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MODULAR MULTIPLICATIVE INVERSES OF FIBONACCI NUMBERS

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Abstract. Let $F_n, n \in \mathbb{N}$ be the $n-th$ Fibonacci number, and let (p, q) be one of ordered pairs (F_{n+2}, F_n) or (F_{n+1}, F_n). Then we show that the multiplicative inverse of q mod p as well as that of p mod q are again Fibonacci numbers. For proof of our claim we make use of well-known Cassini, Catlan and dOcagne identities. As an application, we determine the number $N_{p,q}$ of nonzero term of a polynomial $\Delta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$ through the Carlitz's formula.

1. Preliminaries

Motivation of problems dealt in this paper arose from two intriguing observations made for a torus knots $t(p, q)$ in knot theory where p, q are relative prime positive integers. One is that for each triple of consecutive Fibonacci numbers F_{n+2}, F_{n+1}, F_n, twisting F_{n+1} -parallel strands of a torus knot $t(F_{n+2}, F_n)$, we have a trivial knot. For more details see [2]. The other is that in 1966 an eminent number theorist Carlitz [1] provided a method of computing the number $N_{p,q}$ of non-zero terms of a polynomial $\Delta_{p,q} = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$, which turns out to be the Alexander polynomial of $t(p, q)$. Indeed the Alexander polynomial $\Delta_{p,q}$ is Φ_{pq}, the $pq-$th cyclotomic polynomial if p, q are distinct primes [3]. Explicit knowledge of $N_{p,q}$ is useful for a certain topological construction of $t(p, q)$ dealt in [5].

Definition 1. Let (p, q) be an ordered pair of relative prime positive integers. Then an ordered pair of positive integers (x, v) is said to be a pairwise modular multiplicative inverse of (p, q) if and only if
(1) $xq \equiv 1 \ mod \ p \ (1 \leq x \leq p-1)$ and
(2) $vp \equiv 1 \ mod \ q \ (1 \leq x \leq -1)$.

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Note that a pairwise modular multiplicative inverse of (p, q) is uniquely determined.

From [4, Proposition 2.1] we have.

Lemma 1.1. *Under the notations in Definition 1.1, the following statements are equivalent.*
(1) (x, v) *is the pairwise modular multiplicative inverse of* (p, q)
(2) there exists a uniquely determined quadruple of positive imtegers u, v, x *and* y *such that*

$$xv - yu = 1 \tag{1}$$

$$p = x + y \tag{2}$$

$$q = u + v \tag{3}$$

Remark 1. In Lemma we can replace equation (1.1) by one of following equations.

$$qx - pu = 1 \tag{4}$$

$$pv - qy = 1 \tag{5}$$

From[4, Corollary 2.6] we have.

Lemma 1.2. *The number , denoted by* $N_{p,q}$ *, of all non-zero terms of* $\Delta_{p,q}(t)$ *is equal to* $vx + uy = 2vx - 1$.

For simplicity we assume that $p > q$.

We recall three well known identities which naturally reveals modular multiplicative inverse of a pair of Fibonacci numbers.

A: Cassinis identity

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

is divided to two subcases: for each $k \in \mathbb{N}$

$$F_{2k-1}F_{2k+1} - F_{2k}^2 = 1 \tag{6}$$

$$F_{2k+1}^2 - F_{2k}F_{2k+2} = 1 \tag{7}$$

B: Catalans identity

$$F_n^2 - F_{n-2}F_{n+2} = (-1)^{n-2}$$

is divided to two subcases: for each $k \in \mathbb{N}$

$$F_{2k}^2 - F_{2k-2}F_{2k+2} = 1 \tag{8}$$

$$F_{2k-1}F_{2k+3} - F_{2k+1}^2 = 1 \tag{9}$$

C: dOcagnes identity

$$F_{n+2}F_{n+1} - F_nF_{n+3} = (-1)^n$$

is divided to two subcases: for each $k \in \mathbb{N}$

$$F_{2k+2}F_{2k+1} - F_{2k}F_{2k+3} = 1 \tag{10}$$

$$F_{2k-1}F_{2k+2} - F_{2k+1}F_{2k} = 1 \tag{11}$$

2. The main results

The following theorem shows intriguing applications of the three well known identities among a sequence of Fibonacci numbers to detecting a pairwise modular multiplicative inverse for a suitable two Fibonacci numbers.

Theorem 2.1. *For each* $k \in \mathbb{N}$ *we have:*
(1)(F_{2k-1}, F_{2k-1}) *is the parewise multiplicative inverse of* (F_{2k+1}, F_{2k}).
(2)(F_{2k+1}, F_{2k-1}) *is the parewise multiplicative inverse of* (F_{2k+2}, F_{2k+1}).
(3)(F_{2k}, F_{2k-1}) *is the parewise multiplicative inverse of* (F_{2k+2}, F_{2k}).
(4)(F_{2k+2}, F_{2k-1}) *is the parewise multiplicative inverse of* (F_{2k+3}, F_{2k+1}).

Proof. (1) For $(p, q) = (F_{2k+1}, F_{2k})$ and $(x, u) = (F_{2k-1}, F_{2k-2})$, applying the d'Ocagne's identity (10) to equation (4) we have the pairwise multiplicative inverse (F_{2k-1}, F_{2k-1}) of (F_{2k+1}, F_{2k}). In this case equation (5) corresponds to the Cassinis identity (6), since

$$\begin{aligned} (F_{2k} - F_{2k-2})F_{2k+1} - (F_{2k+1} - F_{2k-1})F_{2k} &= 1; \\ F_{2k-1}F_{2k+1} - F_{2k}^2 &= 1 \end{aligned}$$

.

(2) For $(p, q) = (F_{2k+2}, F_{2k+1})$ and $(x, u) = (F_{2k+1}, F_{2k})$, applying the Cassini identity (7) to equation (4) we have the pairwise multiplicative inverse (F_{2k+1}, F_{2k-1}) of (F_{2k+2}, F_{2k+1}). In this case equation (5) corresponds to dOcagnes identity (11), since

$$\begin{aligned} (F_{2k+1} - F_{2k})F_{2k+2} - (F_{2k+2} - F_{2k+1})F_{2k+1} &= 1; \\ F_{2k-1}F_{2k+2} - F_{2k}F_{2k+1} &= 1 \end{aligned}$$

.

(3) For $(p, q) = (F_{2k+2}, F_{2k})$ and $(x, u) = (F_{2k}, F_{2k-2})$, applying the Caltatans identity (8) to equation (4), we have the pairwise multiplicative inverse (F_{2k}, F_{2k-1}) of (F_{2k+2}, F_{2k}). In this case equation (1.5) corresponds to dOcagnes identity (1.11), since

$$\begin{aligned} (F_{2k} - F_{2k-2})F_{2k+2} - (F_{2k+2} - F_{2k})F_{2k} &= 1; \\ F_{2k-1}F_{2k+2} - F_{2k+1}F_{2k} &= 1. \end{aligned}$$

(4) For $(p, q) = (F_{2k+3}, F_{2k+1})$ and $(x, u) = (F_{2k+2}, F_{2k})$, identifying the dOcagnes identity (10) to equation (4) we have the pairwise multiplicative inverse (F_{2k+2}, F_{2k-1}) of (F_{2k+3}, F_{2k+1}). In this case equation (5) corresponds to the Catalans identity (9), since

$$\begin{aligned} (F_{2k+1} - F_{2k})F_{2k+3} - (F_{2k+3} - F_{2k+2})F_{2k+1} &= 1; \\ F_{2k-1}F_{2k+3} - F_{2k+1}^2 &= 1. \end{aligned}$$

□

As an application, we determine the number $N_{p,q}$ of non-zero term of the Alexander polynomial $\Delta_{p,q}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$ of a torus knot $t(p, q)$ as follows:

Corollary 2.2.

$$\begin{array}{lrcl} (1) & N_{F_{2k+1},F_{2k}} & = & 2F_{2k-1}^2 - 1 \\ (2) & N_{F_{2k+2},F_{2k+1}} & = & 2F_{2k+1}F_{2k-1} - 1 \\ (3) & N_{F_{2k+2},F_{2k}} & = & 2F_{2k}F_{2k-1} - 1 \\ (4) & N_{F_{2k+3},F_{2k+1}} & = & 2F_{2k+2}F_{2k-1} - 1 \end{array}$$

Applying the method introduced in [5] to the Corollary 2.2 , we shall determine (1,1)-diagrams of torus knots $t(F_n, F_{n+2})$ for each $n \geq 3$.

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