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ON THE PROXIMAL POINT METHOD FOR AN INFINITE FAMILY OF EQUILIBRIUM PROBLEMS IN BANACH SPACES

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Abstract. In this paper, we study the convergence analysis of the sequences generated by the proximal point method for an infinite family of pseudo-monotone equilibrium problems in Banach spaces. We first prove the weak convergence of the generated sequence to a common solution of the infinite family of equilibrium problems with summable errors. Then, we show the strong convergence of the generated sequence to a common equilibrium point by some various additional assumptions. We also consider two variants for which we establish the strong convergence without any additional assumption. For both of them, each iteration consists of a proximal step followed by a computationally inexpensive step which ensures the strong convergence of the generated sequence. Also, for this two variants we are able to characterize the strong limit of the sequence: for the first variant it is the solution lying closest to an arbitrarily selected point, and for the second one it is the solution of the problem which lies closest to the initial iterate. Finally, we give a concrete example where the main results can be applied.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$. E^* will denote the topological dual of E. The duality mapping $J : E \to \mathcal{P}(E^*)$ is defined as

$$J(x) = \{v \in E^* : \langle x, v\rangle = \|x\|^2 = \|v\|^2\}.$$

Take a closed and convex set $K \subset E$. $f : K \times K \to \mathbb{R}$ is called a bifunction. An equilibrium problem for f and K, as briefly $\mathrm{EP}(f;K)$, consists of finding $x^* \in K$ such that

$$f(x^*, y) \geq 0, \quad \forall y \in K, \tag{1.1}$$

where x^* is called an equilibrium point. The set of all equilibrium points of (1.1) is denoted by $S(f;K)$.

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The equilibrium problem encompasses, among its particular cases, convex optimization problems, variational inequalities (monotone or otherwise), Nash equilibrium problems, and other problems of interest in many applications.

Equilibrium problems with monotone and pseudo-monotone bifunctions have been studied extensively in Hilbert, Banach, Hadamard as well as in topological vector spaces by many authors (e.g. [3,6,7,10,15,17–19]). Recently the second author and Iusem have studied equilibrium problems and vector equilibrium problems in Banach spaces (see [11,12]).

The study of equilibrium problems goes back to Ky Fan [8]. Subsequently, Brézis, Nirenberg and Stampacchia [5] studied (1.1) with a coercivity assumption on f. Blum and Oettli [4] proved the existence of solutions of (1.1) with monotonicity condition on f. Later, the equilibrium problems were studied extensively for existence of solutions (see for example [10] and references therein). Recently there are many researches devoted to approximation of solutions of an equilibrium problem (see, e.g. [14, 15, 18, 19, 22] and references therein). A popular method for approximating solutions of equilibrium problems is the proximal point algorithm which was first introduced by Martinet [21] and then systematically by Rockafellar [26] for maximal monotone operators. Moudafi [22] applied the following proximal point algorithm for solving monotone equilibrium problems;

$$f(x^k, y) + \lambda_{k-1}\langle y - x^k, x^k - x^{k-1}\rangle \geq 0, \quad \forall y \in K. \tag{1.2}$$

Iusem and Sosa [15] extended his results to pseudo-monotone bifunctions. They proved the existence of the sequence given by (1.2) in Hilbert spaces provided $\lambda_k > \theta$, where $\theta \geq 0$ is the undermonotonicity constant of f (see Proposition 3 of [15]). They also proved the weak convergence of the sequence generated by (1.2) to an equilibrium point of the pseudo-monotone bifunction f (see also [19] for some extensions and more results). Then Iusem and Nasri in [13] extended this result to Banach spaces, i.e., they proved the existence and uniqueness of the sequence given by the following process;

$$f(x^k, y) + \lambda_{k-1}\langle y - x^k, Jx^k - Jx^{k-1}\rangle \geq 0, \qquad \forall y \in K, \tag{1.3}$$

with an initial guess $x_0 \in E$, and also proved the weak convergence of the sequence generated by (1.3) to a solution of (1.1).

In [18], the proximal point algorithm for an infinite family of pseudo-monotone bifunctions was also studied by the authors in Hilbert spaces. They proved the weak and strong convergence of the sequence given by

$$f_k(x^k, y) + \lambda_{k-1}\langle y - x^k, x^k - x^{k-1}\rangle \geq 0, \qquad \forall y \in K$$

to a common solution of the infinite many pseudo-monotone bifunctions $\{f_k\}$ in Hilbert spaces.

In this paper, we first prove the weak convergence of the sequence generated by the proximal point method to a common solution of an infinite family of equilibrium problems in Banach spaces with summable errors. Then we show the strong convergence of the generated sequence to a common equilibrium point by some additional assumptions. To get the strong convergence without additional assumption, we must use some regularization methods like Halpern method (see [9,29]) or hybrid projection method (see [16,28]) that both of them have been applied in this paper. Therefore, we consider two variants for which we establish the strong convergence without any additional assumption.

In this paper we will consider the proximal point method which improves upon (1.3) in four senses:

a) We will deal with an infinite family of pseudo-monotone bifunctions, while [13] only considers a bifunction. The results among others extend the results of [18] from Hilbert spaces to Banach spaces.
b) We will study the strong convergence of the sequences generated by two regularization methods (Halpern and hybrid methods) without any additional assumption. For both of them, each iteration consists of a proximal step followed by a computationally inexpensive step which ensures the strong convergence of the generated sequence.
c) We also study the strong convergence of the generated sequence by imposing some various additional assumptions on the problem, while the method in [13] did not deal with the strong convergence. These results extend the corresponding results of [19] from Hilbert to Banach spaces.
d) Convergence analysis of the method usually requires the weak upper semicontinuity of $f(\cdot, y)$ for all $y \in K$, but we use a condition which is weaker than the weak upper semicontinuity of f with respect to the first argument, as mentioned in Section 3.

The paper is organized as follows. In Section 2, we introduce some preliminary materials related to the geometry of Banach spaces. In Section 3, we prove the weak convergence of the sequence generated by the proximal point method to a common solution of the infinite family of equilibrium problems with summable errors. In Section 4, we prove the strong convergence of the generated sequence by imposing some additional assumptions on the problem. In Section 5, we prove the strong convergence of the sequence generated by a Halpern type regularization of the proximal point method. In Section 6, we consider another variant of regularization method, i.e., hybrid proximal point algorithm, to establish the strong convergence of the sequence without any additional assumption. Finally in Section 7, we give a concrete example where the main results can be applied.

2. Preliminaries

A Banach space E is said to be *strictly convex* if $\left\|\frac{x+y}{2}\right\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be *uniformly convex* if for each $\varepsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and

$\|x-y\| \geq \varepsilon$, it holds that $\left\|\frac{x+y}{2}\right\| < 1-\delta$. It is known that uniformly convex Banach spaces are reflexive and strictly convex.

A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in S = \{z \in E : \|z\| = 1\}$. It is said to be *uniformly smooth* if the limit in (2.1) is attained uniformly for $x, y \in S$. It is well known that the spaces L^p $(1 < p < \infty)$ and the Sobolev spaces $W^{k,p}$ $(1 < p < \infty)$ are both uniformly convex and uniformly smooth.

It is well known that when E is smooth, the duality operator J is single valued. Let E be a smooth Banach space. We define $\phi : E \times E \to \mathbb{R}$ as

$$\phi(x,y) = \|x\|^2 - 2\langle x, J(y)\rangle + \|y\|^2 . \tag{2.2}$$

This function can be seen as a "distance-like" function, better conditioned than the square of the metric distance, namely $\|x-y\|^2$; see e.g. [1], [16] and [25].

It is elementary that

$$0 \leq (\|x\| - \|y\|)^2 \leq \phi(x,y) \tag{2.3}$$

for all $x, y \in E$. In Hilbert spaces, where the duality mapping J is the identity operator, it holds that $\phi(x,y) = \|x-y\|^2$. In the sequel, we will need the following three properties of ϕ, proved in [16].

Proposition 2.1. *Let E be a smooth and uniformly convex Banach space. Take two sequences $\{x^k\}$, $\{y^k\} \subset E$. If $\lim_{k\to\infty} \phi(x^k, y^k) = 0$ and either $\{x^k\}$ or $\{y^k\}$ is bounded, then $\lim_{k\to\infty} \left\|x^k - y^k\right\| = 0$.*

Proposition 2.2. *Let E be a reflexive, strictly convex and smooth Banach space. Take a nonempty, closed and convex set $C \subset E$. Then for all $x \in E$ there exists a unique $x_0 \in C$ such that*

$$\phi(x_0, x) = \inf\{\phi(z,x) : z \in C\}.$$

We define $P_C : E \to C$ by taking as $P_C(x)$ the unique $x_0 \in C$ given by Proposition 2.2. P_C is called the *generalized projection onto C*. When E is Hilbertian, P_C is just the metric, or orthogonal, projection onto C.

The third result taken from [16] is the following.

Proposition 2.3. *Consider a smooth Banach space E, and a closed and convex set $C \subset E$. Take $x \in E$, $x_0 \in C$. Then $x_0 = P_C(x)$ if and only if*

$$\langle z - x_0, J(x) - J(x_0)\rangle \leq 0$$

for all $z \in C$.

Throughout this paper we assume that E is a real Banach space which is uniformly convex and uniformly smooth and $K \subset E$ is a nonempty, closed and convex set, unless otherwise specified. We introduce some standard assumptions on a bifunction $f : K \times K \to \mathbb{R}$, which will be needed.

A_1: $f(x,x)=0$ for all $x\in K$.
A_2: $f(\cdot,y):K\to\mathbb{R}$ is upper semicontinuous for all $y\in K$.
A_3: $f(x,\cdot):K\to\mathbb{R}$ is convex and lower semicontinuous for all $x\in K$.
A_4: $f(x,y)+f(y,x)\le 0$, $\forall x,y\in K$ (f is called monotone).
A_4^*: Whenever $f(x,y)\ge 0$ with $x,y\in K$, it holds that $f(y,x)\le 0$ (f is called pseudo-monotone).
$A_4^\bullet$: There exists $\theta\ge 0$ such that $f(x,y)+f(y,x)\le \frac{\theta}{2}(\phi(x,y)+\phi(y,x))$ for all $x,y\in K$ (f is called θ-undermonotone and θ is the undermonotonicity constant of f).

Let $\{f_k\}$ be a sequence of pseudo-monotone and θ-undermonotone bifunctions. We study the convergence analysis of the sequence generated by the proximal point method which is generated by the following process;

$$f_k(x^k,y)+\lambda_{k-1}\langle y-x^k, Jx^k-Jx^{k-1}\rangle\ge 0,\qquad \forall y\in K, \tag{2.4}$$

where $\{\lambda_k\}$ is a bounded sequence such that $\lambda_{k-1}\in(\theta_k,+\infty)$ for all $k\in\mathbb{N}$ and $x^0\in E$. In order to prove the existence and uniqueness of the sequence $\{x^k\}$ satisfying (2.4), consider the bifunction $\tilde f$ which is defined as

$$\tilde f(x,y)=f(x,y)+\lambda\langle y-x, Jx-J\bar x\rangle, \tag{2.5}$$

where $\bar x\in E$ and f is a bifunction that satisfies A_1, A_2, A_3 and $A_4^\bullet$, and $\lambda>\theta$.

Proposition 2.4. *If f satisfies A_1, A_2, A_3, $A_4^\bullet$ and $\lambda>\theta$, then $EP(\tilde f,K)$ has a unique solution.*

Proof. See [13]. □

Proposition 2.4 ensures the existence and uniqueness of the sequence generated by (2.4). In fact each step of the proximal point algorithm (2.4) is solving an equilibrium problem for $\tilde f$ by replacing f with f_k and $\bar x$ with x_{k-1}.

We end this section with a notational comment: when $\{x^k\}$ is a sequence in E, we denote the strong convergence of $\{x^k\}$ to $x\in E$ by $x^k\to x$ and the weak convergence by $x^k\rightharpoonup x$.

3. Weak convergence

In this section, we study the weak convergence of the sequence generated by the proximal point method for an infinite family of pseudo-monotone equilibrium problems. Since the generated sequence in general weakly convergent, in order to establish optimality of its weak cluster points, we need to assume something akin to the weak upper semicontinuity of f in its first argument. We will use a slightly weaker assumption, which has been introduced in [18]. We recall it in the following:

(3.1)
$$\begin{cases}\text{For each arbitrary sequence } \{z^k\} \text{ and each subsequence } \{z^{k_n}\} \text{ of } \{z^k\},\\ \text{if } z^{k_n}\rightharpoonup z \text{ and } \limsup_{n\to\infty} f_{k_n}(z^{k_n},y)\ge 0,\ \forall y\in K,\ \text{then } z\in\bigcap_k S(f_k;K).\end{cases}$$

When $f_k \equiv f$, the above condition is reduced to the following condition which is strictly weaker than the weak upper semicontinuity of f with respect to the first argument (see [18]).

(3.2)
$$\begin{cases} \text{For each arbitrary sequence } \{z^k\} \subset K \text{ and } z \in K \text{ such that } z^k \rightharpoonup z, \\ \text{and } \limsup_{k\to\infty} f(z^k, y) \geq 0, \ \forall y \in K, \text{ we have } z \in S(f, K). \end{cases}$$

In the following example, we give a family of bifunctions which satisfies (3.1).

Example 3.1. Let $E = \ell^p$ for $1 < p < \infty$, $K = \{\xi = (\xi_1, \xi_2, \xi_3, \ldots) \in \ell^p : \xi_i \geq 0, \forall i \in \mathbb{N}\}$ and $f_k(x, y) = \frac{k}{k+1}(y_1 - x_1)\sum_{i=1}^k (x_i)^p$ for all $k \in \mathbb{N}$. We show that the sequence $\{f_k\}$ satisfies condition (3.1). If $z^k = (z_1^k, z_2^k, \ldots) \rightharpoonup z = (z_1, z_2, \ldots)$ is an arbitrary sequence and $\limsup_{k\to\infty} f_k(z^k, y) \geq 0$ for all $y \in K$, then we have $\limsup_{k\to\infty} \frac{k}{k+1}(y_1 - z_1^k)\sum_{i=1}^k (z_i^k)^p \geq 0$. Since $\lim_{k\to\infty} \frac{k}{k+1}(y_1 - z_1^k) = y_1 - z_1$, we get

$$\limsup_{k\to\infty} f_k(z^k, y) = (y_1 - z_1)\sum_{i=1}^{\infty}(z_i^k)^p \geq 0.$$

Therefore $y_1 \geq z_1$ and we have $f_k(z, y) \geq 0$ for all $y \in K$ and all $k \in \mathbb{N}$, i.e., $\{f_k\}$ satisfies condition (3.1).

We perform a minor modification on Example 3.1 to exhibit a family of bifunctions which satisfies (3.1) but are not weakly upper semicontinuous with respect to the first argument.

Example 3.2. Suppose that E and K are defined similar to Example 3.1 and define $f_k(x, y) = \frac{k}{k+1}(y_1 - x_1)\sum_{i=1}^{\infty}(x_i)^p$ for all $k \in \mathbb{N}$. We show that $f_k(\cdot, y)$ is not weakly upper semicontinuous for all k, but $\{f_k\}$ satisfies condition (3.1). Take $x^k = (x_1^k, x_2^k, x_3^k, \cdots) = (\underbrace{0, \ldots, 0}_{k}, 1, \frac{1}{2}, \frac{1}{3}, \ldots)$. It is easy to see that $x^k \rightharpoonup x^* = (0, 0, 0, \ldots) = 0$ and $x^* \in \bigcap_k S(f_k; K)$. Obviously there is a $y \in K$ such that $\limsup_{k\to\infty} f_n(x^k, y) > 0 = f_n(x^*, y)$ for all $n \in \mathbb{N}$. Therefore $f_k(\cdot, y)$ is not weakly upper semicontinuous for all $k \in \mathbb{N}$. Now, similar to Example 3.1, we can show that $\{f_k\}$ satisfies condition (3.1).

It is clear that for one bifunction f, if f is weakly upper semicontinuous with respect to the first argument, then it satisfies (3.2). It is well known that a concave and upper semicontinuous function is always weakly upper semicontinuous. Thus a bifunction f such that $f(\cdot, y)$ is concave and upper semicontinuous satisfies (3.2). Also, we mention that (3.2) is only relevant in the infinite dimensional case, because in the finite dimensional case it follows from the assumption A_2.

In the sequel, let $f_k : K \times K \to \mathbb{R}$, $k = 1, 2, \ldots$, be a sequence of θ_k-undermonotone bifunctions and $\{\lambda_k\}$ be a bounded sequence such that $\lambda_{k-1} \in (\theta_k, +\infty)$ for all $k \in \mathbb{N}$ and $x^0 \in E$. In order to numerical purposes, it is useful to consider the inexact version of the proximal point algorithm for the infinite family of pseudo-monotone bifunctions $\{f_k\}$, which is formulated as

$$f_k(x^k, y) + \lambda_{k-1}\langle y - x^k, Jx^k - Jx^{k-1}\rangle \geq -\varepsilon_{k-1}, \qquad \forall y \in K, \tag{3.3}$$

where the sequence $\{\varepsilon_k\}$ is nonnegative and $\sum_{k=0}^{\infty} \frac{\varepsilon_k}{\lambda_k} < \infty$. We study the weak convergence of each sequence which satisfies (3.3) to an element of $\bigcap_k S(f_k; K)$ such that $\{f_k\}$ satisfies A_1, A_2, A_3, A_4^*, $A_4^\bullet$, (3.1) and $\bigcap_k S(f_k; K) \neq \varnothing$. Weak convergence of the sequence given by (3.3) is studied in Section 6 of [18] in Hilbert spaces.

Lemma 3.3. *Suppose that $\{f_k\}$ is a sequence of bifunctions, which satisfies A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and $\bigcap_k S(f_k; K) \neq \varnothing$. If the sequence $\{x^k\}$ is generated by* (3.3)*, then $\lim_{k\to\infty} \phi(x^*, x^k)$ exists for all $x^* \in \bigcap_k S(f_k; K)$ and hence $\{x^k\}$ is bounded. In addition $\lim_{k\to\infty} \|x^k - x^{k+1}\| = 0$.*

Proof. Take any $x^* \in \bigcap_k S(f_k; K)$. We know that

$$f_k(x^k, x^*) + \lambda_{k-1}\langle x^* - x^k, Jx^k - Jx^{k-1}\rangle \geq -\varepsilon_{k-1}, \quad \forall k \in \mathbb{N}.$$

By A_4^*, we have $f_k(x^k, x^*) \leq 0$ for all $k \in \mathbb{N}$. Therefore, we get

$$\begin{aligned} \frac{-\varepsilon_{k-1}}{\lambda_{k-1}} &\leq \langle x^* - x^k, Jx^k - Jx^{k-1}\rangle \\ &= \frac{1}{2}(\phi(x^*, x^{k-1}) - \phi(x^*, x^k) - \phi(x^k, x^{k-1})). \end{aligned} \tag{3.4}$$

Therefore for $m, n \in \mathbb{N}$, we have

$$\phi(x^*, x^{m+n+1}) \leq \phi(x^*, x^{m+n}) + \frac{2\varepsilon_{m+n}}{\lambda_{m+n}} \leq \cdots \leq \phi(x^*, x^m) + 2\sum_{k=m}^{m+n} \frac{\varepsilon_k}{\lambda_k}. \tag{3.5}$$

Now, since $\sum_{k=0}^{\infty} \frac{\varepsilon_k}{\lambda_k} < \infty$, (3.5) implies that

$$\limsup_{n\to\infty} \phi(x^*, x^n) \leq \liminf_{m\to\infty} \phi(x^*, x^m).$$

Therefore $\lim_{k\to\infty} \phi(x^*, x^k)$ exists. On the other hand, since $(\|x^*\| - \|x^k\|)^2 \leq \phi(x^*, x^k)$, hence $\{x^k\}$ is bounded. Also we may conclude from (3.4) that $\lim_{k\to\infty} \phi(x^k, x^{k-1}) = 0$. Now, Proposition 2.1 implies that

$$\lim_{k\to\infty} \|x^{k-1} - x^k\| = 0.$$

□

In order to prove the uniqueness of the weak limit point of the sequence in the following theorem, we need the following condition on a Banach space E, i.e., we assume that if $\{y^k\}$ and $\{z^k\}$ are arbitrary sequences in K that converge weakly to y and z, respectively and $y \neq z$, then

$$\liminf_{k\to\infty} |\langle y - z, Jy^k - Jz^k\rangle| > 0. \tag{3.6}$$

For non-Hilbertian examples of such spaces, we can consider ℓ^p spaces for $1 < p < \infty$ that satisfy in the above condition. It is valuable to mention that we need this condition on E just in the following theorem and we will never use it in the sequel, i.e., when we prove the strong convergence theorems in this paper we do not need the above condition.

Theorem 3.4. *Suppose that* $\{f_k\}$ *is a sequence of bifunctions, which satisfy* A_1, A_2, A_3, A_4^*, $A_4^\bullet$ *and* $\bigcap_k S(f_k;K) \neq \varnothing$. *If the sequence* $\{x^k\}$ *is generated by* (3.3) *and the sequence* $\{f_k\}$ *satisfies* (3.1), *then*

(i) *all cluster points of* $\{x^k\}$ *belong to* $\bigcap_k S(f_k;K)$,
(ii) *if in addition either* E *satisfies* (3.6) *or the problem has a unique solution, then the whole sequence* $\{x^k\}$ *is weakly convergent to an element of* $\bigcap_k S(f_k;K)$.

Proof. (i) From Lemma 3.3 we have

$$\lim_{k\to\infty} \|x^{k-1} - x^k\| = 0. \tag{3.7}$$

Uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, we get from (3.7),

$$\lim_{k\to\infty} \|Jx^{k-1} - Jx^k\| = 0. \tag{3.8}$$

Now, let $y \in K$ be arbitrary and fixed. In turn for all $k \in \mathbb{N}$, we have

$$\begin{aligned} -\varepsilon_{k-1} &\le f_k(x^k, y) + \lambda_{k-1}\langle y - x^k, Jx^k - Jx^{k-1}\rangle \\ &\le f_k(x^k, y) + \lambda_{k-1}\|y - x^k\|\|Jx^k - Jx^{k-1}\|. \end{aligned} \tag{3.9}$$

Since $\{x^k\}$ is bounded by Lemma 3.3, take any subsequence $\{x^{k_n}\}$ and $p \in K$ such that $x^{k_n} \rightharpoonup p$. Note that since $\{\lambda_k\}$ is a bounded sequence and $\lim_{k\to\infty} \varepsilon_k = 0$, by replacing k by k_n and then taking limit in (3.9), we have

$$\limsup_{n\to\infty} f_{k_n}(x^{k_n}, y) \ge 0, \qquad \forall y \in K. \tag{3.10}$$

Now, (3.1) implies that $p \in \bigcap_k S(f_k;K)$.

(ii) When the problem has a unique solution, the result is trivial. It remains to be proved that there exists only one weak cluster point of $\{x^k\}$ whenever E satisfies (3.6). Let q be an other weak cluster point of $\{x^k\}$. Then there exists subsequence $\{x^{k_j}\}$ such that $x^{k_j} \rightharpoonup q$. We have already proved that q is the element of $\bigcap_k S(f_k;K)$, also $\lim_{k\to\infty} \phi(p, x^k)$ and $\lim_{k\to\infty} \phi(q, x^k)$ exist. Note that

$$\begin{aligned} 2\langle p - q, Jx^{k_n} - Jx^{k_j}\rangle &= 2\langle p, Jx^{k_n}\rangle - 2\langle q, Jx^{k_n}\rangle - 2\langle p, Jx^{k_j}\rangle + 2\langle q, Jx^{k_j}\rangle \\ &= -\phi(p, x^{k_n}) + \phi(q, x^{k_n}) + \phi(p, x^{k_j}) - \phi(q, x^{k_j}). \end{aligned}$$

(In the second equality the relation (2.2) is used.) Taking limit when $n \to \infty$ and then when $j \to \infty$, we obtain $p = q$, i.e., $\{x^k\}$ converges weakly to a point of $\bigcap_k S(f_k;K)$. □

4. Strong convergence

In this section, we study the strong convergence of the sequence generated by (3.3) to an element of $\bigcap_k S(f_k;K)$ with some various additional assumptions on the problem.

In the following theorem we suppose that the interior of the solution set is nonempty instead of $\bigcap_k S(f_k;K) \neq \varnothing$ in Theorem 3.4 and as we mentioned before, we do not need the condition (3.6) on a Banach space E.

Theorem 4.1. *Suppose that $\{f_k\}$ satisfy A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and*

$$\operatorname{int}(\bigcap_k S(f_k;K)) \neq \varnothing.$$

If $\{x^k\}$ is generated by (3.3) *and $\{f_k\}$ satisfies* (3.1)*, then $\{x^k\}$ converges strongly to an element of $\bigcap_k S(f_k;K)$.*

Proof. Since $\operatorname{int}(\bigcap_k S(f_k;K)) \neq \varnothing$, there exist $r > 0$ and $x^* \in \operatorname{int}(\bigcap_k S(f_k;K))$ such that $\bar{B}_r(x^*) \subset \operatorname{int}(\bigcap_k S(f_k;K))$. Note that since E is uniformly smooth and uniformly convex, E is reflexive and E^* is uniformly smooth, therefore the duality mapping J is single valued, one-to-one and surjective. Also the duality mapping $J^{-1} : E^* \to E$ is single valued, one-to-one, onto and uniformly norm to norm continuous on each bounded subset of E^*. Now, if $\|Jx^k - Jx^{k-1}\| \neq 0$ by letting $\tilde{x} = x^* - r\frac{J^{-1}(Jx^k - Jx^{k-1})}{\|Jx^k - Jx^{k-1}\|}$, we have

$$f_k(x^k, \tilde{x}) + \lambda_{k-1}\langle \tilde{x} - x^k, Jx^k - Jx^{k-1}\rangle \geq -\varepsilon_{k-1}, \quad \forall k \in \mathbb{N}.$$

Since $\tilde{x} \in \bigcap_k S(f_k;K)$, pseudo-monotonicity of f_k implies that $f_k(x^k, \tilde{x}) \leq 0$ for all $k \in \mathbb{N}$. Therefore, we have

$$\langle \tilde{x} - x^k, Jx^k - Jx^{k-1}\rangle \geq \frac{-\varepsilon_{k-1}}{\lambda_{k-1}}$$

and hence

$$\begin{aligned} \frac{-\varepsilon_{k-1}}{\lambda_{k-1}} + r\|Jx^k - Jx^{k-1}\| &= \frac{-\varepsilon_{k-1}}{\lambda_{k-1}} + r\langle \frac{J^{-1}(Jx^k - Jx^{k-1})}{\|Jx^k - Jx^{k-1}\|}, Jx^k - Jx^{k-1}\rangle \\ &\leq \langle x^* - x^k, Jx^k - Jx^{k-1}\rangle \\ &= \frac{1}{2}(\phi(x^*, x^{k-1}) - \phi(x^*, x^k) - \phi(x^k, x^{k-1})). \end{aligned} \tag{4.1}$$

Also, if $\|Jx^k - Jx^{k-1}\| = 0$, then (4.1) is satisfied by (3.4). Summing up (4.1) from $k = 1$ to $k = n$, we get

$$2r\sum_{k=1}^{n} \|Jx^k - Jx^{k-1}\| \leq \phi(x^*, x^0) - \phi(x^*, x^n) + 2\sum_{k=1}^{n} \frac{\varepsilon_{k-1}}{\lambda_{k-1}}. \tag{4.2}$$

Since $\sum_{k=0}^{\infty} \frac{\varepsilon_k}{\lambda_k} < \infty$, therefore we may conclude from (4.2) that

$$\sum_{k=1}^{\infty} \|Jx^k - Jx^{k-1}\| < \infty.$$

It follows that $\{Jx^k\}$ converges strongly to an element in E^*. Since the duality mapping J^{-1} is uniformly norm to norm continuous on each bounded subset of E^*, hence $\{x^k\}$ converges strongly to an element in K. On the other hand, since all cluster points of $\{x^k\}$ belong to $\bigcap_k S(f_k;K)$ by Theorem 3.4(i), $\{x^k\}$ converges strongly to an element of $\bigcap_k S(f_k;K)$. □

Definition 4.2. A bifunction f is called strongly monotone if there exists $\alpha > 0$ such that $f(x,y) + f(y,x) \leq -\alpha\|x-y\|^2$ for all $x,y \in K$.

Also, a bifunction f is called strongly pseudo-monotone if there exists $\beta > 0$ such that whenever $f(x,y) \geq 0$, then $f(y,x) \leq -\beta\|x-y\|^2$ for all $x,y \in K$.

Theorem 4.3. *Suppose that the assumptions of Theorem 3.4 are satisfied. If each of the following conditions is satisfied*:

(i) f_k *is strongly pseudo-monotone except for a finite number of* $\{f_k\}$*, and* $\liminf_{k\to\infty} \beta_k > 0$*, where* β_k *is dependent on* f_k *by the definition of the strongly pseudo-monotone bifunction,*

(ii) $f_k(x,\cdot)$ *is strongly convex for all* $x \in K$ *and* $k \in \mathbb{N}$ *except for a finite number of* $\{f_k\}$*,*

(iii) $f_k(\cdot,y)$ *is strongly concave for all* $y \in K$ *and* $k \in \mathbb{N}$ *except for a finite number of* $\{f_k\}$*,*

then the sequence $\{x^k\}$ *generated by* (3.3) *is strongly convergent to an element of* $\bigcap_k S(f_k;K)$.

Proof. Note that all cluster points of $\{x^k\}$ belong to $\bigcap_k S(f_k;K)$ by Theorem 3.4(i). Now, if $\{x^{k_n}\}$ and $\{x^{k_j}\}$ are arbitrary subsequences of $\{x^k\}$ that converge strongly to p and q, respectively, then

$$\begin{aligned}2\langle p-q, Jx^{k_n} - Jx^{k_j}\rangle &= 2\langle p, Jx^{k_n}\rangle - 2\langle q, Jx^{k_n}\rangle - 2\langle p, Jx^{k_j}\rangle + 2\langle q, Jx^{k_j}\rangle \\ &= -\phi(p,x^{k_n}) + \phi(q,x^{k_n}) + \phi(p,x^{k_j}) - \phi(q,x^{k_j}).\end{aligned}$$

Since $\lim_{k\to\infty}\phi(p,x^k)$ and $\lim_{k\to\infty}\phi(q,x^k)$ exist and uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, taking limit when $n \to \infty$ and then when $j \to \infty$, we obtain $p = q$, i.e., $\{x^k\}$ converges strongly to a point of $\bigcap_k S(f_k;K)$.

Therefore, in each item, it remains to be proved that if $x^{k_n} \rightharpoonup x^*$, then $x^{k_n} \to x^*$. Hence we suppose that $x^{k_n} \rightharpoonup x^*$. Note that we have

$$\begin{aligned}-\varepsilon_{k_n-1} &\leq f_{k_n}(x^{k_n},x^*) + \lambda_{k_n-1}\langle x^* - x^{k_n}, Jx^{k_n} - Jx^{k_n-1}\rangle \\ &\leq f_{k_n}(x^{k_n},x^*) + \lambda_{k_n-1}\|x^*-x^{k_n}\|\|Jx^{k_n} - Jx^{k_n-1}\|.\end{aligned} \tag{4.3}$$

From Lemma 3.3 we have

$$\lim_{n\to\infty}\|x^{k_n-1} - x^{k_n}\| = 0. \tag{4.4}$$

Since J is uniform norm-to-norm continuous on each bounded set of E. Therefore, we get from (4.4),

$$\lim_{n\to\infty}\|Jx^{k_n-1} - Jx^{k_n}\| = 0. \tag{4.5}$$

In the sequel;

(i) Since $f_{k_n}(x^*, x^{k_n}) \geq 0$, by the definition of the strongly pseudo-monotone bifunction, there is a $\beta_{k_n} > 0$ such that, $f_{k_n}(x^{k_n}, x^*) \leq -\beta_{k_n}\|x^{k_n} - x^*\|^2$ for sufficiently large $n \in \mathbb{N}$. Now, by taking liminf from (4.3) and $f_{k_n}(x^{k_n}, x^*) \leq -\beta_{k_n}\|x^{k_n} - x^*\|^2$, we get

$$\begin{aligned} 0 &\leq \liminf_{n\to\infty} f_{k_n}(x^{k_n}, x^*) \\ &\leq \liminf_{n\to\infty}(-\beta_{k_n}\|x^{k_n} - x^*\|^2) \\ &\leq -\liminf_{n\to\infty} \beta_{k_n} \limsup_{n\to\infty} \|x^{k_n} - x^*\|^2 \end{aligned}$$

and hence we deduce that $\limsup_{n\to\infty} \|x^{k_n} - x^*\|^2 \leq 0$, i.e., $x^{k_n} \to x^*$.

(ii) Let $\lambda \in (0,1)$ and set $w^{k_n} = \lambda x^{k_n} + (1-\lambda)x^*$ for all $n \in \mathbb{N}$. Since $f_{k_n}(x^*, \cdot)$ is strongly convex for sufficiently large $n \in \mathbb{N}$, we have

$$\begin{aligned} -\varepsilon_{k_n-1} &\leq f_{k_n}(x^{k_n}, w^{k_n}) + \lambda_{k_n-1}\langle w^{k_n} - x^{k_n}, Jx^{k_n} - Jx^{k_n-1}\rangle \\ &\leq \lambda f_{k_n}(x^{k_n}, x^{k_n}) + (1-\lambda)f_{k_n}(x^{k_n}, x^*) - \lambda(1-\lambda)\|x^{k_n} - x^*\|^2 \\ &\quad + (1-\lambda)\lambda_{k_n-1}\langle x^* - x^{k_n}, Jx^{k_n} - Jx^{k_n-1}\rangle. \end{aligned}$$

Hence, we have

$$\begin{aligned} -\varepsilon_{k_n-1} + \lambda(1-\lambda)\|x^{k_n} - x^*\|^2 &\leq (1-\lambda)\lambda_{k-1}\langle x^* - x^{k_n}, Jx^{k_n} - Jx^{k_n-1}\rangle \\ &\leq (1-\lambda)\lambda_{k-1}\|x^* - x^{k_n}\|\|Jx^{k_n} - Jx^{k_n-1}\|. \end{aligned}$$

Since $\lim_{k\to\infty} \varepsilon_k = 0$, boundedness of $\{x^{k_n}\}$ and (4.5) imply that $\|x^{k_n} - x^*\| \to 0$.

(iii) Let $\lambda \in (0,1)$ and set $w^{k_n} = \lambda x^{k_n} + (1-\lambda)x^*$ for all $n \in \mathbb{N}$. Then since $f_{k_n}(\cdot, x^*)$ is strongly concave for sufficiently large $n \in \mathbb{N}$, we have

$$\lambda f_{k_n}(x^{k_n}, x^*) + (1-\lambda)f_{k_n}(x^*, x^*) + \lambda(1-\lambda)\|x^{k_n} - x^*\|^2 \leq f_{k_n}(w^{k_n}, x^*) \leq 0.$$

Therefore, we get $f_{k_n}(x^{k_n}, x^*) \leq -(1-\lambda)\|x^{k_n} - x^*\|^2$. Next, similar to (i), we have

$$0 \leq \liminf_{n\to\infty} f_{k_n}(x^{k_n}, x^*) \leq -(1-\lambda)\limsup_{n\to\infty} \|x^{k_n} - x^*\|^2$$

and hence $x^{k_n} \to x^*$. □

5. Halpern regularization method

This section and the next one are devoted to two regularization methods ensure the strong convergence. First in this section we consider the Halpern type regularization of the proximal point algorithm for an infinite family $\{f_k\}$ of bifunctions that satisfy A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and $\bigcap_k S(f_k; K) \neq \varnothing$ (see [9], [29]). Consider the sequence $\{x^k\}$ given by the following process

$$(5.1) \qquad \begin{cases} f_k(y^k, y) + \lambda_k\langle y - y^k, Jy^k - Jx^k\rangle \geq 0, & \forall y \in K, \\ x^{k+1} = J^{-1}(\alpha_k Ju + (1-\alpha_k)Jy^k), \end{cases}$$

where $x^1, u \in E$, $\{\lambda_k\}$ is a bounded sequence such that $\lambda_k \in (\theta_k, \infty)$, the sequence $\alpha_k \in (0,1)$ satisfies $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$. We prove the strong convergence of the sequence generated by (5.1) to the generalized projection of u on $\bigcap_k S(f_k; K)$.

In order to establish the strong convergence of the sequence generated by (5.1), we need two intermediate results. The first one establishes an elementary property of real sequences.

Lemma 5.1. *Consider sequences $\{s_k\} \subset [0,\infty), \{t_k\} \subset \mathbb{R}$ and $\{\gamma_k\} \subset (0,1)$ satisfying $\sum_{k=1}^{\infty} \gamma_k = \infty$. Suppose that $s_{k+1} \leq (1-\gamma_k)s_k + \gamma_k t_k$ for all $k \geq 1$. If $\limsup_{m\to\infty} t_{k_m} \leq 0$ for all subsequence $\{s_{k_m}\}$ of $\{s_k\}$ satisfying $\liminf_{m\to\infty}(s_{k_m+1} - s_{k_m}) \geq 0$, then $\lim_{k\to\infty} s_k = 0$.*

Proof. See [27]. □

Next we make use of the map $V : E \times E^* \to \mathbb{R}$ studied in [1] and defined as

$$V(x,v) = \|x\|^2 - 2\langle x, v\rangle + \|v\|^2. \tag{5.2}$$

Note that $V(x,v) = \phi(x, J^{-1}v)$ for all $(x,v) \in E \times E^*$.

Lemma 5.2 ([20])**.** *Let E be a strictly convex, smooth, and reflexive Banach space, and let V be as in* (5.2)*. Then*

$$V(x,v) \leq V(x, v+w) - 2\langle J^{-1}(v) - x, w\rangle \tag{5.3}$$

for all $x \in E$ and $v, w \in E^*$.

Theorem 5.3. *Assume that f_k for all k satisfy A_1, A_2, A_3, A_4^* and $A_4^\bullet$. Suppose that the sequence $\{x^k\}$ is generated by* (5.1) *such that $\{f_k\}$ satisfies* (3.1)*. If $\bigcap_k S(f_k;K) \neq \varnothing$ and the sequence $\alpha_k \in (0,1)$ satisfies $\lim_{k\to\infty} \alpha_k = 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$, then $\{x^k\}$ converges strongly to $P_{\bigcap_k S(f_k;K)} u$.*

Proof. Since $\bigcap_k S(f_k;K)$ is closed and convex, we set $x^* := P_{\bigcap_k S(f_k;K)} u$. Note that

$$f_k(y^k, x^*) + \lambda_k \langle x^* - y^k, Jy^k - Jx^k\rangle \geq 0.$$

Since $x^* \in \bigcap_k S(f_k;K)$, hence A_4^* implies that $f_k(y^k, x^*) \leq 0$ for all $k \in \mathbb{N}$. Therefore we obtain

$$0 \leq \langle x^* - y^k, Jy^k - Jx^k\rangle = \frac{1}{2}(\phi(x^*, x^k) - \phi(x^*, y^k) - \phi(y^k, x^k)). \tag{5.4}$$

Therefore, we have

$$\phi(x^*, y^k) \leq \phi(x^*, x^k). \tag{5.5}$$

By (5.5) and (5.1), we have

$$\begin{aligned}
\phi(x^*, x^{k+1}) &= \phi(x^*, J^{-1}(\alpha_k Ju + (1-\alpha_k)Jy^k)) \\
&= V(x^*, \alpha_k Ju + (1-\alpha_k)Jy^k) \\
&\leq \alpha_k V(x^*, Ju) + (1-\alpha_k)V(x^*, Jy^k) \\
&= \alpha_k \phi(x^*, u) + (1-\alpha_k)\phi(x^*, y^k)
\end{aligned}$$

$$\leq \alpha_k \phi(x^*, u) + (1-\alpha_k)\phi(x^*, x^k)$$
$$\leq \max\{\phi(x^*, u), \phi(x^*, x^k)\} \leq \cdots \leq \max\{\phi(x^*, u), \phi(x^*, x^1)\},$$

which implies that $\{x^k\}$ is bounded by virtue of (2.3). Also $\{y^k\}$ is bounded by (5.5). On the other hand, by Lemma 5.2, we have

$$\begin{aligned}
\phi(x^*, x^{k+1}) &= V(x^*, \alpha_k Ju + (1-\alpha_k)Jy^k) \\
&\leq V(x^*, \alpha_k Ju + (1-\alpha_k)Jy^k - \alpha_k(Ju - Jx^*)) \\
&\quad - 2\langle J^{-1}(\alpha_k Ju + (1-\alpha_k)Jy^k) - x^*, -\alpha_k(Ju - Jx^*)\rangle \\
&= V(x^*, (1-\alpha_k)Jy^k + \alpha_k Jx^*) + 2\langle x^{k+1} - x^*, \alpha_k(Ju - Jx^*)\rangle \\
&\leq (1-\alpha_k)V(x^*, Jy^k) + \alpha_k V(x^*, Jx^*) + 2\alpha_k\langle x^{k+1} - x^*, Ju - Jx^*\rangle \\
&= (1-\alpha_k)\phi(x^*, y^k) + 2\alpha_k\langle x^{k+1} - x^*, Ju - Jx^*\rangle \\
&\leq (1-\alpha_k)\phi(x^*, x^k) + 2\alpha_k\langle x^{k+1} - x^*, Ju - Jx^*\rangle.
\end{aligned}$$

In the sequel, we show $\phi(x^*, x^k) \to 0$. By Lemma 5.1, it suffices to show that

$$\limsup_{m\to\infty}\langle x^{k_m+1} - x^*, Ju - Jx^*\rangle \leq 0 \tag{5.6}$$

for every subsequence $\{\phi(x^*, x^{k_m})\}$ of $\{\phi(x^*, x^k)\}$ satisfying

$$\liminf_{m\to\infty}(\phi(x^*, x^{k_m+1}) - \phi(x^*, x^{k_m})) \geq 0.$$

Consider such a subsequence. We have

$$\begin{aligned}
0 &\leq \liminf_{m\to\infty}(\phi(x^*, x^{k_m+1}) - \phi(x^*, x^{k_m})) \\
&= \liminf_{m\to\infty}(V(x^*, \alpha_{k_m} Ju + (1-\alpha_{k_m})Jy^{k_m}) - \phi(x^*, x^{k_m})) \\
&\leq \liminf_{m\to\infty}(\alpha_{k_m} V(x^*, Ju) + (1-\alpha_{k_m})V(x^*, Jy^{k_m}) - \phi(x^*, x^{k_m})) \\
&= \liminf_{m\to\infty}(\alpha_{k_m}\phi(x^*, u) + (1-\alpha_{k_m})\phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) \\
&= \liminf_{m\to\infty}(\alpha_{k_m}(\phi(x^*, u) - \phi(x^*, y^{k_m})) + \phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) \\
&\leq \limsup_{m\to\infty}\alpha_{k_m}(\phi(x^*, u) - \phi(x^*, y^{k_m})) + \liminf_{m\to\infty}(\phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) \\
&= \liminf_{m\to\infty}(\phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) \\
&\leq \limsup_{m\to\infty}(\phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) \leq 0.
\end{aligned}$$

This follows

$$\lim_{m\to\infty}(\phi(x^*, y^{k_m}) - \phi(x^*, x^{k_m})) = 0. \tag{5.7}$$

Combining (5.4) and (5.7), we get

$$\lim_{m\to\infty}\phi(y^{k_m}, x^{k_m}) = 0. \tag{5.8}$$

In the sequel, by Proposition 2.1, we have

$$\lim_{m\to\infty}\|y^{k_m}-x^{k_m}\|=0. \tag{5.9}$$

On the other hand, there exist a subsequence $\{x^{k_{m_t}}\}$ of $\{x^{k_m}\}$ and $p\in E$ such that $x^{k_{m_t}}\rightharpoonup p$ and

$$\begin{aligned}\limsup_{m\to\infty}\langle x^{k_m}-x^*,Ju-Jx^*\rangle &= \lim_{t\to\infty}\langle x^{k_{m_t}}-x^*,Ju-Jx^*\rangle\\ &=\langle p-x^*,Ju-Jx^*\rangle.\end{aligned} \tag{5.10}$$

Note that

$$\begin{aligned}\phi(y^{k_m},x^{k_m+1}) &= V(y^{k_m},\alpha_{k_m}Ju+(1-\alpha_{k_m})Jy^{k_m})\\ &\le \alpha_{k_m}V(y^{k_m},Ju)+(1-\alpha_{k_m})V(y^{k_m},Jy^{k_m})\\ &=\alpha_{k_m}\phi(y^{k_m},u).\end{aligned}$$

Next, taking limit, we get

$$\lim_{m\to\infty}\phi(y^{k_m},x^{k_m+1})=0,$$

now Proposition 2.1 implies that

$$\lim_{m\to\infty}\|y^{k_m}-x^{k_m+1}\|=0. \tag{5.11}$$

Finally (5.9) and (5.11) imply that

$$\lim_{m\to\infty}\|x^{k_m}-x^{k_m+1}\|=0. \tag{5.12}$$

On the other hand, we have

$$\begin{aligned}0 &\le f_k(y^k,y)+\lambda_k\langle y-y^k,Jy^k-Jx^k\rangle\\ &\le f_k(y^k,y)+\lambda_k\|y-y^k\|\|Jy^k-Jx^k\|.\end{aligned} \tag{5.13}$$

Also, note that uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, since $\lim_{m\to\infty}\|y^{k_m}-x^{k_m}\|=0$ by (5.9), we have

$$\lim_{k\to\infty}\|Jy^{k_m}-Jx^{k_m}\|=0. \tag{5.14}$$

Now, since $y^{k_{m_t}}\rightharpoonup p$, replacing k by k_{m_t} in (5.13) and taking limsup, we have

$$\limsup_{t\to\infty}f_{k_{m_t}}(y^{k_{m_t}},y)\ge 0,\quad \forall y\in K.$$

Now, since $\{f_k\}$ satisfies (3.1), hence $p\in\bigcap_k S(f_k;K)$. Therefore, by (5.10) and Proposition 2.3, we have

$$\limsup_{m\to\infty}\langle x^{k_m}-x^*,Ju-Jx^*\rangle=\langle p-x^*,Ju-Jx^*\rangle\le 0. \tag{5.15}$$

Hence, by using (5.12), we have

$$\limsup_{m\to\infty}\langle x^{k_m+1}-x^*,Ju-Jx^*\rangle = \limsup_{m\to\infty}\langle x^{k_m+1}-x^{k_m}+x^{k_m}-x^*,Ju-Jx^*\rangle$$

$$\leq \limsup_{m\to\infty}\langle x^{k_m+1} - x^{k_m}, Ju - Jx^*\rangle$$
$$+ \limsup_{m\to\infty}\langle x^{k_m} - x^*, Ju - Jx^*\rangle$$
$$= 0 + \langle p - x^*, Ju - Jx^*\rangle \leq 0.$$

Therefore, by Lemma 5.1, we have

$$\phi(x^*, x^k) \to 0.$$

Now, Proposition 2.1 implies that $x^k \to x^* = P_{\bigcap_k S(f_k;K)}u$. □

6. Hybrid proximal point algorithm

In this section, we study the strong convergence of the sequence generated by the hybrid proximal point method for an infinite family of bifunctions. This section extends [16] which states the hybrid proximal point method for a monotone inclusion problem. First we propose the algorithm, then we show the sequence generated by the algorithm converges strongly to a common equilibrium point of an infinite family of bifunctions $\{f_n\}$ that satisfy A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and $\bigcap_k S(f_k;K) \neq \varnothing$.

Algorithm 6.1.
Initialize: $x^0 \in E$, $n := 0$, $\theta_k < \lambda_k \leq \bar{\lambda}$ for some $\bar{\lambda}$, $\gamma_k \in [\varepsilon, \frac{1}{2}]$ for some $\varepsilon \in (0, \frac{1}{2}]$ and $k = 0, 1, 2, \ldots$.
Step 1: Let z^n be the equilibrium point of $f_n(x, y) + \lambda_n\langle y - x, Jx - Jx^n\rangle$ by Proposition 2.4, i.e.,

$$f_n(z^n, y) + \lambda_n\langle y - z^n, Jz^n - Jx^n\rangle \geq 0, \quad \forall y \in K. \tag{6.1}$$

Step 2: Determine the next approximation x^{n+1} as the projection of x^0 onto $H_n \cap W_n$, $x^{n+1} = P_{H_n\cap W_n}(x^0)$, where

$$H_n = \{z \in E : \langle z - x^n, Jx^n - Jz^n\rangle \leq -\gamma_n\phi(x^n, z^n)\},$$
$$W_n = \{z \in E : \langle z - x^n, Jx^0 - Jx^n\rangle \leq 0\}.$$

Step 3: Set $n := n + 1$ and go back Step 1.

In order to prove the strong convergence of the sequence $\{x^n\}$ generated by Algorithm 6.1, we need the following lemmas.

Lemma 6.2. *If Algorithm 6.1 reaches to the iteration step n, then $\bigcap_k S(f_k;K) \subset H_n \cap W_n$ and x^{n+1} is well defined.*

Proof. Note that $\bigcap_k S(f_k;K)$, H_n and W_n are closed and convex. We first show that $\bigcap_k S(f_k;K) \subset H_n \cap W_n$ for all $n \geq 0$. Putting

$$C_n = \{z \in E : \phi(z, z^n) \leq \phi(z, x^n)\}.$$

A straightforward calculation leads to

$$C_n = \{z \in E : \langle z - x^n, Jx^n - Jz^n\rangle \leq -\frac{1}{2}\phi(x^n, z^n)\}.$$

By $\gamma_n \in [\varepsilon, \frac{1}{2}]$, $C_n \subseteq H_n$. Letting $y = x^* \in \bigcap_k S(f_k; K)$ in (6.1), since f_n is pseudo monotone, we have $f_n(z^n, x^*) \leq 0$. Hence $\langle x^* - z^n, Jz^n - Jx^n \rangle \geq 0$ which shows that

$$\phi(x^*, z^n) \leq \phi(x^*, x^n) \tag{6.2}$$

for all $x^* \in \bigcap_k S(f_k; K)$. Therefore $\bigcap_k S(f_k; K) \subset C_n$ for all $n \geq 0$, which implies that $\bigcap_k S(f_k; K) \subset H_n$ for all $n \geq 0$. Next, by induction, we show that $\bigcap_k S(f_k; K) \subset H_n \cap W_n$ for all $n \geq 0$. Indeed, we have $\bigcap_k S(f_k; K) \subset H_0 \cap W_0$, because $W_0 = E$. Assume that $\bigcap_k S(f_k; K) \subset H_n \cap W_n$ for some $n \geq 0$. Since $x^{n+1} = P_{H_n \cap W_n}(x^0)$, we have by Proposition 2.3 that

$$\langle x^{n+1} - z, Jx^0 - Jx^{n+1} \rangle \geq 0, \quad \forall z \in H_n \cap W_n.$$

Since $\bigcap_k S(f_k; K) \subset H_n \cap W_n$,

$$\langle x^{n+1} - z, Jx^0 - Jx^{n+1} \rangle \geq 0, \quad \forall z \in \bigcap_k S(f_k; K).$$

Now, since $\langle x^{n+1} - z, Jx^0 - Jx^{n+1} \rangle \geq 0, \ \forall z \in \bigcap_k S(f_k; K)$, hence the definition of W_{n+1} implies that $\bigcap_k S(f_k; K) \subset W_{n+1}$, and so $\bigcap_k S(f_k; K) \subset H_n \cap W_n$ for all $n \geq 0$. Finally, since $\bigcap_k S(f_k; K)$ is nonempty, hence $H_n \cap W_n$ is nonempty, therefore x^{n+1} is well defined. □

Lemma 6.3. *If $\{x^n\}$ and $\{z^n\}$ are the sequences generated by Algorithm 6.1, then*

$$\lim_{n \to \infty} \|x^{n+1} - x^n\| = \lim_{n \to \infty} \|z^n - x^n\| = 0.$$

Proof. From the definition of W_n, we have $x^n = P_{W_n}(x^0)$. For each $u \in \bigcap_k S(f_k; K) \subset W_n$, since P_{W_n} is the projection map onto W_n, we have $\langle x^n - u, Jx^0 - Jx^n \rangle \geq 0$ by Proposition 2.3, this implies that

$$\phi(x^n, x^0) \leq \phi(u, x^0). \tag{6.3}$$

Thus, the sequence $\{x^n\}$ is bounded by (2.3). Moreover, $x^{n+1} = P_{H_n \cap W_n}(x^0)$ implies that $x^{n+1} \in W_n$. Similar to the previous case, we can obtain

$$\phi(x^n, x^0) \leq \phi(x^{n+1}, x^0).$$

So, the sequence $\{\phi(x^n, x^0)\}$ is non-decreasing. Hence, $\lim_{n \to \infty} \phi(x^n, x^0)$ exists. By $x^{n+1} \in W_n$ and $x^n = P_{W_n}(x^0)$, we also have

$$\phi(x^{n+1}, x^n) \leq \phi(x^{n+1}, x^0) - \phi(x^n - x^0).$$

Passing to the limit in the above inequality as $n \to \infty$, one gets

$$\lim_{n \to \infty} \phi(x^{n+1}, x^n) = 0.$$

Now, by Proposition 2.1, we have

$$\lim_{n \to \infty} \|x^{n+1} - x^n\| = 0.$$

Since $x^{n+1} \in H_n$, from the definition of H_n, we have

$$\gamma_n \phi(x^n, z^n) \leq \langle x^n - x^{n+1}, Jx^n - Jz^n \rangle.$$

Thus $\gamma_n \phi(x^n, z^n) \leq \|x^n - x^{n+1}\| \|Jx^n - Jz^n\|$. Since $\{z^n\}$ is bounded by (6.2), $\lim_{n\to\infty} \|x^{n+1} - x^n\| = 0$ and $\gamma_n \geq \varepsilon > 0$, we have $\lim_{n\to\infty} \phi(x^n, z^n) = 0$. Therefore Proposition 2.1 implies that

$$\lim_{n\to\infty} \|z^n - x^n\| = 0.$$

□

Theorem 6.4. *Assume that* $\{f_n\}$ *satisfy* A_1, A_2, A_3, A_4^*, $A_4^\bullet$ *and* (3.1). *In addition* $\bigcap_k S(f_k; K)$ *is nonempty. Then, the sequence* $\{x^n\}$ *generated by Algorithm 6.1 converges strongly to* $P_{\bigcap_k S(f_k;K)}(x^0)$.

Proof. It is clear that $\bigcap_k S(f_k; K)$ is closed and convex, hence we define $\bar{x} = P_{\bigcap_k S(f_k;K)}(x^0)$. Besides, the sequence $\{x^n\}$ is bounded. Assume that p is any weak limit point of the sequence $\{x^n\}$. Then, there exists a subsequence of $\{x^n\}$ converging weakly to p. We denote this subsequence by $\{x^{n_k}\}$ such that $x^{n_k} \rightharpoonup p$ as $k \to \infty$. By Lemma 6.3, we have $z^{n_k} \rightharpoonup p$ as $k \to \infty$. Now, note that uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, again from Lemma 6.3 we have

$$\lim_{n\to\infty} \|Jz^n - Jx^n\| = 0. \tag{6.4}$$

By setting $n = n_k$ in (6.1) and taking limsup, we can obtain

$$\limsup_{k\to\infty} f_{n_k}(z^{n_k}, y) \geq 0, \quad \forall y \in K. \tag{6.5}$$

Now (3.1) implies that $p \in \bigcap_k S(f_k; K)$.

In the sequel, we first prove that there exists only one weak cluster point of $\{x^n\}$. Finally, we show that $x^n \to \bar{x} = P_{\bigcap_k S(f_k;K)}(x^0)$. From the definition of W_n, we have $x^n = P_{W_n}(x^0)$. For $\bar{x} \in \bigcap_k S(f_k; K) \subset W_n$, since P_{W_n} is the projection map onto W_n, we have $\langle \bar{x} - x^n, Jx^0 - Jx^n \rangle \leq 0$ by Proposition 2.3, this implies that $\phi(x^n, x^0) \leq \phi(\bar{x}, x^0)$. Therefore we have

$$\|x^n\|^2 - 2\langle x^n, Jx^0 \rangle + \|x^0\|^2 \leq \phi(\bar{x}, x^0). \tag{6.6}$$

Note that since $x^{n_k} \rightharpoonup p$, by the weak lower semicontinuity of the norm $\|\cdot\|$ and replacing n by n_k in (6.6), we have

$$\begin{aligned}\phi(p, x^0) = \|p\|^2 - 2\langle p, Jx^0 \rangle + \|x^0\|^2 &\leq \liminf_{k\to\infty} (\|x^{n_k}\|^2 - 2\langle x^{n_k}, Jx^0 \rangle + \|x^0\|^2) \\ &\leq \phi(\bar{x}, x^0).\end{aligned}$$

By the definition of $\bar{x}$, we have $\bar{x} = p$. In the sequel, since $\{x^{n_k}\}$ is an arbitrary subsequence of $\{x^n\}$, we have $x^n \rightharpoonup \bar{x}$ and $\lim_{n\to\infty} \|x^n\| = \|\bar{x}\|$. Now, note that

$$\lim_{n\to\infty} \phi(x^n, \bar{x}) = \lim_{n\to\infty} (\|x^n\|^2 - 2\langle x^n, J\bar{x} \rangle + \|\bar{x}\|^2) = 0.$$

Therefore by Proposition 2.1, we have $x^n \to \bar{x} = P_{\bigcap_k S(f_k;K)}(x^0)$. □

In the sequel, we use Theorem 6.4 to prove the strong convergence of the sequence generated by the hybrid proximal point algorithm to a common equilibrium point of a finite family of bifunctions g_i, $i = 0, 1, \ldots, m-1$, which satisfy A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and (3.2).

Algorithm 6.5 (The hybrid proximal point algorithm for a finite family of bifunctions)**.**
Initialize: $x^0 \in E$, $n := 0$, $\theta_k < \lambda_k \leq \bar{\lambda}$ for some $\bar{\lambda}$, $\gamma_k \in [\varepsilon, \frac{1}{2}]$ for some $\varepsilon \in (0, \frac{1}{2}]$ and $k = 0, 1, 2, \ldots$.
Step 1: Let z^n be the equilibrium point of $g_{[n]}(x, y) + \lambda_n \langle y - x, Jx - Jx^n \rangle$ by Proposition 2.4, i.e.,

$$g_{[n]}(z^n, y) + \lambda_n \langle y - z^n, Jz^n - Jx^n \rangle \geq 0, \quad \forall y \in K, \tag{6.7}$$

where $[n] \in \{0, \ldots, m-1\}$, in other words $[n] \equiv n \pmod{m}$.
Step 2: Determine the next approximation x^{n+1} as the projection of x^0 onto $H_n \cap W_n$, $x^{n+1} = P_{H_n \cap W_n}(x^0)$, where

$$H_n = \{z \in E : \langle z - x^n, Jx^n - Jz^n \rangle \leq -\gamma_n \phi(x^n, z^n)\};$$
$$W_n = \{z \in E : \langle z - x^n, Jx^0 - Jx^n \rangle \leq 0\}.$$

Step 3: Set $n := n + 1$ and go back Step 1.

Corollary 6.6. *Assume that the bifunctions g_i, $0 \leq i \leq m-1$, satisfy A_1, A_2, A_3, A_4^*, $A_4^\bullet$ and* (3.2)*. In addition $\bigcap_{i=0}^{m-1} S(g_i; K)$ is nonempty. Then, the sequence $\{x^n\}$ generated by Algorithm 6.5 converges strongly to*

$$P_{\bigcap_{i=0}^{m-1} S(g_i;K)}(x^0).$$

Proof. We first define $\{f_n\}$ as $f_{km} \equiv g_0$, $f_{km+1} \equiv g_1, \ldots, f_{km+m-1} \equiv g_{m-1}$ for any $k \in \mathbb{N} \cup \{0\}$. By (6.3) and (2.3), the sequence $\{x^n\}$ is bounded, hence the subsequences $\{x^{km+i}\}$ of $\{x^n\}$ are bounded for $i = 0, 1, \ldots, m-1$. Consider the subsequence $\{x^{km}\}$. Assume that p_0 is any weak limit point of the sequence $\{x^{km}\}$. Then, there exists a subsequence $\{x^{k_j m}\}$ such that $x^{k_j m} \rightharpoonup p_0$ as $j \to \infty$. By Lemma 6.3, $x^{k_j m+i} \rightharpoonup p_0$ for $i = 0, 1, \ldots, m-1$. Also, by Lemma 6.3, we have $z^{k_j m+i} \rightharpoonup p_0$ as $j \to \infty$ for $i = 0, 1, \ldots, m-1$. Now, note that uniform smoothness of E implies uniform norm-to-norm continuity of J on each bounded set of E. Therefore, again from Lemma 6.3 we have

$$\lim_{n \to \infty} \|Jz^n - Jx^n\| = 0. \tag{6.8}$$

Now, we show that $p_0 \in \bigcap_{i=0}^{m-1} S(g_i; K)$. By setting $n = k_j m + i$ and taking liminf in (6.7), we have

$$\liminf_{j \to \infty} f_{k_j m+i}(z^{k_j m+i}, y) \geq 0, \quad \forall y \in K \text{ and } 0 \leq i \leq m-1. \tag{6.9}$$

Now, since $z^{k_j m+i} \rightharpoonup p_0$ as $j \to \infty$ for $i = 0, 1, \ldots, m-1$, therefore (3.2) implies that $p_0 \in \bigcap_{i=0}^{m-1} S(g_i; K)$. The remaining proof is similar to the proof of Theorem 6.4. □

7. A concrete example

There are some concrete examples of Nash equilibrium problems which are modeled as equilibrium problems for infinitely many bifunctions. We refer the reader to [18] (see also [19, 23, 24]).

In this section, we give an example in Banach spaces.

Example 7.1. Consider a finite set of points $\{z^1, z^2, \dots, z^n\}$ of a Banach space E. We define the geometric median as

$$M(z^1, \dots, z^n) := \operatorname{Argmin} \left\{ \sum_{i=1}^{n} \|x - z^i\| : x \in E \right\}$$

and its Fréchet mean as

$$\sigma(z^1, \dots, z^n) := \operatorname{Argmin} \left\{ \sum_{i=1}^{n} \|x - z^i\|^2 : x \in E \right\}.$$

It is well known that only in Hilbert spaces $\sigma(z^1, \dots, z^n) = \frac{z^1 + \cdots + z^n}{n}$ (see [2]). To compute the mean and the median of $z^1, \dots, z^n$ in general Banach spaces, the proximal point algorithm can be applied for the bifunctions $f, g : E \times E \to \mathbb{R}$ defined by $f(x, y) = \phi(y) - \phi(x)$ and $g(x, y) = \psi(y) - \psi(x)$, where $\phi(x) = \sum_{i=1}^{n} \|x - z^i\|$ and $\psi(x) = \sum_{i=1}^{n} \|x - z^i\|^2$. It is obvious that f and g satisfy A_1, A_2, A_3, A_4 and also any equilibrium point of $EP(f, E)$ (resp. $EP(g, E)$) is a minimum point of ϕ (resp. ψ) and vice versa.

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