling measures. A quasi-metric is a generalization of a metric. More precisely, quasi-metric satisfies all the usual metric conditions with replacing the triangle inequality by weakened version:

$$\rho(x, y) \leq A_0 \left(\rho(x, z) + \rho(z, y)\right) \tag{1.1}$$

for all $x, y, z \in X$. A measure μ is said to be doubling when it satisfies that there exist some constant A_1 such that for every $x \in X$ and for every $r > 0$, the following inequality holds:

$$\mu\left(B(x, 2r)\right) \leq A_1 \mu\left(B(x, r)\right) , \tag{1.2}$$

where $B(x, r) := \{y \in X \,:\, \rho(y, x) < r\}$ we mean the open ball with respect to the quasi-metric ρ centered at $x \in X$ with radius $r > 0$. Because the main operators in our interests are dyadic operator it is natural to ask what is the dyadic structure in the spaces of homogeneous type. In fact we adapt the construction of a dyadic lattice on spaces of homogeneous type which are introduced by the authors of [8]. Although M. Christ first introduced the construction of a dyadic

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lattice on SHTs in his 1990 paper [4], his construction had a few weaknesses such as: it is not over the entirety of X, omitted infinitely many points and the proof relied on the use of the axiom of choice to locate the dyadic centers. However, Christ's theorem is widely cited and is the base of the modern theorem for dyadic cubes over SHTs such as the cubes of T. Hytönen and A. Karema [8].

In this paper, we consider the two weight estimate for the dyadic paraproduct and provide a quantitative estimates in the SHTs. The authors in [3] proposed the new class of functions for b to obtain the boundedness of the dyadic paraproduct π_b from $L^2_u(\mathbb{R})$ into $L^2_v(\mathbb{R})$ called the two weight Carleson class, $Carl_{u,v}$. Therefore we will adapt their idea associated with function b and generalize the results to more general space, the spaces of homogeneous types. It is also shown in [3] that $Carl_{u,v}$ coincides with BMO^d when when $u = v$ belongs to the Muckenhoupt weights class A^d_2 and the author in [6] use $Carl_{u,v}$ to provide quantitative two weight estimates for the commutator of the Hilbert transform. We prove the following theorem.

Theorem 1.1 (Main Result). *Let (X, ρ, μ) be a space of homogeneous type and (u, v) be a pair of μ-measurable functions on X such that v and u^{-1} are weights on (X, ρ, μ), and such that*

(1) $(u, v) \in \mathcal{A}^d_2$.

(2) *there is a constant $\mathcal{D}_{u,v} > 0$ such that*

$$\frac{1}{\mu(R)} \sum_{Q\in\mathscr{D}(R)} |\Delta_Q v|^2 \mu(Q)\langle u^{-1}\rangle_Q \leq \mathcal{D}_{u,v}\langle v\rangle_R \quad \textit{for all } R \in \mathscr{D}.$$

Assume that $b \in Carl_{u,v}(X)$, that is $b \in L^1_{loc}(X)$ and there is a constant $\mathcal{B}_{u,v} > 0$ such that

$$\frac{1}{\mu(R)} \sum_{Q\in\mathscr{D}(R)} \frac{|\langle b, h_Q\rangle|^2}{\langle v\rangle_Q} \leq \mathcal{B}_{u,v}\langle u^{-1}\rangle_R \quad \textit{for all } R \in \mathscr{D}.$$

Then there is a constant $C > 0$ such that for all $f \in L^2_u(X)$

$$\|\pi_b f\|_{L^2_v(X)} \leq C\sqrt{[u, v]_{\mathcal{A}^d_2}\mathcal{B}_{u,v}}\left(\sqrt{[u, v]_{\mathcal{A}^d_2}} + \sqrt{\mathcal{D}_{u,v}}\right)\|f\|_{L^2_u(X)}.$$

By $\|f\|_{L^2_u(X)}$ we mean $\left(\int_X f(x)^2u(x)d\mu(x)\right)^{1/2}$.

Here we use the notation $\langle f\rangle_Q$ to denote the μ-average of f:

$$\langle f\rangle_Q := \frac{1}{\mu(Q)}\int_Q f(x)d\mu(x)$$

and for $f, g \in L^2(X)$, we write $\langle f, g\rangle$ to mean the $L^2(X)$ inner product, i.e,

$$\langle f, g\rangle := \int_X f(x)g(x)d\mu(x).$$

Since we didn't know the number of dyadic children for a cube Q in SHTs, it is not clear yet to define the operator Δ_Q where Δ_Q is an operator that takes the difference of the average in the children. We will define the operator Δ_Q after we present a dyadic system of X in Section 2, but one can use the obvious definition in $\mathbb{R}$ i.e, $\Delta_Q v = \langle v\rangle_{Q_-} - \langle v\rangle_{Q_+}$.

This paper organized as follows. Definitions and frequently used theorems are collected in Section 2, including joint $\mathcal{A}_2^{\mathscr{D}}$ condition, regular and weighted Haar functions in SHTs, honest dyadic structure, v-Carleson sequences, the class $Carl_{u,v}$, the weighted Carleson Lemma, and the known weighted estimates. We provide the proof of quantitative two weight estimates for the dyadic paraproduct in SHTs in Section 3. After which we give remarks with the two weight estimates relation between dyadic square function and dyadic paraproduct in Section 4.

2. Notations and Useful known results

Throughout the paper a constant C will be a numerical constant that may change from line to line. Given a μ-measurable set E in X, $\mu(E)$ will stand for its μ-measure, $\int_X \mathbb{1}(x)d\mu(x)$, where the characteristic function $\mathbb{1}_E(x) = 1$ if $x \in E$, zero otherwise. We say a function $v : X \to \mathbb{R}$ is a weight if v is a locally integrable function which is positive almost everywhere with respect to μ. For a given weight v, the v-measure of a μ-measurable set E, denoted by $v(E)$, is $v(E) = \int_E v(x)d\mu(x)$.

We say that a pair of weights (u, v) satisfies the joint $\mathcal{A}_2^d$ condition if and only if both v and u are weights and

$$[u, v]_{\mathcal{A}_2^d} := \sup_{Q\in\mathscr{D}} \langle u^{-1}\rangle_Q \langle v\rangle_Q < \infty\,.$$

We now define the set of Haar functions. Given a SHT (X, ρ, μ) with dyadic grid $\mathscr{D}$, we can construct a system of Haar functions which form an orthonormal basis of L^2_X. Given a cube $Q \in \mathscr{D}$, ch(Q) denotes the collection of dyadic children of Q, and $N(Q)$ denotes its cardinality, i.e. Q has $N(Q)$ children. For an enumeration of the children of Q, we define recursively subsets of Q that are unions of children of Q. Precisely, at each step we remove one child according to the given enumeration, that is a bijection $u_Q : \{1, 2, \dots, N(Q)\} \to \mathrm{ch}(Q)$. Let $E^1_Q := Q$, given $E^k_Q \subset Q$, let $E^{k+1}_Q = E^k_Q/u_Q(k)$ for $k = 1, 2, \dots, N(Q) - 1$. We split each of these sets into two disjoint pieces, $E^i_Q := E^{i,+}_Q \cup E^{i,-}_Q$ where $E^{i,+}_Q = u_Q(i)$, the child removed and $E^{i,+}_Q = E^{i+1}_Q$. With these notations, we can define the Haar functions associated to the cube Q by

$$h^i_Q(x) := a\mathbb{1}_{E^{i,+}_Q}(x) - b\mathbb{1}_{E^{i,-}_Q}(x), \quad 1 \le i \le N(Q) - 1\,,$$

where the constants a and b can be chosen to enforce L^2 normalization and mean zero. Indeed, one can show that a and b as follows,

$$a = \sqrt{\mu(E_Q^{i,-})/\left(\mu(E_Q^i)\mu(E_Q^{i,+})\right)}, \quad b = \sqrt{\mu(E_Q^{i,+})/\left(\mu(E_Q^i)\mu(E_Q^{i,-})\right)}.$$

We refer [4, 10] for more detailed arguments for the construction in SHTs. One can also find the similar construction of the Haar system in [5, 11]. Since the measure μ is assumed to be Borel regular, the Lebesgue differentiation theorem holds. While we construct the Haar system in SHTs similar to the one in $\mathbb{R}^n$, one of the most difficult issues when we work with dyadic operators in SHTs is that the number of children in each cube is not promised. In t$\mathbb{R}^n$, we know that each cube has the same number of children, 2^n . But this nice property fails in SHTs and difficulties occur due to this failure when we generalize the result of $\mathbb{R}^n$ into the SHT. However, we can get help from the following lemma which appeared in [10]. Before we state it, we define overlapping grids.

Definition 1. Let X be a SHT and let $\mathscr{D}$ and $\mathscr{D}'$ be two dyadic grids on X. If for every $Q \in \mathscr{D}$, it is also true that $Q \in \mathscr{D}'$ then we say that $\mathscr{D}'$ overlaps $\mathscr{D}$.

Then the lemma we refer to here is the following.

Lemma 2.1. *Let $X = (X, \rho)$ denote a quasi-metric which is geometrically doubling and let $\mathscr{D}$ be a dyadic grid over X . There exists a dyadic structure $\widetilde{\mathscr{D}}$ on X which overlaps $\mathscr{D}$ and is honest, that is, that each cube in $\widetilde{\mathscr{D}}$ has no more than two children cubes.*

One can find the proof and some examples of the Lemma 2.1 in [10]. Although the honest dyadic systems are not strictly necessary to derive any results, they provide a nice simplification and remove complicate notations. In fact, for any dyadic grids $\mathscr{D}$, Lemma 2.1 guarantees the existence of an honest dyadic grid $\widetilde{\mathscr{D}}$. Thus, by using honest systems, we always know the number of children each cube has. We can now define the set of Haar functions in SHTs with honest dyadic structure.

Definition 2. Let (X, ρ, μ) be a SHT with any dyadic structure $\mathscr{D}$ and let $\widetilde{\mathscr{D}}$ be an honest dyadic structure which overlaps $\mathscr{D}$. For any honest cube $Q \in \widetilde{\mathscr{D}}$ we define the function

$$h_Q(x) := \begin{cases} \lambda_Q^+ \cdot \mathbb{1}_{Q_+}(x) - \lambda_Q^- \cdot \mathbb{1}_{Q_-}(x) & \text{if } N(Q) = 2 \\ 0 & \text{if } N(Q) = 1 \end{cases}$$

where the $\lambda_Q^\pm$ are normalization constants given by

$$\lambda_Q^+ = \sqrt{\frac{\mu(Q_-)}{\mu(Q_+) \cdot \mu(Q)}} \text{ and } \lambda_Q^- = \sqrt{\frac{\mu(Q_+)}{\mu(Q_-) \cdot \mu(Q)}} .$$

Then we use the following theorem to obtain our main result which is the generalization of the quantitative two weight estimate of the dyadic paraproduct in $\mathbb{R}$ presented in [3]. We also refer [10] for the proof of the following theorem.

Theorem 2.2. *Let X be a SHT with honest dyadic structure $\widetilde{\mathscr{D}}$. The set $\{h_Q\}_{Q\in\mathscr{D}}$ forms a complete orthonormal basis for $L^2(X)$ and each of the following conditions holds:*

1. $\int_X h_Q = 0$ *for all* $Q \in \mathscr{D}$.
2. *For each* $Q \in \mathscr{D}$, h_Q *is supported on* Q.
3. *If* h_Q *is supported on* Q, *then* h_Q *is constant on each of* Q*'s children.*
4. *If* h_Q *is supported on* Q, *then* h_Q *is positively valued on exactly half of* Q*'s children, and negatively valued on the other half.*

Then we now give definitions and useful known results.

Definition 3. Let X be a SHT with dyadic structure $\mathscr{D}$, and let $v : X \to \mathbb{R}$ be a weight. A sequence of non-negative real numbers $\{\alpha_Q\}_{Q\in\mathscr{D}}$ is called a v-Carleson sequence with intensity B if and only if there exists a constant B such that for every $R \in \mathscr{D}$,

$$\sum_{Q\in\mathscr{D}(R)} \alpha_Q \le B \cdot \int_R v(x)d\mu(x)\,.$$

We now define a class of objects that will take the place of the BMO class in the two weighted case. It is called the two weight Carleson class.

Definition 4. Let (X, ρ, μ) be a space of homogeneous type. Given a pair of weights (u, v) on X, we say that a locally integrable function b belongs to the two weight Carleson class, $Carl_{u,v}(X)$ if $\{b_Q^2/\langle v\rangle_Q\}_{Q\in\mathscr{D}}$ is a u^{-1}-Carleson sequence where $b_Q = \langle b, h_Q\rangle$.

We refer to [3] for the properties of the class $Carl_{u,v}$ and its relations to the BMO class. We now introduce the Weighted Carleson Lemma for SHTs which will be used frequently throughout this paper. The lemma was stated first in [9] for $\mathbb{R}$. One can also find its proofs in [7, 10] for $\mathbb{R}$ and SHTs respectively. We recall that a weight v is dyadic doubling if there exists a constant $D \ge 1$ such that for all cubes Q, $v(\hat{Q}) \le Dv(Q)$.

Theorem 2.3. *[Weighted Carleson Lemma for SHTs]. Let v be a dyadic doubling weight with respect to $\mathscr{D}$. Then $\{\alpha_Q\}_{Q\in\mathscr{D}}$ is a v-Carleson sequence with intensity B if and only if for all non-negative σ-measurable function F on the line,*

$$\sum_{Q\in\mathscr{D}} (\inf_{x\in Q} F(x))\alpha_Q \le B\int_X F(x)v(x)d\mu(x)\,, \tag{2.1}$$

where the measure σ is defined as

$$\sigma(E) := \int_E v d\mu\,.$$

We now list the definitions of dyadic operators, such as dyadic paraproduct which is our main subject in this paper, dyadic weighted maximal function, and the dyadic square functions.

Definition 5. We formally define the dyadic paraproduct π_b associated to $b \in L^1_{loc}(X)$ as follows, for functions f which are at least locally integrable:

$$\pi_b f(x) := \sum_{Q \in \mathcal{D}} \langle f \rangle_Q \, \langle b, h_Q \rangle h_Q(x)$$

where $\{h_Q\}_{Q \in \mathscr{D}}$ are the Haar functions constructed in Definition 2.

It is a well know fact that the dyadic paraproduct is bounded not only on L^p_{dx} but also in L^p_v when $b \in BMO^d$ and $v \in A^d_p$. Beznosova proved in [2] that the $L^2_v(\mathbb{R})$ norm of the dyadic paraproduct depends linearly on both $[v]_{A^d_2}$ and $\|b\|_{BMO^d}$ and in [5, 10] the authors extended this result into $\mathbb{R}^n$ and SHTs, respectively. Also, [3] provides the quantitative two weight estimate for the dyadic paraproduct.

Definition 6. Let X be a SHT and v be a weight over X. If $\mathscr{D}$ is a dyadic lattice over X, we can define dyadic weighted maximal function,

$$M_v^{\mathscr{D}} f(x) := \sup_{\substack{Q \ni x \\ Q \in \mathscr{D}}} \frac{1}{v(Q)} \int_Q |f(y)| \, v(y) d\mu(y)\,,$$

where the supremum is taken over all cubes $Q \in \mathscr{D}$ which contain the point x.

The weighted maximal function M_v is defined analogously, only taking the supremum over all intervals not just dyadic intervals. A very important fact about the weighted maximal function is that the $L^p_v(\mathbb{R})$ norm of $M_v^{\mathscr{D}}$ only depends on $p' = p/(p-1)$ and not on the weight v. Here we state SHT version of this theorem.

Theorem 2.4. *Let X be a SHT, $\mathscr{D}$ a dyadic lattice over X. Let v be a locally integrable function defined on X such that $v > 0$ a.e. Then for all $1 < p < \infty$, $M_v^{\mathscr{D}}$ is bounded in $L^p_v(X)$. Moreover, for all $f \in L^p_v(X)$*

$$\|M_v^{\mathscr{D}} f\|_{L^p_v(X)} \leq p' \|f\|_{L^p_v(X)}\,.$$

Definition 7. Let X be a SHT and $\mathscr{D}$ is a dyadic lattice over X. We define the dyadic square function as follows

$$S^{\mathscr{D}} f(x) := \left(\sum_{Q \in \mathscr{D}} |\langle f \rangle_Q - \langle f \rangle_{\hat{Q}}|^2 \mathbb{1}_Q(x) \right)^{1/2}$$

where the notation $\hat{Q}$ stands for the parent of Q.

The following two weight characterization was introduced by Wilson in [11], see also [9, 3]. In the following Theorem, since we work with an honest dyadic lattice, the same definition with $\mathbb{R}$ applies for the Δ_Q operator.

Theorem 2.5. *Let X be a SHT, $\mathscr{D}$ be an honest dyadic lattice. $(u, v) \in A_2^{\mathscr{D}}$ and $\{|\Delta_Q u^{-1}|^2 \mu(Q) \langle v \rangle_Q\}_{Q \in \mathscr{D}}$ is a u^{-1}-Carleson sequence with intensity $C_{v^{-1},u^{-1}}$ if* *and only if the dyadic square function $S^{\mathscr{D}}$ is bounded from $L^2_u(X)$ into $L^2_v(X)$ i.e. there is a constant $C > 0$ such that*

$$\|S^{\mathscr{D}}\|_{L^2_u(X)\to L^2_v(X)} \le C([u,v]_{A_2^{\mathscr{D}}} + \mathcal{C}_{v^{-1},u^{-1}})^{1/2}\,.$$

Lastly, we define the weighted Haar function over a SHT.

Definition 8. The weighted Haar function h^v_Q with honest dyadic grid $\mathscr{D}$ is defined as

$$h^v_Q := \kappa^+_Q \mathbb{1}_{Q_+} - \kappa^-_Q \mathbb{1}_{Q_-}$$

where

$$\kappa^{\pm}_Q = \sqrt{\frac{v(Q_{\mp})}{v(Q)\cdot v(Q_{\pm}}}$$

One can easily check that $\kappa^{\pm}_Q = \lambda^{\pm}_Q$ when $v = 1$ from Definition 2. Since $\{h^v_Q\}_{Q\in\mathscr{D}}$ forms an orthonormal family for $L^2_v(X)$,

$$\sum_{Q\in\mathscr{D}} |\langle f, h^v_Q\rangle_Q|^2 \le \|f\|^2_{L^2_v(X)}$$

where $\langle f, g\rangle_v$ to be the $L^2_v(X)$ inner product. We can give the relation between the weighted and unweighted Haar functions over an SHT as follows:

Proposition 2.6. *For any weight v and any honest dyadic cube Q, there exists α^v_Q and β^v_Q such that*

$$h_Q(x) = \alpha^v_Q \cdot h^v_Q(x) + \beta^v_Q \frac{\mathbb{1}_Q(x)}{\sqrt{\mu(Q)}}$$

where

$$|\alpha^v_Q| \le \sqrt{\langle v\rangle_Q} \quad \textit{and} \quad |\beta^v_Q| \le \frac{|\Delta_Q w|}{\langle w\rangle_Q}\,.$$

We refer [10] for the proof of Proposition 2.6, but it can be easily checked, after rewriting the weighted Haar function

$$h^v_Q = \frac{h_Q(x)}{\alpha^v_Q} - \frac{\beta^v_Q}{\alpha^v_Q}\cdot\frac{\mathbb{1}_Q(x)}{\sqrt{\mu(Q)}}\,,$$

by using axioms for Haar functions and the fact that the geometric mean is bounded by the arithmetic mean.

3. The dyadic paraproduct and its two-weight estimates in SHTs

To prove Theorem 1.1, we can essentially repeat the proof for the $\mathbb{R}$ version due to Lemma 2.1.

Proof of Theorem 1.1. If $\|f\|_{L^2_u(X)} = \infty$ then the inequality is trivially true. Fix $f \in L^2_{u^{-1}}(X)$ and $g \in L^2_v(X)$. Note that $fu^{-1} \in L^2_u(X)$ and $\|fu^{-1}\|_{L^2_u(X)} = \|f\|_{L^2_{u^{-1}}(X)}$, $gv \in L^2_{v^{-1}}(X)$ and $\|gv\|_{L^2_{v^{-1}}(X)} = \|g\|_{L^2_v(X)}$, $\pi_b(fu^{-1})$ is expected to be in $L^2_v(X)$, then $gv \in L^2_{v^{-1}}(X)$ is in the right space for the pairing. By duality, it is enough to show that

$$|\langle \pi_b(fu^{-1}), gv\rangle| \le C\sqrt{[u,v]_{\mathcal{A}_2^{\mathscr{D}}}]\mathcal{B}_{u,v}}\left(\sqrt{[u,v]_{\mathcal{A}_2^{\mathscr{D}}}} + \sqrt{\mathcal{D}_{u,v}}\right)\|f\|_{L^2_{u^{-1}}(X)}\|g\|_{L^2_v(X)}\,. \tag{3.1}$$

Expanding the left hand side of (3.1) gives

$$|\langle \pi_b(fu^{-1}), gv\rangle| = \left|\left\langle \sum_{Q\in\mathscr{D}} b_Q\langle fu^{-1}\rangle_Q h_Q, gv\right\rangle\right|$$

where $b_Q = \langle b, h_Q\rangle$. Replacing $h_Q = \alpha_Q h^v_Q + \beta_Q \frac{\mathbb{1}_Q}{\sqrt{\mu(Q)}}$ where $\alpha_Q = \alpha^v_Q$ and $\beta_Q = \beta^v_Q$ as described in Proposition 2.6, we get

$$|\langle \pi_b(fu^{-1}), gv)\rangle| \le \sum_{Q\in\mathscr{D}} |b_Q|\langle |f|u^{-1}\rangle_Q \left|\left\langle gv, \alpha_Q h^v_Q + \beta_Q \frac{\mathbb{1}_Q}{\sqrt{\mu(Q)}}\right\rangle\right| \tag{3.2}$$

Use the triangle inequality to separate the sum in (3.2) into two summands.

$$\begin{aligned}|\langle \pi_b(fu^{-1}), gv\rangle| \le &\sum_{Q\in\mathscr{D}} |b_Q||\alpha_Q|\langle |f|u^{-1}\rangle_Q\langle gv, h^v_Q\rangle| \\ &+ \sum_{Q\in\mathscr{D}} |b_Q|\frac{|\beta_Q|}{\sqrt{\mu(Q)}}\langle |f|u^{-1}\rangle_Q|\langle gv, \mathbb{1}_Q\rangle|\,.\end{aligned}$$

Using the estimate $|\alpha_Q| \le \sqrt{\langle v\rangle_Q}$, and $|\beta_Q| \le \frac{|\Delta_Q v|}{\langle v\rangle_Q}$, we have

$$|\langle \pi_b(fu^{-1}), gv\rangle| \le \Sigma_1 + \Sigma_2,$$

where

$$\Sigma_1 := \sum_{Q\in\mathscr{D}} |b_Q|\langle |f|u^{-1}\rangle_Q|\langle gv, h^v_Q\rangle|\sqrt{\langle v\rangle_Q}$$

$$\Sigma_2 := \sum_{Q\in\mathscr{D}} |b_Q|\langle |f|u^{-1}\rangle_Q|\langle gv, \mathbb{1}_Q\rangle|\frac{|\Delta_Q v|}{\langle v\rangle_Q}\frac{1}{\sqrt{\mu(Q)}}\,.$$

We will now estimate Σ_1 and Σ_2 separately.

Estimating Σ_1: First using that $\langle gv, f\rangle = \langle g, f\rangle_v$ and second using that $M^{\mathscr{D}}_{u^{-1}}f(x) \ge \langle |f|u^{-1}\rangle_Q/\langle u^{-1}\rangle_Q$ for all $x \in Q$ and that $\langle u^{-1}\rangle_Q\langle v\rangle_Q \le [u,v]_{\mathcal{A}_2^d}$, we get

$$\begin{aligned}\Sigma_1 &\le \sum_{Q\in\mathscr{D}} \frac{|b_Q|}{\sqrt{\langle v\rangle_Q}}\left(\inf_{x\in Q}(M^{\mathscr{D}}_{u^{-1}}f)(x)\right)|\langle g, h^v_Q\rangle_v|\langle u^{-1}\rangle_Q\langle v\rangle_Q \\ &\le [u,v]_{\mathcal{A}_2^d}\sum_{Q\in\mathscr{D}} \frac{b_Q}{\sqrt{\langle v\rangle_Q}}\left(\inf_{x\in I}(M^{\mathscr{D}}_{u^{-1}}f)(x)\right)|\langle g, h^v_Q\rangle_v|\,.\end{aligned}$$

Applying Cauchy-Schwarz gives that

$$\Sigma_1 \leq [u,v]_{\mathcal{A}_2^d} \Bigg(\sum_{Q\in\mathscr{D}} \frac{|b_Q|^2}{\langle v\rangle_Q} \inf_{x\in I} (M^{\mathscr{D}}_{u^{-1}} f)(x)^2 \Bigg)^{1/2} \Bigg(\sum_{Q\in\mathscr{D}} |\langle g, h^v\rangle_v|^2 \Bigg)^{1/2} .$$

Since $\{h^v_Q\}_{Q\in\mathscr{D}}$ forms an orthonormal family for $L^2_v(X)$

$$\Bigg(\sum_{Q\in\mathscr{D}} |\langle g, h^v\rangle_v|^2 \Bigg)^{1/2} \leq \|g\|_{L^2_v(X)} .$$

We apply the weighted Carleson Lemma 2.3 for SHTs (Theorem 2.3), with $F(x) = (M^{\mathscr{D}}_{u^{-1}} f)(x)^2$, and $\alpha_Q = |b_Q|^2/\langle v\rangle_Q$, which is a u^{-1}-Carleson sequence with intensity $\mathcal{B}_{u,v}$, by assumption. Thus we have

$$\Bigg(\sum_{Q\in\mathscr{D}} \frac{|b_Q|^2}{\langle v\rangle_Q} \inf_{x\in I} (M^{\mathscr{D}}_{u^{-1}} f)(x)^2 \Bigg)^{1/2} \leq \sqrt{\mathcal{B}_{u,v}} \Bigg(\int_X (M^{\mathscr{D}}_{u^{-1}} f)(x)^2 u^{-1} d\mu(x) \Bigg)^{1/2} .$$

Then together with the fact that $M^{\mathscr{D}}_{u^{-1}}$ is bounded in $L^2_{u^{-1}}(X)$ with bound not dependent on u^{-1} , we get

$$\Sigma_1 \leq C [u,v]_{\mathcal{A}_2^d} \sqrt{\mathcal{B}_{u,v}} \|f\|_{L^2_{u^{-1}}(X)} \|g\|_{L^2_v(X)} . \tag{3.3}$$

Estimating Σ_2: We start with

$$\begin{aligned} \Sigma_2 &= \sum_{Q\in\mathscr{D}} |b_Q| \langle |f| u^{-1}\rangle_Q |\langle g v, \mathbb{1}_Q\rangle| \frac{|\Delta_Q v|}{\langle v\rangle_Q} \frac{1}{\sqrt{\mu(Q)}} \\ &= \sum_{Q\in\mathscr{D}} |b_Q| \frac{\langle |f| u^{-1}\rangle_Q}{\langle u^{-1}\rangle_Q} \frac{\langle |g| v\rangle_Q}{\langle v\rangle_Q} |\Delta_Q v| \sqrt{\mu(Q)} \langle u^{-1}\rangle_Q . \end{aligned}$$

Using similar arguments as the ones used for Σ_1 , we conclude that,

$$\begin{aligned} \Sigma_2 &\leq \sum_{Q\in\mathscr{D}} |b_Q| \left(\inf_{x\in Q} M^{\mathscr{D}}_{u^{-1}} f(x) M^{\mathscr{D}}_v g(x) \right) |\Delta_Q v| \sqrt{\mu(Q)} \langle u^{-1}\rangle_Q \\ &= \sum_{Q\in\mathscr{D}} \frac{|b_Q|}{\sqrt{\langle v\rangle_Q}} \left(\inf_{x\in Q} M^{\mathscr{D}}_{u^{-1}} f(x) M^{\mathscr{D}}_v g(x) \right) |\Delta_Q v| \sqrt{\mu(Q) \langle v\rangle_Q} \langle u^{-1}\rangle_Q \\ &\leq [u,v]^{1/2}_{\mathcal{A}_2^d} \sum_{Q\in\mathscr{D}} \frac{|b_Q|}{\sqrt{\langle v\rangle_Q}} \left(\inf_{x\in Q} M^{\mathscr{D}}_{u^{-1}} f(x) \right) |\Delta_Q v| \sqrt{\mu(Q) \langle u^{-1}\rangle_Q} \left(\inf_{x\in Q} M^{\mathscr{D}}_v g(x) \right) \\ &\leq [u,v]^{1/2}_{\mathcal{A}_2^d} \Bigg(\sum_{Q\in\mathscr{D}} \frac{|b_Q|^2}{\langle v\rangle_Q} \inf_{x\in I} (M^{\mathscr{D}}_{u^{-1}} f)(x)^2 \Bigg)^{1/2} \Bigg(\sum_{I\in\mathcal{C}} |\Delta_Q v|^2 \langle u^{-1}\rangle_Q \mu(Q) \inf_{x\in I} (M^{\mathscr{D}}_v g)(x)^2 \Bigg)^{1/2} . \end{aligned}$$

By hypothesis $\{|b_Q|^2/\langle v\rangle_Q\}_{Q\in\mathscr{D}}$ is a u^{-1}-Carleson sequence and $\{|\Delta_Q v|^2 \langle u^{-1}\rangle_Q \mu(Q)\}_{Q\in\mathscr{D}}$ is a v-Carleson sequence with intensity $\mathcal{B}_{u,v}$ and $\mathcal{D}_{u,v}$ respectively. By Theorem

2.3,

$$\begin{aligned}\Sigma_2 &\leq \sqrt{[u,v]_{\mathcal{A}_2^d}\mathcal{B}_{u,v}\mathcal{D}_{u,v}}\left(\int_{\mathbb{R}}(M_{u^{-1}}^{\mathscr{D}}f)(x)^2u^{-1}(x)dx\right)^{1/2}\left(\int_{\mathbb{R}}(M_v^{\mathscr{D}}g)(x)^2v(x)dx\right)^{1/2}\\ &\leq \sqrt{[u,v]_{\mathcal{A}_2^d}\mathcal{B}_{u,v}\mathcal{D}_{u,v}}\|M_{u^{-1}}f\|_{L^2(u^{-1})}\|M_vg\|_{L^2(v)}\\ &\leq 4\sqrt{[u,v]_{\mathcal{A}_2^d}\mathcal{B}_{u,v}\mathcal{D}_{u,v}}\|f\|_{L^2(u^{-1})}\|g\|_{L^2(v)}\,.\end{aligned}$$

This estimate together with estimate (3.3) give (3.1), which complete the proof. □

4. Remarks

Remark 1. We can replace the condition on the pair (u,v) by boundedness of the dyadic square function, which is defined in Definition 7, to deduce boundedness of the dyadic paraproduct. We state this as the following corollary

Corollary 4.1. *Let (X,ρ,μ) be a space of homogeneous type and $b\in L^1_{loc}X$ and (u,v) be a pair of function such that v and u^{-1} are weights and $\{|b_Q|^2/\langle v\rangle_Q\rangle_{Q\in\mathscr{D}}$ is a u^{-1}-Carleson sequence ($b\in Carl_{u,v}$) with intensity $\mathcal{B}_{u,v}$. If the dyadic square function $S^{\mathscr{D}}$ is bounded from $L^2_{v^{-1}}(X)$ into $L^2_{u^{-1}}(X)$ then the dyadic paraproduct π_b is bounded from $L^2_u(X)$ into $L^2_v(X)$.*

Proof. Assume that $S^{\mathscr{D}}$ is bounded from $L^2_{v^{-1}}(X)$ into $L^2_{u^{-1}}(X)$, Theorem 2.5 implies that $(v^{-1},u^{-1})\in\mathcal{A}_2^{\mathscr{D}}$ and $\{|\Delta_Qv|^2\mu(Q)\langle u^{-1}\rangle_Q\}_{Q\in\mathscr{D}}$ is v-Carleson sequence. Also $(v^{-1},u^{-1})\in\mathcal{A}_2^{\mathscr{D}}$ is equivalent to $(u,v)\in\mathcal{A}_2^{\mathscr{D}}$. These two facts with the hypothesis that $\{|b_Q|^2/\langle v\rangle_Q\}_{Q\in\mathscr{D}}$ is a u^{-1}-Carleson sequence imply,by Theorem 1.1, that π_b is bounded from $L^2_u(X)$ into $L^2_v(X)$. □

Remark 2. In this paper we generalize the quantitative two weight estimate for the dyadic paraproduct to the space of homogeneous type. In general, the practical cost of the generalization is an unfortunate increase in the amount of bookkeeping when dealing with dyadic focused proofs. However the honest dyadic system presented in [10] makes the generalization process easy and it allows us to remove complicate notations in the proofs.

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Daewon Chung
Faculty of Basic Sciences, Mathematics Major, Keimyung University, 1095 Dalgubeol-Daero, Daegu, 704-701, Daegu, Korea
E-mail address: dwchung@kmu.ac.kr