

GENERALIZED FORMS OF SWIATAK'S FUNCTIONAL EQUATIONS WITH INVOLUTION

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ABSTRACT. In this paper, we study two functional equations with two unknown functions from an Abelian group into a commutative ring without zero divisors. The two equations are generalizations of Swiatak's functional equations with an involution. We determine the general solutions of the two functional equations and the properties of the general solutions of the two functional equations under three different hypotheses, respectively. For one of the functional equations, we establish the Hyers-Ulam stability in the case that the unknown functions are complex valued.

1. Introduction

Let G be an abelian group with the identity element e and K a commutative ring without zero divisors. Let $\sigma : G \rightarrow G$ be an involution, that is, σ is an endomorphism of G satisfying $\sigma(\sigma(x)) = x$ for all $x \in G$. We consider the following two functional equations

$$(1.1) \quad f(x+y) + f(x+\sigma(y)) = 2f(x) + 2f(y) + g(x)g(y),$$
$$f(x+y+z) + f(x+\sigma(y)+z) + f(x+y+\sigma(z)) + f(\sigma(x)+y+z)$$
$$(1.2) \quad = 4f(x) + 4f(y) + 4f(z) + 2g(x)g(y) + 2g(x)g(z) + 2g(y)g(z)$$

for all $x, y, z \in G$ and f, g take values in K . When $\sigma(x) = -x$ for all $x \in G$, then the above two equations are respectively reduced to

$$(1.3) \quad f(x+y) + f(x-y) = 2f(x) + 2f(y) + g(x)g(y),$$
$$f(x+y+z) + f(x-y+z) + f(x+y-z) + f(-x+y+z)$$
$$(1.4) \quad = 4f(x) + 4f(y) + 4f(z) + 2g(x)g(y) + 2g(x)g(z) + 2g(y)g(z)$$

Received June 22, 2018; Accepted January 8, 2019.

2010 *Mathematics Subject Classification.* 39B82, 39B52.

Key words and phrases. abelian group, Hyers-Ulam stability, quadratic functional equation, Swiatak's functional equation.

The author is supported by the National Natural Science Foundation of China (Grant Nos. 11401190) and 17YJA790098.

which are connected with the quadratic functional equation, and these two functional equations were studied by Swiatak in [20]. She determined the solutions $f, g : G \rightarrow K$ of (1.3) and (1.4) when $g(e) \neq 0$. Later, the equation (1.3) was completely solved by Chung, Ebank, Ng and Sahoo [5] on arbitrary groups when the unknown functions $f : G \rightarrow \mathbb{C}$ was abelian. Recall that a function f is abelian if $f(xyz) = f(xzy)$ for all $x, y, z \in G$. Without the restriction $g(e) \neq 0$, the general solutions $f, g : G \rightarrow K$ of (1.4) were determined by Chung, Ebanks and Sahoo in [6] on arbitrary groups when f is abelian. Recently, Belaid, Elhoucien and Rassias [2] proved the generalized Hyers-Ulam stability of (1.3) in the case that $f, g : G \rightarrow \mathbb{C}$ are complex-valued functions and $g(e) \neq 0$. However, the Hyers-Ulam stability of the equation (1.4) is unknown.

In this paper, we consider the functional equations (1.1) and (1.2) under three different hypotheses:

(H1): $g(e) \neq 0$ and $(K, +)$ is a uniquely 2-divisible group (i.e., for each $x \in K$ there is a unique $y \in K$ such that $x = 2y$).

(H2): $g(e) = 0$ and for each $k \in K$, $2k = 0$ if and only if $k = 0$.

(H2s): $g : G \rightarrow K$ satisfies

$$(1.5) \quad g(x + y) + g(x + \sigma(y)) = 2g(x) + 2g(y)$$

and for each $k \in K$, $2k = 0$ if and only if $k = 0$.

Note that in **(H2s)** the condition (1.5) implies $g(e) = 0$. So **(H2)** is weaker than **(H2s)**. If $(K, +)$ is a uniquely 2-divisible group, then we have $2k = 0$ if and only if $k = 0$ for each $k \in K$. We first present the relationship between the general solutions of (1.1) (resp. (1.2)) and the quadratic functional equation with an involution under **(H1)**. Then we characterize properties of the general solutions of (1.1) and (1.2) under **(H2)** and prove that (1.1) is equivalent to (1.2) under the special hypotheses **(H2s)**. In this paper, we also discuss the Hyers-Ulam stability of the equation (1.1) in the case that $f, g : G \rightarrow \mathbb{C}$ satisfying $g(e) \neq 0$.

2. The general solutions and properties of the general solutions of (1.1) and (1.2)

In this section, we discuss the functional equations (1.1) and (1.2) under hypotheses **(H1)**, **(H2)** and **(H2s)**, respectively.

Proposition 2.1. *Suppose **(H1)** holds. Then the general solution (f, g) of the equation (1.1) (resp. (1.2)) is given by*

$$(2.1) \quad f(x) = \varphi(x) + A, \quad g(x) = B,$$

for all $x \in G$, where φ is an arbitrary solution of the following quadratic functional equation with an involution

$$(2.2) \quad \varphi(x + y) + \varphi(x + \sigma(y)) = 2\varphi(x) + 2\varphi(y),$$

$A := f(e)$ and $B := g(e) \neq 0$ satisfy $2A + B^2 = 0$ (resp. $4A + 3B^2 = 0$).

Remark 1 (see [19, Theorem 3]). Let $(S, +)$ be an additive semigroup, and let H be a uniquely 2-divisible Abelian group. Note that the general solution of the functional equation (2.2) is given by

$$\varphi(x) = B(x, x) + \phi(x), \forall x \in S,$$

where $B : S \times S \rightarrow H$ is an arbitrary symmetric biadditive function with $B(\sigma(x), y) = -B(x, y)$ and ϕ is an arbitrary additive function with $\phi(\sigma(x)) = \phi(x)$.

Proof. It is easy to check that any function (f, g) of the form (2.1) satisfies the equation (1.1) (resp. (1.2)).

Conversely, assume that (f, g) is a solution of the equation (1.1) (resp. (1.2)). We first consider the equation (1.1). Setting $x = y = e$ in (1.1), we obtain $2f(e) + g(e)^2 = 0$, i.e.,

$$(2.3) \quad 2A + B^2 = 0.$$

Putting $y = e$ in (1.1), we obtain

$$(2.4) \quad 2A + Bg(x) = 0.$$

From (2.3) and (2.4), we get $B(g(x) - B) = 0$, which implies that

$$(2.5) \quad g(x) \equiv B,$$

since the ring K has no zero divisors and $B \neq 0$. Subtracting $2A$ from both sides of (1.1) and using (2.3) and (2.5), we get

$$(f(x + y) - A) + (f(x + \sigma(y)) - A) = 2(f(x) - A) + 2(f(y) - A).$$

This implies that $f(x) - A$ is a solution of (2.2).

Next, we consider the equation (1.2). Putting $x = y = z = e$ in (1.2), we get $8f(e) + 6g(e)^2 = 0$. Since $(K, +)$ is a uniquely 2-divisible group, we have

$$(2.6) \quad 4A + 3B^2 = 0.$$

Let $\varphi(x) := f(x) - A, \gamma(x) := g(x) - B$. Then $\varphi(e) = \gamma(e) = 0$, and

$$\begin{aligned} & \varphi(x + y + z) + \varphi(x + \sigma(y) + z) + \varphi(x + y + \sigma(z)) + \varphi(\sigma(x) + y + z) \\ &= 4\varphi(x) + 4\varphi(y) + 4\varphi(z) + 2\gamma(x)\gamma(y) + 2\gamma(x)\gamma(z) + 2\gamma(y)\gamma(z) \\ & \quad + 4B[\gamma(x) + \gamma(y) + \gamma(z)] + 8A + 6B^2 \end{aligned}$$

for all $x, y, z \in G$. It follows from (2.6) that

$$(2.7) \quad \begin{aligned} & \varphi(x + y + z) + \varphi(x + \sigma(y) + z) + \varphi(x + y + \sigma(z)) + \varphi(\sigma(x) + y + z) \\ &= 4\varphi(x) + 4\varphi(y) + 4\varphi(z) + 2\gamma(x)\gamma(y) + 2\gamma(x)\gamma(z) + 2\gamma(y)\gamma(z) \\ & \quad + 4B[\gamma(x) + \gamma(y) + \gamma(z)]. \end{aligned}$$

Putting $y = z = e$ in (2.7), we get

$$(2.8) \quad \varphi(\sigma(x)) = \varphi(x) + 4B\gamma(x).$$

Hence, we have

$$(2.9) \quad \varphi(x) = \varphi(\sigma(x)) + 4B\gamma(\sigma(x))$$

for all $x \in G$. By (2.8) and (2.9), we obtain

$$4B[\gamma(x) + \gamma(\sigma(x))] = 0$$

for all $x \in G$. Since $(K, +)$ is a uniquely 2-divisible group, we have

$$B[\gamma(x) + \gamma(\sigma(x))] = 0.$$

Since the ring K has no zero divisors and $B \neq 0$, we have

$$(2.10) \quad \gamma(x) + \gamma(\sigma(x)) = 0$$

for all $x \in G$. Putting $z = e$ in (2.7), we obtain

$$(2.11) \quad \begin{aligned} & 2\varphi(x+y) + \varphi(x+\sigma(y)) + \varphi(\sigma(x)+y) \\ &= 4\varphi(x) + 4\varphi(y) + 2\gamma(x)\gamma(y) + 4B[\gamma(x) + \gamma(y)]. \end{aligned}$$

Replacing x and y by $\sigma(x)$ and $\sigma(y)$ in (2.11), respectively, we get

$$(2.12) \quad \begin{aligned} & 2\varphi(\sigma(x)+y) + \varphi(\sigma(x)+\sigma(y)) + \varphi(x+\sigma(y)) \\ &= 4\varphi(\sigma(x)) + 4\varphi(\sigma(y)) + 2\gamma(\sigma(x))\gamma(\sigma(y)) + 4B[\gamma(\sigma(x)) + \gamma(\sigma(y))]. \end{aligned}$$

Put $\tilde{\varphi}(x) = \varphi(x) + \varphi(\sigma(x))$. It follows from (2.11) and (2.12) that

$$(2.13) \quad \begin{aligned} & 2\tilde{\varphi}(x+y) + 2\tilde{\varphi}(x+\sigma(y)) \\ &= 4\tilde{\varphi}(x) + 4\tilde{\varphi}(y) + 2[\gamma(x)\gamma(y) + \gamma(\sigma(x))\gamma(\sigma(y))] \\ & \quad + 4B[\gamma(x) + \gamma(\sigma(x)) + \gamma(y) + \gamma(\sigma(y))]. \end{aligned}$$

By (2.10), we have

$$(2.14) \quad \tilde{\varphi}(x+y) + \tilde{\varphi}(x+\sigma(y)) = 2\tilde{\varphi}(x) + 2\tilde{\varphi}(y) + 2\gamma(x)\gamma(y).$$

Replacing y by $\sigma(y)$ in (2.14), and noting that $\tilde{\varphi}(x) = \tilde{\varphi}(\sigma(x))$, we obtain

$$(2.15) \quad \tilde{\varphi}(x+\sigma(y)) + \tilde{\varphi}(x+y) = 2\tilde{\varphi}(x) + 2\tilde{\varphi}(y) + 2\gamma(x)\gamma(\sigma(y)).$$

From (2.14) and (2.15) we get

$$2\gamma(x)[\gamma(y) - \gamma(\sigma(y))] = 0.$$

Consequently,

$$(2.16) \quad \gamma(x)[\gamma(y) - \gamma(\sigma(y))] = 0.$$

By **(H1)**, (2.10) and (2.16), we obtain $\gamma(x)\gamma(y) = 0$, which implies that

$$\gamma(x) \equiv 0, \quad \text{i.e., } g(x) \equiv B.$$

Now, (2.8) implies that $\varphi(\sigma(x)) = \varphi(x)$. Then (2.11) can be expressed as the functional equation

$$\varphi(x+y) + \varphi(x+\sigma(y)) = 2\varphi(x) + 2\varphi(y)$$

for all $x \in G$. This completes the proof of the proposition. \square

Proposition 2.2. *Suppose (H2) holds. Then the general solution (f, g) of the equation (1.1) satisfies*

$$f(e) = 0, f(x) = f(\sigma(x)) \text{ and } g(x) = g(\sigma(x)).$$

Proof. Setting $x = y = e$ in (1.1), we get $2f(e) + g(e)^2 = 0$. According to hypotheses (H2), we get $f(e) = 0$.

Putting $x = e$ in (1.1), we get $f(y) = f(\sigma(y))$. Replacing y by $\sigma(y)$ in (1.1), and noting that $f(y) = f(\sigma(y))$, we obtain

$$(2.17) \quad f(x + y) + f(x + \sigma(y)) = 2f(x) + 2f(y) + g(x)g(\sigma(y)).$$

Hence by (1.1), we have $g(x)g(y) = g(x)g(\sigma(y))$. Since the ring K has no zero divisors, we have

$$g(y) = g(\sigma(y)).$$

This completes the proof of the proposition. □

Remark 2. Without the second part of hypotheses (H2), Proposition 2.2 remains true if the condition $g(e) = 0$ is replaced with that $f(e) = 0$, i.e., we have the following result: *If $f(e) = 0$, then the general solution (f, g) of the equation (1.1) satisfies $g(e) = 0, f(x) = f(\sigma(x))$ and $g(x) = g(\sigma(x))$.*

Proposition 2.3. *Suppose (H2) holds. Then the general solution (f, g) of the equation (1.2) satisfies $f(e) = 0, f(x) = f(\sigma(x)), g(x) = g(\sigma(x))$ and*

$$[g(x + y) - g(x + \sigma(y))][g(x + y) + g(x + \sigma(y)) - 2g(x) - 2g(y)] = 0.$$

Proof. Putting $x = y = z = e$ in (1.2) and noting that $g(e) = 0$, we obtain $8f(e) = 0$. By hypotheses (H2), we obtain $f(e) = 0$.

Putting $y = z = e$ in (1.2), we get $f(x) = f(\sigma(x))$. Furthermore, if we put $z = e$ in (1.2), then the equation (1.2) yields the equation (1.1). Hence, similar to the proof of Proposition 2.2, we have $g(x) = g(\sigma(x))$.

Putting $y = \sigma(x), z = e$ in (1.2), we obtain

$$2f(2x) + 2f(x + \sigma(x)) = 8f(x) + 2g(x)^2.$$

Then

$$(2.18) \quad f(2x) + f(x + \sigma(x)) = 4f(x) + g(x)^2.$$

Letting $z = x + y$ in (1.2) and noting that $f(e) = 0$, we obtain

$$(2.19) \quad \begin{aligned} & f(2(x + y)) + f(x + \sigma(y) + x + y) \\ & \quad + f(x + y + \sigma(x) + \sigma(y)) + f(\sigma(x) + y + x + y) \\ & = 4f(x) + 4f(y) + 4f(x + y) + 2g(x)g(y) \\ & \quad + 2g(x)g(x + y) + 2g(y)g(x + z). \end{aligned}$$

Replacing x by $\sigma(x)$ in (2.19), we have

$$\begin{aligned} & f(2(\sigma(x) + y)) + f(\sigma(x) + \sigma(y) + \sigma(x) + y) \\ & \quad + f(\sigma(x) + y + x + \sigma(y)) + f(x + y + \sigma(x) + y) \end{aligned}$$

$$(2.20) \quad = 4f(x) + 4f(y) + 4f(\sigma(x) + y) + 2g(x)g(y) \\ + 2g(x)g(\sigma(x) + y) + 2g(y)g(\sigma(x) + y).$$

From (2.19) and (2.20), we get

$$(2.21) \quad [f(2(x+y)) + f(x+y+\sigma(x)+\sigma(y)) \\ - [f(2(\sigma(x)+y)) + f(\sigma(x)+y+x+\sigma(y))]] \\ = [4f(x+y) - 4f(\sigma(x)+y)] + 2g(x)[g(x+y) - g(\sigma(x)+y)] \\ + 2g(y)[g(x+y) - g(\sigma(x)+y)].$$

It follows from (2.17) that

$$g(x+y)^2 - g(\sigma(x)+y)^2 = 2[g(x)+g(y)][g(x+y) - g(\sigma(x)+y)],$$

i.e.,

$$[g(x+y) - g(\sigma(x)+y)][g(x+y) + g(\sigma(x)+y) - 2g(x) - 2g(y)] = 0.$$

This completes the proof of the proposition. \square

Proposition 2.4. *Suppose (H2s) holds. Then the equation (1.1) is equivalent to the equation (1.2).*

Proof. Setting $x = y = e$ in (1.5), we obtain $g(e) = 0$. Putting $x = e$ in (1.5), we get $g(y) = g(\sigma(y))$. And by the proof of Proposition 2.3, one can prove that the equation (1.2) implies the equation (1.1).

In order to prove that the equation (1.1) implies the equation (1.2), putting $x = y = e$ in (1.1), we get $f(e) = 0$. Putting $x = e$ in (1.1), we obtain $f(y) = f(\sigma(y))$. Replacing x by $x+z$ in (1.1), we get

$$(2.22) \quad f(x+y+z) + f(x+\sigma(y)+z) = 2f(x+z) + 2f(y) + g(x+z)g(y).$$

Replacing x by $\sigma(x)$ in (2.22), we obtain

$$(2.23) \quad f(\sigma(x)+y+z) + f(\sigma(x)+\sigma(y)+z) \\ = 2f(\sigma(x)+z) + 2f(y) + g(\sigma(x)+z)g(y).$$

It follows from (2.22) and (2.23) that

$$(2.24) \quad f(x+y+z) + f(x+\sigma(y)+z) \\ + f(\sigma(x)+y+z) + f(\sigma(x)+\sigma(y)+z) \\ = 2[f(x+z) + f(\sigma(x)+z)] + 4f(y) + [g(x+z) + g(\sigma(x)+z)]g(y).$$

Since

$$f(\sigma(x)+\sigma(y)+z) = f(x+y+\sigma(z)), \\ f(x+z) + f(\sigma(x)+z) = f(x+z) + f(x+\sigma(z)) \\ = 2f(x) + 2f(z) + g(x)g(z).$$

And by hypotheses (H2s) and $g(y) = g(\sigma(y))$, we have

$$g(x+z) + g(\sigma(x)+z) = g(x+z) + g(x+\sigma(z)) = 2g(x) + 2g(z).$$

Thus, we conclude that the equation (2.24) implies the equation (1.2), and the proof of the proposition is now complete. \square

3. Stability of the equation (1.1)

Given an operator T and a solution class $\{u\}$ with the property that $T(u) = 0$, when does $\|T(v)\| \leq \varepsilon$ for an $\varepsilon > 0$ imply that $\|u - v\| \leq \delta(\varepsilon)$ for some u and for some $\delta > 0$? This problem is called the stability of the functional transformation. A great deal of work has been done in connection with the ordinary and partial differential equations. If f is a function from a normed vector space into a Banach space and satisfies $\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$, Hyers [9] proved that there exists an additive function A such that $\|f(x) - A(x)\| \leq \varepsilon$. If $f(x)$ is a real continuous function of x over \mathbb{R} , and $|f(x+y) - f(x) - f(y)| \leq \varepsilon$, it was shown by Hyers and Ulam [12] that there exists a constant k such that $|f(x) - kx| \leq 2\varepsilon$. Taking these results into account, we say that the additive Cauchy equation $f(x + y) = f(x) + f(y)$ is stable in the sense of Hyers and Ulam. The stability of many other functional equations have been studied in the sense of Hyers-Ulam as well as Hyers-Ulam-Rassias after Hyers paper [9]. The interested reader should refer to the books [10, 13, 18, 21] and papers [1, 4, 7, 8, 11, 14–17] and references therein for an in depth account on the subject of stability of functional equations.

In this section, we discuss the Hyers-Ulam stability of the functional equation (1.1) in the case of $K = \mathbb{C}$.

Theorem 3.1. *Suppose that $f, g : G \rightarrow \mathbb{C}$ with $g(e) \neq 0$ satisfy the inequality*

$$(3.1) \quad |f(x + y) + f(x + \sigma(y)) - 2f(x) - 2f(y) - g(x)g(y)| \leq \delta$$

for all $x, y \in G$, where $\delta > 0$ is a constant. Then

$$(3.2) \quad |g(x) - g(e)| \leq \frac{2\delta}{|g(e)|}$$

for all $x \in G$, and there exists a unique solution $q : G \rightarrow \mathbb{C}$ of the equation (2.2) such that

$$(3.3) \quad |f(x) - q(x) - f(e)| \leq \frac{2\delta^2}{|g(e)|^2} + 2\delta$$

for all $x \in G$.

Proof. Let (f, g) be a solution of inequality (3.1). Define

$$(3.4) \quad \theta(x, y) = g(x)g(y) + 2f(x) + 2f(y) - f(x + y) - f(x + \sigma(y))$$

for all $x, y \in G$. It follows from (3.1) that

$$|\theta(x, y)| \leq \delta$$

for all $x, y \in G$. Putting $x = y = e$ in (3.4), we have $\theta(e, e) = g(e)^2 + 2f(e)$. Letting $y = e$ in (3.4), we obtain $\theta(x, e) = g(x)g(e) + 2f(e)$ for all $x \in G$. Hence, we get

$$(3.5) \quad \theta(x, e) - \theta(e, e) = g(e)(g(x) - g(e))$$

for all $x, y \in G$. Since $g(e) \neq 0$, it follows that

$$(3.6) \quad |g(x) - g(e)| \leq \frac{2\delta}{|g(e)|}$$

which establishes (3.2) for all $x \in G$. By (3.4), we get

$$\begin{aligned} & (f - f(e))(x + y) + (f - f(e))(x + \sigma(y)) \\ &= 2(f - f(e))(x) + 2(f - f(e))(y) + g(x)g(y) - g(x)g(e) \\ & \quad + \theta(x, e) - \theta(x, y) \\ &= 2(f - f(e))(x) + 2(f - f(e))(y) + g(x) \frac{\theta(y, e) - \theta(e, e)}{g(e)} \\ & \quad + \theta(x, e) - \theta(x, y). \end{aligned}$$

By using the inequality (3.6), we obtain

$$\left| g(x) \frac{\theta(y, e) - \theta(e, e)}{g(e)} \right| \leq 2\delta + \frac{4\delta^2}{|g(e)|^2}$$

for all $x \in G$, and hence, we have

$$\begin{aligned} & |(f - f(e))(x + y) + (f - f(e))(x + \sigma(y)) \\ & \quad - 2(f - f(e))(x) - 2(f - f(e))(y)| \\ & \leq \frac{4\delta^2}{|g(e)|^2} + 4\delta \end{aligned}$$

for all $x, y \in G$. By [3, Theorem 2.1], there exists a unique solution $q : G \rightarrow \mathbb{C}$ of the equation (2.2) such that

$$|f(x) - q(x) - f(e)| \leq \frac{2\delta^2}{|g(e)|^2} + 2\delta$$

for all $x \in G$. This completes the proof of the theorem. \square

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