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STABILITY OF PEXIDERIZED JENSEN AND JENSEN TYPE FUNCTIONAL EQUATIONS ON RESTRICTED DOMAINS

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Abstract. In this paper, using the Baire category theorem we investigate the Hyers-Ulam stability problem of pexiderized Jensen functional equation

$$2f\left(\frac{x+y}{2}\right) - g(x) - h(y) = 0$$

and pexiderized Jensen type functional equations

$$f(x+y) + g(x-y) - 2h(x) = 0,$$
$$f(x+y) - g(x-y) - 2h(y) = 0$$

on a set of Lebesgue measure zero. As a consequence, we obtain asymptotic behaviors of the functional equations.

1. Introduction

Throughout the paper, we denote by $\mathbb{R}$, X and Y be the set of real numbers, a real normed space and a real Banach space, respectively, $d > 0$ and $\epsilon \geq 0$ be fixed. A mapping $f : X \to Y$ is called *the Jensen functional equation*

$$2f\left(\frac{x+y}{2}\right) - f(x) - f(y) = 0 \tag{1.1}$$

for all $x, y \in X$. A mapping $f : X \to Y$ is called *the Jensen type functional equation* if f satisfies one of the functional equations

$$f(x+y) + f(x-y) - 2f(x) = 0, \tag{1.2}$$
$$f(x+y) - f(x-y) - 2f(y) = 0 \tag{1.3}$$

for all $x, y \in X$. A mapping $f : X \to Y$ is called *an additive function* if f satisfies

$$f(x+y) - f(x) - f(y) = 0$$

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for all $x, y \in X$. The stability problems for functional equations have been originated by Ulam in 1940 (see [32]). One of the first assertions to be obtained is the following result, essentially due to Hyers [18] that gives an answer to the question of Ulam.

Theorem 1.1. *Let $\epsilon > 0$ be fixed. Suppose that $f : X \to Y$ satisfies the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X$. Then there exists a unique additive mapping $A : X \to Y$ satisfying

$$\|f(x) - A(x)\| \leq \epsilon$$

for all $x \in X$.

Among the numerous results on Ulam-Hyers stability theorem for functional equations (e.g. [4, 18–21, 25, 27–31]) there are various interesting results which deal with the stability of functional equations in restricted domains ([1–3, 5–17, 22, 24, 26, 27]). In particular, J. Chung ([9]) prove the Ulam-Hyers stability of the Jensen functional equation (1.1) and C.-K. Choi and B. Lee ([8]) prove the Ulam-Hyers stability of the Jensen type functional equations (1.2) and (1.3).

In this paper, generalizing the functional equations (1.1) $\sim$ (1.3) we consider the Ulam-Hyers stability of *the pexiderized Jensen functional equation*

$$2f\left(\frac{x+y}{2}\right) - g(x) - h(y) = 0 \tag{1.4}$$

for all $x, y \in X$, where $f, g, h : X \to Y$, and *the pexiderized Jensen type functional equations*

$$f(x+y) + g(x-y) - 2h(x) = 0, \tag{1.5}$$

$$f(x+y) - g(x-y) - 2h(y) = 0 \tag{1.6}$$

for all $x, y \in X$, where $f, g, h : X \to Y$ in restricted domains $\Omega \subset X \times X$ satisfying the condition (C):

Let $(\gamma_j, \lambda_j) \in \mathbb{R}^2$, $j = 1, 2, \dots, r$, with $\gamma_j^2 + \lambda_j^2 \neq 0$ for all $j = 1, 2, \dots, r$, be given.

(C) For any $p_j,\ q_j \in X$, $j = 1, 2, \dots, r$, there exists $t \in X$ such that

$$\{(p_j + \gamma_j t,\ q_j + \lambda_j t) : j = 1, 2, \dots, r\} \subset \Omega.$$

Remark 1.2. Functional equation (1.3) has only zero solutions. (If $x = y = 0$, then $f(0) = 0$; if $x = y$, then $f(2x) = 2f(x)$; if $y = -x$, then $-f(2x) = 2f(x)$, whence $f(x) = 0$.) The more so that (1.6) is just a case of (1.5) with g replaced by $-g$.

Secondly, using the Baire category theorem, we prove the stability of the functional equations (1.4) $\sim$ (1.6) on restricted domains of form $\mathcal{H}^2 \cap \{(x, y) \in$

$X^2 : \|x\| + \|y\| \ge d\}$ with $d > 0$, where $\mathcal{H}$ is a subset of X such that $\mathcal{H}^c$ is of the first category. Constructing a subset Ω_d of $\{(x, y) \in \mathbb{R}^2 : |x| + |y| \ge d\}$ of 2-dimensional Lebesgue measure zero satisfying the condition (C) we obtain measure zero stability problems of the functional equations (1.4) $\sim$ (1.6) when $X = \mathbb{R}$.

As consequences of the results we also prove that if $f, g, h : \mathbb{R} \to Y$ satisfy the asymptotic conditions

$$\left\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\right\| \to 0,$$
$$\|f(x+y) + g(x-y) - 2h(x)\| \to 0,$$
$$\|f(x+y) - g(x-y) - 2h(y)\| \to 0$$

as $|x| + |y| \to \infty$ only for (x, y) in a set of Lebesgue measure zero in $\mathbb{R}$.

2. Abstract approach

Throughout this section we assume that $\Omega \subset X \times X$ satisfies the conditions $(C_1) \sim (C_6)$: For given $(x, y) \in X$, there exists $t \in X$ such that

(C_1) $\{(2x+t, 2y-t), (2x-2y+t, 2y-t), (2x+t, -2x+2y-t), (2x-2y+t, -2x+2y-t)\} \subset \Omega$,

(C_2) $\{(x+y, t), (x, t), (y, x+t), (0, x+t)\} \subset \Omega$,

(C_3) $\{(t, x+y), (t, x), (x+t, y), P(x+t, 0)\} \subset \Omega$,

(C_4) $\left\{(x-t, y+t), (x-t, t), \left(\frac{1}{2}x - t, -\frac{1}{2}x + y + t\right), \left(\frac{1}{2}x - t, -\frac{1}{2}x + t\right)\right\} \subset \Omega$,

(C_5) $\left\{(x+t, -y+t), (x+t, t), \left(\frac{1}{2}x + t, -\frac{1}{2}x - y + t\right), \left(\frac{1}{2}x + t, \frac{1}{2}x + t\right)\right\} \subset \Omega$,

(C_6) $\{(x+y, -x+y+t), (x, -x+t), (y, y+t), (0, t)\} \subset \Omega$,

respectively. We prove the Ulam-Hyers stability of (1.4) $\sim$ (1.6) in Ω.

Theorem 2.1. *Suppose that $f, g, h : X \to Y$ satisfies the functional inequality*

$$\left\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\right\| \le \epsilon \tag{2.1}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon, \tag{2.2}$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon, \tag{2.3}$$
$$\|h(x) - A(x) - h(0)\| \le 4\epsilon \tag{2.4}$$

for all $x \in X$.

Proof. Let $P(x, y) = 2f\left(\frac{x+y}{2}\right) - g(x) - h(y)$. Since Ω satisfies the condition (C_1), it follows from (2.1) that for given $x, y \in X$, there exists $t \in X$ such that

$$\|P(2x+t, 2y-t)\| = \|2f(x+y) - g(2x+t) - h(2y-t))\| \le \epsilon, \tag{2.5}$$

$$\|P(2x-2y+t,2y-t)\| = \|2f(x) - g(2x-2y+t) - h(2y-t)\| \le \epsilon,$$
$$\|P(2x+t,-2x+2y-t)\| = \|2f(y) - g(2x+t) - h(-2x+2y-t)\| \le \epsilon,$$

$$\begin{aligned}(2.6)\qquad &\|P(2x-2y+t,-2x+2y-t)\| \\ &= \|2f(0) - g(2x-2y+t) - h(-2x+2y-t)\| \le \epsilon.\end{aligned}$$

Thus, using the triangle inequality we have

$$\begin{aligned}(2.7)\quad &\|f(x+y) - f(x) - f(y) + f(0)\| \\ &\le \left\| \frac{1}{2}P(2x+t,2y-t) - \frac{1}{2}P(2x-2y+t,2y-t) \right. \\ &\quad \left. -\frac{1}{2}P(2x+t,-2x+2y-t) + \frac{1}{2}P(2x-2y+t,-2x+2y-t) \right\| \le 2\epsilon\end{aligned}$$

for all $x, y \in X$. Since Ω satisfies the condition (C_2), it follows from (2.1) that for given $x, y \in X$, there exists $t \in X$ such that

$$\|P(x+y,t)\| = \left\| 2f\left(\frac{x+y+t}{2}\right) - g(x+y) - h(t) \right\| \le \epsilon,$$
$$\|P(x,t)\| = \left\| 2f\left(\frac{x+t}{2}\right) - g(x) - h(t) \right\| \le \epsilon,$$
$$\|P(y,x+t)\| = \left\| 2f\left(\frac{x+y+t}{2}\right) - g(y) - h(x+t) \right\| \le \epsilon,$$
$$\|P(0,x+t)\| = \left\| 2f\left(\frac{x+t}{2}\right) - g(0) - h(x+t) \right\| \le \epsilon.$$

Thus, using the triangle inequality we have

$$\begin{aligned}(2.8)\qquad &\|g(x+y) - g(x) - g(y) + g(0)\| \\ &\le \|P(x+y,t) - P(x,t) - P(y,x+t) + P(0,x+t)\| \le 4\epsilon\end{aligned}$$

for all $x, y \in X$. Since Ω satisfies the condition (C_3), it follows from (2.1) that for given $x, y \in X$, there exists $t \in X$ such that

$$\|P(t,x+y)\| = \left\| 2f\left(\frac{x+y+t}{2}\right) - g(t) - h(x+y) \right\| \le \epsilon,$$
$$\|P(t,x)\| = \left\| 2f\left(\frac{x+t}{2}\right) - g(t) - h(x) \right\| \le \epsilon,$$
$$\|P(x+t,y)\| = \left\| 2f\left(\frac{x+y+t}{2}\right) - g(x+t) - h(y) \right\| \le \epsilon,$$
$$\|P(x+t,0)\| = \left\| 2f\left(\frac{x+t}{2}\right) - g(x+t) - h(0) \right\| \le \epsilon.$$

Thus, using the triangle inequality we have

$$(2.9)\qquad \|h(x+y) - h(x) - h(y) + h(0)\|$$

$$\leq \|P(t, x+y) - P(t, x) - P(x+t, y) + P(x+t, 0)\| \leq 4\epsilon$$

for all $x, y \in X$. By Theorem 1.1 with (2.7) $\sim$ (2.9), there exist additive functions $A_1, A_2, A_3 : X \to Y$ such that

$$\|f(x) - A_1(x) - f(0)\| \leq 2\epsilon, \tag{2.10}$$

$$\|g(x) - A_2(x) - g(0)\| \leq 4\epsilon, \tag{2.11}$$

$$\|h(x) - A_3(x) - h(0)\| \leq 4\epsilon \tag{2.12}$$

for all $x \in X$. Replacing x by $2x-2y+t$, y by $2y$ in (2.8) and x by $-2x+2y-t$, y by $2x$ in (2.9) we have

$$\|g(2x+t) - g(2x-2y+t) - g(2y) + g(0)\| \leq 4\epsilon, \tag{2.13}$$

$$\|h(2y-t) - h(-2x+2y-t) - h(2x) + h(0)\| \leq 4\epsilon \tag{2.14}$$

for all $x, y \in X$. Using the triangle inequality with (2.5), (2.6), (2.13) and (2.14) we have

$$\|2f(x+y) - g(2y) - h(2x) - 2f(0) + g(0) + h(0)\| \leq 10\epsilon \tag{2.15}$$

for all $x, y \in X$. Replacing x by $x+y$ in (2.10), x by $2y$ in (2.11) and x by $2x$ in (2.12) we have

$$\|f(x+y) - A_1(x+y) - f(0)\| \leq 2\epsilon, \tag{2.16}$$

$$\|g(2y) - A_2(2y) - g(0)\| \leq 4\epsilon, \tag{2.17}$$

$$\|h(2x) - A_3(2x) - h(0)\| \leq 4\epsilon \tag{2.18}$$

for all $x, y \in X$. Using the triangle inequality with (2.15) $\sim$ (2.18) we have

$$\|2A_1(x+y) - A_2(2y) - A_3(2x)\| \leq 22\epsilon \tag{2.19}$$

for all $x, y \in X$. Putting $y = 0$ in (2.19) and using the additivity of A_j, $j = 1, 2, 3$, we have $A_1 = A_2$. Similarly, putting $x = 0$ in (2.19) we have $A_1 = A_3$. Thus, we have $A_1 = A_2 = A_3 (:= A)$. Hence, there exists a unique additive function $A : X \to Y$ such that (2.2) $\sim$ (2.4) for all $x \in X$. This completes the proof. □

Theorem 2.2. *Suppose that $f, g, h : X \to Y$ satisfies the functional inequality*

$$\|f(x+y) + g(x-y) - 2h(x)\| \leq \epsilon \tag{2.20}$$

for all $(x, y) \in \Omega$. Then there exists a unique additive function $A : X \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 4\epsilon, \tag{2.21}$$

$$\|g(x) - A(x) - g(0)\| \leq 4\epsilon, \tag{2.22}$$

$$\|h(x) - A(x) - h(0)\| \leq 2\epsilon \tag{2.23}$$

for all $x \in X$.

Proof. Let $Q(x,y)=f(x+y)+g(x-y)-2h(x)$. Since Ω satisfies the condition (C_4), it follows from (2.20) that for given $x,y\in X$, there exists $t\in X$ such that

$$\|Q(x-t,y+t)\|=\|f(x+y)+g(x-y-2t)-2h(x-t)\|\le\epsilon, \tag{2.24}$$
$$\|Q(x-t,t)\|=\|f(x)+g(x-2t)-2h(x-t)\|\le\epsilon,$$
$$\left\|Q\left(\frac{1}{2}x-t,-\frac{1}{2}x+y+t\right)\right\|=\left\|f(y)+g(x-y-2t)-2h\left(\frac{1}{2}x-t\right)\right\|\le\epsilon,$$

$$\left\|Q\left(\frac{1}{2}x-t,-\frac{1}{2}x+t\right)\right\|=\left\|f(0)+g(x-2t)-2h\left(\frac{1}{2}x-t\right)\right\|\le\epsilon. \tag{2.25}$$

Thus, using the triangle inequality we have

$$\begin{aligned}&\|f(x+y)-f(x)-f(y)+f(0)\| \\ &\le\left\|Q(x-t,y+t)-Q(x-t,t)-Q\left(\frac{1}{2}x-t,-\frac{1}{2}x+y+t\right)\right. \\ &\qquad\left.+Q\left(\frac{1}{2}x-t,-\frac{1}{2}x+t\right)\right\|\le 4\epsilon\end{aligned} \tag{2.26}$$

for all $x,y\in X$. Since Ω satisfies the condition (C_5), it follows from (2.20) that for given $x,y\in X$, there exists $t\in X$ such that

$$\|Q(x+t,-y+t)\|=\|f(x-y+2t)+g(x+y)-2h(x+t)\|\le\epsilon,$$
$$\|Q(x+t,t)\|=\|f(x+2t)+g(x)-2h(x+t)\|\le\epsilon,$$
$$\left\|Q\left(\frac{1}{2}x+t,-\frac{1}{2}x-y+t\right)\right\|=\left\|f(x-y+2t)+g(y)-2h\left(\frac{1}{2}x+t\right)\right\|\le\epsilon,$$
$$\left\|Q\left(\frac{1}{2}x+t,\frac{1}{2}x+t\right)\right\|=\left\|f(x+2t)+g(0)-2h\left(\frac{1}{2}x+t\right)\right\|\le\epsilon.$$

Thus, using the triangle inequality we have

$$\begin{aligned}&\|g(x+y)-g(x)-g(y)+g(0)\| \\ &\le\left\|Q(x+y,-y+t)-Q(x+t,t)-Q\left(\frac{1}{2}x+t,-\frac{1}{2}x-y+t\right)\right. \\ &\qquad\left.+Q\left(\frac{1}{2}x+t,\frac{1}{2}x+t\right)\right\|\le 4\epsilon\end{aligned} \tag{2.27}$$

for all $x,y\in X$. Since Ω satisfies the condition (C_6), it follows from (2.20) that for given $x,y\in X$, there exists $t\in X$ such that

$$\|Q(x+y,-x+y+t)\|=\|f(2y+t)+g(2x-t)-2h(x+y)\|\le\epsilon,$$
$$\|Q(x,-x+t)\|=\|f(t)+g(2x-t)-2h(x)\|\le\epsilon,$$
$$\|Q(y,y+t)\|=\|f(2y+t)+g(-t)-2h(y)\|\le\epsilon,$$
$$\|Q(0,t)\|=\|f(t)+g(-t)-2h(0)\|\le\epsilon.$$

Thus, using the triangle inequality we have

$$\begin{aligned}(2.28)\quad &\|h(x+y)-h(x)-h(y)+h(0)\|\\ &\le \left\|\frac{1}{2}Q(x+y,-x+y+t)-\frac{1}{2}Q(x,-x+t)-\frac{1}{2}Q(y,y+t)+\frac{1}{2}Q(0,t)\right\|\\ &\le 2\epsilon\end{aligned}$$

for all $x, y \in X$. By Theorem 1.1 with (2.26) $\sim$ (2.28), there exist additive functions $A_1, A_2, A_3 : X \to Y$ such that

$$\|f(x)-A_1(x)-f(0)\| \le 4\epsilon, \tag{2.29}$$

$$\|g(x)-A_2(x)-g(0)\| \le 4\epsilon, \tag{2.30}$$

$$\|h(x)-A_3(x)-h(0)\| \le 2\epsilon \tag{2.31}$$

for all $x \in X$. Replacing x by $x-2t$, y by $-y$ in (2.27) and x by $\frac{1}{2}x-t$, y by $\frac{1}{2}x$ in (2.28) we have

$$\|g(x-y-2t)-g(x-2t)-g(-y)+g(0)\| \le 4\epsilon, \tag{2.32}$$

$$\left\|h(x-t)-h\left(\frac{1}{2}x-t\right)-h\left(\frac{1}{2}x\right)+h(0)\right\| \le 2\epsilon \tag{2.33}$$

for all $x, y \in X$. Using the triangle inequality with (2.24), (2.25), (2.32) and (2.33) we have

$$\left\|f(x+y)+g(-y)-2h\left(\frac{1}{2}x\right)-f(0)-g(0)+2h(0)\right\| \le 10\epsilon \tag{2.34}$$

for all $x, y \in X$. Replacing x by $x+y$ in (2.29), x by $-y$ in (2.30) and x by $\frac{1}{2}x$ in (2.31) we have

$$\|f(x+y)-A_1(x+y)-f(0)\| \le 4\epsilon, \tag{2.35}$$

$$\|g(-y)-A_2(-y)-g(0)\| \le 4\epsilon, \tag{2.36}$$

$$\left\|h\left(\frac{1}{2}x\right)-A_3\left(\frac{1}{2}x\right)-h(0)\right\| \le 2\epsilon \tag{2.37}$$

for all $x, y \in X$. Using the triangle inequality with (2.34) $\sim$ (2.37) we have

$$\left\|A_1(x+y)+A_2(-y)-2A_3\left(\frac{1}{2}x\right)\right\| \le 22\epsilon \tag{2.38}$$

for all $x, y \in X$. Putting $y = 0$ in (2.38) and using the additivity of $A_j, j = 1, 2, 3$, we have $A_1 = A_3$. Similarly, putting $x = 0$ in (2.38) we have $A_1 = A_2$. Thus, we have $A_1 = A_2 = A_3 (:= A)$. Hence, there exists a unique additive function $A : X \to Y$ such that (2.21) $\sim$ (2.23) for all $x \in X$. This completes the proof. □

Theorem 2.3. *Suppose that $f, g, h : X \to Y$ satisfies the functional inequality*

$$\|f(x+y)-g(x-y)-2h(y)\| \le \epsilon$$

for all $(x, y) \in \Omega$. *Then there exists a unique additive function* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 2\epsilon$$

for all $x \in X$.

It is obvious that the set $\{(x, y) \in X^2 : \|x\| + \|y\| \ge d\}$ satisfies the condition $(C_1) \sim (C_6)$. Thus, as direct consequences of Theorem 2.1 $\sim$ Theorem 2.3 we obtain the results following.

Corollary 2.4. *Let* $d > 0$. *Suppose that* $f, g, h : X \to Y$ *satisfies the functional inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - g(x) - h(y) \right\| \le \epsilon$$

for all $(x, y) \in X$ *with* $\|x\| + \|y\| \ge d$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 2\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 4\epsilon$$

for all $x \in X$.

Corollary 2.5. *Let* $d > 0$. *Suppose that* $f, g, h : X \to Y$ *satisfies the functional inequality*

$$\|f(x + y) + g(x - y) - 2h(x)\| \le \epsilon$$

for all $(x, y) \in X$ *with* $\|x\| + \|y\| \ge d$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 2\epsilon$$

for all $x \in X$.

Corollary 2.6. *Let* $d > 0$. *Suppose that* $f, g, h : X \to Y$ *satisfies the functional inequality*

$$\|f(x + y) - g(x - y) - 2h(y)\| \le \epsilon$$

for all $(x, y) \in X$ *with* $\|x\| + \|y\| \ge d$. *Then there exists a unique additive mapping* $A : X \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 2\epsilon$$

for all $x \in X$.

3. Main results

Throughout this section we assume that X is complete. By constructing subsets $\Omega \subset X \times X$ satisfying the three conditions $(\mathrm{C}_1) \sim (\mathrm{C}_6)$ we prove the Hyers-Ulam stability of the functional equations (1.4) $\sim$ (1.6) satisfied on restricted domains of form $\mathcal{H}^2 \cap \{(x,y) \in X^2 : \|x\| + \|y\| \geq d\}$ with $d > 0$, where $\mathcal{H}$ is a subset of X such that $\mathcal{H}^c$ is of the first category. As a consequence we obtain a stability theorem of the functional equations on a set of Lebesgue measure zero when $X = \mathbb{R}$.

Recall that a subset K of a topological space E is said to be of the first category if K is a countable union of nowhere dense subsets of E, and otherwise it is said to be of the second category. As named *Baire category theorem* it is well known that every nonempty open subset of a compact Hausdorff space or a complete metric space is of the second category.

The proof of the following lemmas can be found in [15]. For the reader we give the proof.

Lemma 3.1. *Let $\mathcal{H}$ be a subset of X such that $\mathcal{H}^c := X \setminus \mathcal{H}$ is of the first category. Then, for any countable subsets $U \subset X$, $\Gamma \subset \mathbb{R} \setminus \{0\}$ and $M > 0$, there exists $t \in X$ with $\|t\| \geq M$ such that*

$$U + \Gamma t = \{u + \gamma t : u \in U, \gamma \in \Gamma\} \subset \mathcal{H}.$$

From now on we identify $\mathbb{R}^2$ with $\mathbb{C}$.

Lemma 3.2. *Let $P = \{(p_j + \gamma_j t, q_j + \lambda_j t) : j = 1, 2, \ldots, r\}$, where $p_j, q_j, t \in X, \gamma_j, \lambda_j \in \mathbb{R}$ with $\gamma_j^2 + \lambda_j^2 \neq 0$ for all $j = 1, 2, \ldots, r$. Then there exists a $\theta \in [0, 2\pi)$ such that $e^{-i\theta} P := \{(p'_j + \gamma'_j t, q'_j + \lambda'_j t) : j = 1, 2, \ldots, r\}$ satisfies $\gamma'_j \lambda'_j \neq 0$ for all $j = 1, 2, \ldots, r$.*

Lemma 3.3. *Let $\mathcal{H}$ be a subset of X such that $\mathcal{H}^c$ is of the first category. Then there exists a $\theta \in [0, 2\pi)$ such that $\Omega_{\theta,d} := (e^{i\theta}\mathcal{H}^2) \cap \{(x,y) \in X^2 : \|x\| + \|y\| \geq d\}$ satisfies the conditions $(\mathrm{C}_1) \sim (\mathrm{C}_6)$ for all $d > 0$.*

Remark 3.4. The set $\mathbb{R}$ of real numbers can be partitioned as follows:

$$\mathbb{R} = \mathcal{K} \cup (\mathbb{R} \setminus \mathcal{K}),$$

where $\mathcal{K}$ is of Lebesgue measure zero and $\mathbb{R} \setminus \mathcal{K}$ is of the first category [23, Theorem 1.6]. Thus, in view of Lemma 3.3, $\Omega_d := (e^{i\theta}\mathcal{K}^2) \cap \{(x,y) \in \mathbb{R}^2 : |x| + |y| \geq d\}$ is of Lebesgue measure zero satisfying $(\mathrm{C}_1) \sim (\mathrm{C}_6)$.

Now, we obtain the following results.

Theorem 3.5. *Suppose that $f, g, h : \mathbb{R} \to Y$ satisfies the functional inequality*

$$\left\| 2f\left(\frac{x+y}{2}\right) - g(x) - h(y) \right\| \leq \epsilon$$

for all $(x,y) \in \Omega_d$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$\|f(x) - A(x) - f(0)\| \leq 2\epsilon,$$

$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 4\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 3.6. *Suppose that* $f, g, h : \mathbb{R} \to Y$ *satisfies the functional inequality*

$$\|f(x+y) + g(x-y) - 2h(x)\| \le \epsilon$$

for all $(x, y) \in \Omega_d$. *Then there exists a unique additive mapping* $A : \mathbb{R} \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 2\epsilon$$

for all $x \in \mathbb{R}$.

Theorem 3.7. *Suppose that* $f, g, h : \mathbb{R} \to Y$ *satisfies the functional inequality*

$$\|f(x+y) - g(x-y) - 2h(y)\| \le \epsilon$$

for all $(x, y) \in \Omega_d$. *Then there exists a unique additive mapping* $A : \mathbb{R} \to Y$ *such that*

$$\|f(x) - A(x) - f(0)\| \le 4\epsilon,$$
$$\|g(x) - A(x) - g(0)\| \le 4\epsilon,$$
$$\|h(x) - A(x) - h(0)\| \le 2\epsilon$$

for all $x \in \mathbb{R}$.

As a consequence of Theorem 3.5 we obtain the asymptotic behavior of f, g, h satisfying

$$\left\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\right\| \to 0 \tag{3.1}$$

as $|x| + |y| \to \infty$ only for $(x, y) \in \Omega \subset \mathbb{R}^2$ with $m(\Omega) = 0$.

Corollary 3.8. *Suppose that* $f, g, h : \mathbb{R} \to Y$ *satisfies the condition* (3.1). *Then there exists a unique additive mapping* $A : \mathbb{R} \to Y$ *such that*

$$f(x) = A(x) + f(0), \tag{3.2}$$
$$g(x) = A(x) + g(0), \tag{3.3}$$
$$h(x) = A(x) + h(0) \tag{3.4}$$

for all $x \in \mathbb{R}$.

Proof. The condition (3.1) implies that for each $n \in \mathbb{N}$, there exists $d_n > 0$ such that

$$\left\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\right\| \le \frac{1}{n}$$

for all $(x, y) \in \Omega_{d_n}$. By Theorem 3.5, there exists a unique additive mapping $A_n : \mathbb{R} \to Y$ such that

$$\|f(x) - A_n(x) - f(0)\| \le \frac{2}{n}, \tag{3.5}$$

$$\|g(x) - A_n(x) - g(0)\| \le \frac{4}{n}, \tag{3.6}$$

$$\|h(x) - A_n(x) - h(0)\| \le \frac{4}{n} \tag{3.7}$$

for all $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Replacing n by $m \in \mathbb{N}$ in (3.5) we have

$$\|f(x) - A_m(x) - f(0)\| \le \frac{2}{m} \tag{3.8}$$

for all $m \in \mathbb{N}$ and $x \in \mathbb{R}$. Using the triangle inequality with (3.5) and (3.8) we have

$$\|A_m(x) - A_n(x)\| \le \frac{2}{m} + \frac{2}{n} \le 4 \tag{3.9}$$

for all $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$. From the additivity of A_m, A_n, it follows that $A_m = A_n$ for all $m, n \in \mathbb{N}$. Letting $n \to \infty$ in (3.9) we get (3.2). Similarly, replacing n by $m \in \mathbb{N}$ in (3.6) and (3.7), respectively, we have

$$\|A_m(x) - A_n(x)\| \le \frac{4}{m} + \frac{4}{n} \le 8 \tag{3.10}$$

for all $m, n \in \mathbb{N}$ and $x \in \mathbb{R}$. From the additivity of A_m, A_n, it follows that $A_m = A_n$ for all $m, n \in \mathbb{N}$. Letting $n \to \infty$ in (3.10) we get (3.3) and (3.4). This completes the proof. □

Similarly, using Theorem 3.6 and Theorem 3.7 we have the following.

Corollary 3.9. *Suppose that $f, g, h : \mathbb{R} \to Y$ satisfies the condition*

$$\|f(x + y) + g(x - y) - 2h(x)\| \to 0$$

as $|x| + |y| \to \infty$ only for $(x, y) \in \Omega \subset \mathbb{R}^2$ with $m(\Omega) = 0$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$f(x) = A(x) + f(0),$$
$$g(x) = A(x) + g(0),$$
$$h(x) = A(x) + h(0)$$

for all $x \in \mathbb{R}$.

Corollary 3.10. *Suppose that $f, g, h : \mathbb{R} \to Y$ satisfies the condition*

$$\|f(x + y) - g(x - y) - 2h(y)\| \to 0$$

as $|x| + |y| \to \infty$ only for $(x, y) \in \Omega \subset \mathbb{R}^2$ with $m(\Omega) = 0$. Then there exists a unique additive mapping $A : \mathbb{R} \to Y$ such that

$$f(x) = A(x) + f(0),$$
$$g(x) = A(x) + g(0),$$

$$h(x) = A(x) + h(0)$$

for all $x \in \mathbb{R}$.

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