Bull. Korean Math. Soc. **56** (2019), No. 3, pp. 745–756
https://doi.org/10.4134/BKMS.b180540
pISSN: 1015-8634 / eISSN: 2234-3016

CHARACTERIZATIONS FOR THE FOCK-TYPE SPACES

Hong Rae Cho, Jeong Min Ha, and Kyesook Nam

Abstract. We obtain Lipschitz type characterization and double integral characterization for Fock-type spaces with the norm

$$\|f\|^p_{F^p_{m,\alpha,t}} = \int_{\mathbb{C}^n} \left| f(z)e^{-\alpha|z|^m} \right|^p \frac{dV(z)}{(1+|z|)^t},$$

where $\alpha > 0$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$. The results of this paper are the extensions of the classical weighted Fock space $F^p_{2,\alpha,t}$.

1. Introduction

For a fixed positive integer n, let $H(\mathbb{C}^n)$ be the space of all entire functions on the complex n-space $\mathbb{C}^n$. For $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, we write

$$z \cdot w = z_1\overline{w}_1 + \cdots + z_n\overline{w}_n, \quad |z| = \sqrt{|z_1|^2 + \cdots + |z_n|^2}.$$

For $\alpha > 0$, $t \in \mathbb{R}$, and $m \in \mathbb{N}$, we define $dG_{m,\alpha,t}$ the t-weighted (m,α)-Gaussian measure on $\mathbb{C}^n$ by

$$dG_{m,\alpha,t}(z) = C_{m,\alpha,t}e^{-\alpha|z|^m} \frac{dV(z)}{(1+|z|)^t},$$

where dV is the volume measure on $\mathbb{C}^n$ and $C_{m,\alpha,t}$ is the positive constant to be the normalized volume measure. For $0 < p < \infty$, we consider the Fock-type space $F^p_{m,\alpha,t}(\mathbb{C}^n)$ consisting of all $f \in H(\mathbb{C}^n)$, the class of all entire functions on $\mathbb{C}^n$, where the norm is defined by

$$\|f\|^p_{F^p_{m,\alpha,t}} = C_{m,\alpha,t} \int_{\mathbb{C}^n} \left| f(z)e^{-\alpha|z|^m} \right|^p \frac{dV(z)}{(1+|z|)^t}.$$

Then $F^p_{m,\alpha,t}(\mathbb{C}^n) = L^p(dG_{m,\alpha p,t}) \cap H(\mathbb{C}^n)$. When $m = 2$, $F^p_{2,\alpha,t}(\mathbb{C}^n)$ is the classical weighted Fock space.

Received June 4, 2018; Accepted August 13, 2018.
2010 *Mathematics Subject Classification.* 32A37, 30H20.
Key words and phrases. Fock-type space, Lipschitz condition, double integral condition.
The first author was supported by NRF of Korea (NRF-2016R1D1A1B03933740) and the third author was supported by NRF of Korea (NRF-2013R1A1A2057890).

In this paper, we characterize the Fock-type space $F^p_{m,\alpha,t}(\mathbb{C}^n)$. One of the main results is Lipschitz type characterization for $F^p_{m,\alpha,t}(\mathbb{C}^n)$ stated in Theorem 1.1 as follows.

Theorem 1.1. *Let $\alpha > 0$, $0 < p < \infty$, $s \geq 0$, $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on $\mathbb{C}^n$:*

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

(b) *There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that*

$$\frac{|f(z)-f(w)|}{|z-w|} \leq (1+|z|^{m-1}+|w|^{m-1})^{1+s}(g(z)+g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \neq w$.

For $f \in H(\mathbb{C}^n)$, we define

$$Lf(z,w) = f(z) - f(w).$$

Let m be an even positive integer and $s \in \mathbb{R}$. As a local type, we define

$$L^s_r f(z,w) = [f(z)-f(w)]e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}\chi_{E_r(z)}(w),$$

where $\chi_{E_r(z)}$ denotes the characteristic function in the Euclidean ball $E_r(z) = \left\{ w \in \mathbb{C}^n : |w-z| < \frac{r}{1+|z|^{m-1}} \right\}$ for $r > 0$. Hence, the other main result is the double integral characterization for $F^p_{m,\alpha,t}$ as follows.

Theorem 1.2. *Let $\alpha > 0$, $0 < p < \infty$, and $t \in \mathbb{R}$. Let m be an even positive integer. For $s \geq 0$, let $\beta = \frac{s+\alpha}{2}$. Then the following statements are equivalent for entire functions f on $\mathbb{C}^n$:*

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

(b) $Lf \in L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})$.

(c) $L^s_r f \in L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})$, *where* $\delta = \frac{t}{2} - n(m-1)$.

In Theorem 1.2, (a) and (b) are equivalent when $m \in \mathbb{N}$. On the other hand, (a) and (c) are equivalent when m is an even positive integer. An even positive integer m is necessary in order that the function $L^s_r f(z,w)$ is holomrphic with respect to w.

Lipschitz type characterization for weighted Bergman spaces with standard weights on the unit disc in the complex plane $\mathbb{C}$ in terms of the Euclidean, hyperbolic, and pseudo-hyperbolic metrics was introduced by Hasi Wulan and Kehe Zhu [5]. Moreover, they generalized these results to the unit ball in $\mathbb{C}^n$. As an application, they proved the boundedness of the symmetric lifting operator in [4]. Moreover, double integral characterizations for weighted Bergman spaces in the unit ball in $\mathbb{C}^n$ were proved in [3] and [4]. Our results were motivated by Lipschitz type characterization and double integral characterization for the weighted Fock spaces $F^p_{2,\alpha,t}(\mathbb{C}^n)$ in [1]. It's well known that $F^p_{2,\alpha,t}(\mathbb{C}^n)$ is closed in $L^p(G_{2,\alpha p,t})$. In particular, $F^2_{2,\alpha,t}(\mathbb{C}^n)$ is a Hilbert space and the reproducing kernel K_z at $z \in \mathbb{C}^n$ for $F^2_{2,1/2,0}(\mathbb{C}^n)$ is given by

$$K_z(w) = e^{\overline{z}\cdot w}.$$

See [6]. In [1], the reproducing kernel K_z was used to get a norm estimate for $F^p_{2,\alpha,t}(\mathbb{C}^n)$. In this paper, we have the same norm estimate for the Fock-type space $F^p_{m,\alpha,t}(\mathbb{C}^n)$ without using K_z.

For nonnegative quantities X and Y, the notation $X \lesssim Y$ means that there exists a positive constant C such that $X \leq CY$. The constant C is independent of the relevant variables. Furthermore, the notation $X \approx Y$ means that $X \lesssim Y$ and $Y \lesssim X$. In this case, we say that X and Y are equivalent.

2. Preliminaries

In this section, we suppose $m \in \mathbb{N}$.

For $r > 0$, $E_r(z)$ is defined by

$$E_r(z) = \left\{ w \in \mathbb{C}^n : |w - z| < \frac{r}{1 + |z|^{m-1}} \right\}. \tag{2.1}$$

Lemma 2.1. *Let $m \in \mathbb{N}$ and $r > 0$. For any $w \in E_r(z)$, there exists a positive constant $C = C(m, r)$ such that*

$$C^{-1} \leq \frac{e^{|w|^m}}{e^{|z|^m}} \leq C. \tag{2.2}$$

Moreover, there exists a positive constant $C = C(m, r)$ such that

$$C^{-1} \leq \frac{1 + |w|^{m-1}}{1 + |z|^{m-1}} \leq C. \tag{2.3}$$

Proof. Note that the asserted inequalities are clear when $m = 1$. So, let $m \geq 2$. For $w \in E_r(z)$, we have

$$|w| < \frac{r}{1 + |z|^{m-1}} + |z|$$

so that

$$\begin{aligned} |w|^m &< |z|^m + \sum_{k=1}^{m} \binom{m}{k} r^k \frac{|z|^{m-k}}{(1 + |z|^{m-1})^k} \\ &< |z|^m + (1 + r)^m. \end{aligned} \tag{2.4}$$

Similarly, we get

$$1 + |w|^{m-1} < (1 + |z|^{m-1})(1 + r)^{m-1} \tag{2.5}$$

which implies

$$\frac{r}{1 + |z|^{m-1}} < \frac{r(1 + r)^{m-1}}{1 + |w|^{m-1}}.$$

Thus, we have

$$E_r(z) \subset E_{2r(1+r)^{m-1}}(w), \quad w \in E_r(z). \tag{2.6}$$

Consequently, (2.4) and (2.6) give us (2.2). Moreover, (2.5) and (2.6) imply (2.3). The proof is complete. □

We denote a multi-index $M = (m_1, \dots, m_n)$ which is an n-tuple of non-negative integers and use the following notation $|M| = m_1 + \cdots + m_n$, $M! = m_1! \cdots m_n!$ and $\partial^M = \partial_1^{m_1} \cdots \partial_n^{m_n}$ where ∂_j denotes partial differentiation with respect to the j-th component.

Lemma 2.2. *Let $r > 0$, $m \in \mathbb{N}$ and $\alpha, t \in \mathbb{R}$. Given a multi-index M, there is a positive constant $C = C(\alpha, m, r, t, M)$ such that*

$$\frac{|\partial^M g(z)|^p e^{-\alpha|z|^m}}{(1+|z|^{m-1})^{p|M|+2n+\frac{t}{m-1}}} \le C \int_{E_r(z)} |g(w)|^p \frac{e^{-\alpha|w|^m}}{(1+|w|)^t}\, dV(w), \quad z \in \mathbb{C}^n,$$

for $0 < p < \infty$ and $g \in H(\mathbb{C}^n)$.

Proof. Let $0 < p < \infty$ and $g \in H(\mathbb{C}^n)$. Let $z \in \mathbb{C}^n$. By subharmonicity we have

$$\begin{aligned} |g(z)|^p &\le \frac{1}{V[E_r(z)]} \int_{E_r(z)} |g(w)|^p \, dV(w) \\ &= \frac{(1+|z|^{m-1})^{2n}}{\omega_n r^{2n}} \int_{E_r(z)} |g(w)|^p \, dV(w), \end{aligned}$$

where ω_n is the volume of the unit ball of $\mathbb{C}^n$. And hence the Cauchy Estimates over the ball $E_{\frac{r}{2}}(z)$ implies that

$$|\partial^M g(z)|^p \le C(1+|z|^{m-1})^{p|M|+2n} \int_{E_r(z)} |g(w)|^p \, dV(w)$$

for some $C = C(m, r, M) > 0$. Moreover, for $z \in \mathbb{C}^n$, we know

$$1 + |z|^{m-1} \approx (1+|z|)^{m-1}.$$

So, using (2.2) and (2.3), we complete the proof. □

For a function $f \in H(\mathbb{C}^n)$, we define the radial derivative $\mathcal{R}f$ of f at z by

$$\mathcal{R}f(z) = \sum_{j=1}^n z_j \frac{\partial f}{\partial z_j}(z).$$

The complex gradient of f at z is defined by

$$|\nabla f(z)| = \left[\sum_{j=1}^n \left| \frac{\partial f}{\partial z_j}(z) \right|^2 \right]^{1/2}.$$

For a radial function $\phi(r)$, we deduce Proposition 2.5 from Definition 2.3.

Definition 2.3 ([2])**.** Assume $\phi : [0,\infty) \to \mathbb{R}^+$ is twice continuously differentiable and there exists $\rho > 0$ such that $\phi'(r) \neq 0$ for $r \geq \rho$. We say that ϕ is in the class $\mathcal{W}_p$ if it satisfies the following conditions:

$$\lim_{r\to\infty} \frac{re^{-p\phi(r)}}{\phi'(r)} = 0,$$
$$\limsup_{r\to\infty} \frac{1}{r}\left(\frac{r}{\phi'(r)}\right)' < \frac{p}{n},$$
$$\liminf_{r\to\infty} \frac{1}{r}\left(\frac{r}{\phi'(r)}\right)' > -\infty.$$

Lemma 2.4 ([2])**.** *Let $0 < p < \infty$ and $\phi \in \mathcal{W}_p$. Then for $f \in H(\mathbb{C}^n)$,*

$$\int_{\mathbb{C}^n} |f(z)|^p e^{-p\phi(|z|)} dV(z) \approx |f(0)|^p + \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1+|z|\phi'(|z|))^p} e^{-p\phi(|z|)}\, dV(z). \tag{2.7}$$

Note that constants functions are contained in $F^p_{m,\alpha,t}(\mathbb{C}^n)$ so we have $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$ if and only if $f - f(0) \in F^p_{m,\alpha,t}(\mathbb{C}^n)$. Furthermore Lemma 2.4 implies that $F^p_{m,\alpha,t}(\mathbb{C}^n)$-norm of $f - f(0)$ is equivalent to the norm of the product of its radial derivative and $\frac{1}{1+|z|^m}$ in the n-dimensional complex space. We know that this factor $\frac{1}{1+|z|^m}$ is the supplementary amount to control the growth of the radial derivative. See the following.

Proposition 2.5. *Let $\alpha > 0$, $0 < p < \infty$, $m \in \mathbb{N}$, and $t \in \mathbb{R}$. Then the following norms*

$$\|f - f(0)\|_{F_{m,\alpha,t}}, \quad \left\|\frac{\mathcal{R}f(z)}{1+|z|^m}\right\|_{L^p(G_{m,\alpha p,t})}, \quad \left\|\frac{\nabla f(z)}{1+|z|^{m-1}}\right\|_{L^p(G_{m,\alpha p,t})}$$

are comparable to one another for $f \in H(\mathbb{C}^n)$.

Proof. Let $\phi(|z|) = \alpha|z|^m + \frac{t}{p}\ln(1+|z|)$ for $z \in \mathbb{C}^n$. Then ϕ is in the class $\mathcal{W}_p$ when $m \in \mathbb{N}$ and $t \in \mathbb{R}$. Also, by simple calculation, there exists a constant $C = C(\alpha, m, r, t) > 0$ such that

$$C^{-1} \leq \frac{1+|z|^m}{1+|z|\phi'(|z|)} \leq C.$$

Thus, applying Lemma 2.4 to $f - f(0)$, we obtain

$$\int_{\mathbb{C}^n} |f(z) - f(0)|^p e^{-\alpha p|z|^m} \frac{dV(z)}{(1+|z|)^t} \approx \int_{\mathbb{C}^n} \frac{|\mathcal{R}f(z)|^p}{(1+|z|^m)^p} e^{-\alpha p|z|^m} \frac{dV(z)}{(1+|z|)^t}.$$

Also, due to $|\mathcal{R}f(z)| \leq |z||\nabla f(z)|$, this estimate implies

$$\|f - f(0)\|_{F_{m,\alpha,t}} \lesssim \left\|\frac{\nabla f(z)}{1+|z|^{m-1}}\right\|_{L^p(G_{m,\alpha p,t})}.$$

Now, it remains to show

$$\left\| \frac{\nabla f(z)}{1+|z|^{m-1}} \right\|_{L^p(G_{m,\alpha p,t})} \lesssim \|f-f(0)\|_{F_{m,\alpha,t}}.$$

It follows from Lemma 2.2 that

$$\frac{|\nabla f(z)|^p}{(1+|z|^{m-1})^p} \lesssim (1+|z|^{m-1})^{2n} \int_{E_r(z)} |f(w)-f(0)|^p \, dV(w).$$

Let $r>0$ and $r_1 := 2r(1+r)^{m-1}$. Integrating both sides of the above against $dG_{m,\alpha p,t}$, we have from (2.6), (2.2) and (2.3)

$$\begin{aligned} &\left\| \frac{\nabla f(z)}{1+|z|^{m-1}} \right\|^p_{L^p(G_{m,\alpha p,t})} \\ &\lesssim \int_{\mathbb{C}^n} \int_{E_r(z)} |f(w)-f(0)|^p \, dV(w) \, dG_{m,\alpha p,t-2n(m-1)}(z) \\ &\lesssim \int_{\mathbb{C}^n} \int_{E_{r_1}(w)} |f(w)-f(0)|^p \, dG_{m,\alpha p,t-2n(m-1)}(z) \, dV(w) \\ &\approx \|f-f(0)\|^p_{F_{m,\alpha,t}} \end{aligned}$$

as desired. □

3. Lipschitz type characterization

In this section, we prove our first result Theorem 1.1. For $r>0$, we set

$$\Omega_r := \left\{ (z,w) : |w-z|(1+|z|^{m-1}+|w|^{m-1}) < r \right\}.$$

Theorem 3.1. *Let $\alpha>0$, $0<p<\infty$, $s\geq 0$, $m\in\mathbb{N}$, and $t\in\mathbb{R}$. Then the following statements are equivalent for entire functions f on $\mathbb{C}^n$:*

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

(b) *There exists a nonnegative continuous function $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ such that*

$$\frac{|f(z)-f(w)|}{|z-w|} \leq (1+|z|^{m-1}+|w|^{m-1})^{1+s}(g(z)+g(w))$$

for each $z,w\in\mathbb{C}^n$ with $z\neq w$.

Proof. First, we assume that (b) holds. Fixing z and taking the limits $w\to z$ along the directions parallel to the coordinate axes,

$$|\partial_j f(z)| \lesssim (1+|z|^{m-1})^{1+s} g(z)$$

for each j. Thus, we have

$$\frac{|\nabla f(z)|}{1+|z|^{m-1}} \lesssim (1+|z|^{m-1})^s g(z), \quad z\in\mathbb{C}^n$$

and thus

$$\int_{\mathbb{C}^n} \frac{|\nabla f(z)|^p}{(1+|z|^{m-1})^p} e^{-\alpha p|z|^m} \frac{dV(z)}{(1+|z|)^t} \lesssim \int_{\mathbb{C}^n} |g(z)|^p e^{-\alpha p|z|^m} \frac{dV(z)}{(1+|z|)^{t-sp(m-1)}}.$$

Since $g(z) \in L^p(G_{m,\alpha p,t-sp(m-1)})$, by Proposition 2.5, we conclude

$$f \in F^p_{m,\alpha,t}(\mathbb{C}^n).$$

Second, we assume that (a) holds. Fix any $r > 0$. We consider $(z, w) \in \Omega_r$. Then $w \in E_r(z)$ and

$$1 + |z|^{m-1} + |w|^{m-1} \approx 1 + |z|^{m-1}.$$

By the fundamental theorem of calculus, we get

$$|f(z) - f(w)| \le |z - w| \int_0^1 |\nabla f(\rho z + (1-\rho)w)|\, d\rho.$$

Since $\rho z + (1-\rho)w$ in $E_r(z)$, it follows

$$|f(z) - f(w)| \le |z - w| \sup_{\zeta \in E_r(z)} |\nabla f(\zeta)|. \tag{3.1}$$

Furthermore, we note

$$\begin{aligned} |\nabla f(\zeta)| &\approx (1 + |z|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}} \\ &\approx (1 + |z|^{m-1} + |w|^{m-1})^{1+s} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}} \end{aligned} \tag{3.2}$$

for $\zeta \in E_r(z)$. Let

$$h_s(z) := \sup_{\zeta \in E_r(z)} \frac{|\nabla f(\zeta)|}{(1 + |\zeta|^{m-1})^{1+s}}.$$

Then we have by (3.1) and (3.2)

$$|f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(h_s(z) + h_s(w))$$

for $(z, w) \in \Omega_r$.

Next, we consider $(z, w) \notin \Omega_r$. Then $|w - z|(1 + |z|^{m-1} + |w|^{m-1}) \ge r$. Therefore, for $s \ge 0$, we obtain

$$\begin{aligned} &|f(z) - f(w)| \\ &\le \frac{|z - w|(1 + |z|^{m-1} + |w|^{m-1})}{r}(|f(z)| + |f(w)|) \\ &\le \frac{|z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}}{r} \left(\frac{|f(z)|}{(1 + |z|^{m-1})^s} + \frac{|f(w)|}{(1 + |w|^{m-1})^s} \right). \end{aligned}$$

Hence, by setting $g(z) := h_s(z) + \frac{|f(z)|}{r(1+|z|^{m-1})^s}$ for $z \in \mathbb{C}^n$, we have

$$|f(z) - f(w)| \lesssim |z - w|(1 + |z|^{m-1} + |w|^{m-1})^{1+s}(g(z) + g(w))$$

for each $z, w \in \mathbb{C}^n$ with $z \ne w$. Note that the constant suppressed above depends only on m, r and s. Also, the function $g(z)$ is continuous on $\mathbb{C}^n$. It remains for us to show the function $g(z)$ belongs to $L^p(G_{m,\alpha p,t-sp(m-1)})$. It is clear that $\frac{|f(z)|}{r(1+|z|^{m-1})^s}$ is in $L^p(G_{m,\alpha p,t-sp(m-1)})$ for $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

Now, we claim h_s is in $L^p(G_{m,\alpha p,t-sp(m-1)})$. Let $\zeta \in E_r(z)$. Then $E_{r_0}(\zeta) \subset E_{r_1}(z)$ by (2.6) where $r_0 = 2r(1+r)^{m-1}$ and $r_1 = 2r_0(1+r_0)^{m-1}$. By Lemma 2.2 and (2.3), we get

$$\begin{aligned}\frac{|\nabla f(\zeta)|^p}{(1+|\zeta|^{m-1})^{(1+s)p}} &\lesssim (1+|\zeta|^{m-1})^{2n-sp} \int_{E_{r_0}(\zeta)} |f(w)|^p \, dV(w) \\ &\lesssim (1+|z|^{m-1})^{2n-sp} \int_{E_{r_1}(z)} |f(w)|^p \, dV(w).\end{aligned}$$

Taking the supremum over $\zeta \in E_r(z)$, we have

$$|h_s(z)|^p \lesssim (1+|z|^{m-1})^{2n-sp} \int_{E_{r_1}(z)} |f(w)|^p \, dV(w)$$

for all $z \in \mathbb{C}^n$. Let $r_2 = 2r_1(1+r_1)^{m-1}$. By integrating both sides of the above against the measure $dG_{m,\alpha p,t-sp(m-1)}(z)$, it follows

$$\begin{aligned}&\|h_s\|^p_{L^p(G_{m,\alpha p,t-sp(m-1)})} \\ &\lesssim \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} \int_{E_{r_1}(z)} |f(w)|^p \, dV(w) \, dG_{m,\alpha p,t}(z) \\ &= \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_1}(z)}(w) \, dG_{m,\alpha p,t}(z) \, dV(w) \\ &< \int_{\mathbb{C}^n} \int_{\mathbb{C}^n} (1+|z|^{m-1})^{2n} |f(w)|^p \chi_{E_{r_2}(w)}(z) \, dG_{m,\alpha p,t}(z) \, dV(w) \\ &= \int_{\mathbb{C}^n} |f(w)|^p \int_{E_{r_2}(w)} (1+|z|^{m-1})^{2n} \, dG_{m,\alpha p,t}(z) \, dV(w),\end{aligned}$$

where χ denotes the characteristic function in its subscripted set. For $z \in E_{r_2}(w)$, we know (2.2) and (2.3). Hence, it follows that

$$\begin{aligned}&\|h_s\|^p_{L^p(G_{m,\alpha p,t-sp(m-1)})} \\ &\lesssim \int_{\mathbb{C}^n} |f(w)|^p (1+|w|^{m-1})^{2n} \int_{E_{r_2}(w)} dV(z) \, dG_{m,\alpha p,t}(w) \\ &\lesssim \int_{\mathbb{C}^n} |f(w)|^p (1+|w|^{m-1})^{2n} \frac{\omega_n r_2^{2n}}{(1+|w|^{m-1})^{2n}} \, dG_{m,\alpha p,t}(w) \\ &\lesssim \|f\|^p_{F^p_{m,\alpha,t}}.\end{aligned}$$

This finishes the proof. □

From the proof of Theorem 3.1, we have the following local version of Theorem 3.1 for arbitrary s real.

Theorem 3.2. *Let $\alpha > 0$, $0 < p < \infty$, $r > 0$, $m \in \mathbb{N}$, and $s, t \in \mathbb{R}$. Then the following statements are equivalent for entire functions f on $\mathbb{C}^n$:*

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

(b) *There exists a nonnegative continuous function* $g \in L^p(G_{m,\alpha p,t-sp(m-1)})$ *such that*

$$\frac{|f(z)-f(w)|}{|z-w|} \leq (1+|z|^{m-1}+|w|^{m-1})^{1+s}(g(z)+g(w))$$

for $(z,w) \in \Omega_r$ *with* $z \neq w$.

4. Double integral characterization

In this section, we prove the main Theorem 1.2. By the same proof of Proposition 3.1 in [1], we have the following.

Theorem 4.1. *Let* $0 < p < \infty$ *and* $\phi \in \mathcal{W}_p$. *Then the estimate*

$$\begin{aligned}&\int_{\mathbb{C}^n} |f(z)-f(0)|^p e^{-p\phi(|z|)} dV(z)\\ \approx &\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |Lf(z,w)|^p e^{-p\phi(|z|)} dV(z)\, e^{-p\phi(|w|)} dV(w)\end{aligned}$$

holds for $f \in H(\mathbb{C}^n)$.

Letting $\phi(|z|) = \alpha|z|^m + \frac{t}{p}\ln(1+|z|)$ for $z \in \mathbb{C}^n$, Theorem 4.1 gives us the following characterization.

Theorem 4.2. *Let* $\alpha > 0$, $0 < p < \infty$, $m \in \mathbb{N}$, *and* $t \in \mathbb{R}$. *Then the following statements are equivalent for entire functions* f *on* $\mathbb{C}^n$:

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.
(b) $Lf \in L^p(G_{m,\alpha p,t} \times G_{m,\alpha p,t})$.

Moreover, the norms

$$\|f-f(0)\|_{F^p_{m,\alpha,t}} \quad \text{and} \quad \|Lf\|_{L^p(G_{m,\alpha p,t}\times G_{m,\alpha p,t})}$$

are comparable to each other.

Lemma 4.3. *Let* $0 < p < \infty$ *and* $s \geq 0$. *Let* m *be an even positive integer. For* $w \in E_r(z)$, *there exists a positive constant* C *such that* $|e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \lesssim e^{sp|z|^m}$.

Proof. We prove that

$$\begin{aligned}|e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p = e^{spRe[(\overline{z}\cdot w)^{\frac{m}{2}}]} &\leq e^{sp|z|^{\frac{m}{2}}|w|^{\frac{m}{2}}}\\ &\leq e^{\frac{sp(|z|^m+|w|^m)}{2}}\\ &\lesssim e^{sp|z|^m}\end{aligned}$$

by the Cauchy-Schwarz inequality, arithmetic-geometric mean inequality and (2.2) in Lemma 2.1 for $w \in E_r(z)$ in turn. □

Theorem 4.4. *Let* $\alpha, r > 0$, $0 < p < \infty$, *and* $t \in \mathbb{R}$. *Let* m *be an even positive integer. For* $s \geq 0$, *let* $\beta := \frac{s+\alpha}{2}$. *Then the following statements are equivalent for entire functions* f *on* $\mathbb{C}^n$:

(a) $f \in F^p_{m,\alpha,t}(\mathbb{C}^n)$.

(b) $L_r^s f \in L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})$, *where* $\delta = \frac{t}{2} - n(m-1)$.

Moreover, the norms

$$\|f - f(0)\|_{F^p_{m,\alpha,t}} \quad \textit{and} \quad \|L_r^s f\|_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})}$$

are comparable to each other.

Proof. We assume that (b) holds. Fix $f \in H(\mathbb{C}^n)$ and let $z \in \mathbb{C}^n$. Define a function

$$g_z(w) = [f(w) - f(z)]e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}.$$

Then $g_z \in H(\mathbb{C}^n)$ and $\nabla g_z(z) = \nabla f(z) e^{s|z|^m}$. By applying Lemma 2.2, (2.2) and (2.3), we get

$$\begin{aligned}
&\left(\frac{|\nabla f(z)|e^{s|z|^m}}{1+|z|^{m-1}}\right)^p \\
&\lesssim (1+|z|^{m-1})^{2n} \int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \, dV(w) \\
&\approx e^{\frac{(s+\alpha)p}{2}|z|^m}(1+|z|^{m-1})^n \\
&\quad \times \int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p e^{-\frac{(s+\alpha)p}{2}|w|^m}(1+|w|^{m-1})^n \, dV(w).
\end{aligned}$$

By integrating both sides of the above against $dG_{m,(s+\alpha)p,t}(z)$ and applying (2.3) for $w \in E_r(z)$, we have

$$\begin{aligned}
&\int_{\mathbb{C}^n} \left(\frac{|\nabla f(z)|}{1+|z|^{m-1}}\right)^p dG_{m,\alpha p,t}(z) \\
\lesssim &\int_{\mathbb{C}^n}\int_{E_r(z)} |f(w) - f(z)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \, dG_{m,\beta p,-n(m-1)}(w)\, dG_{m,\beta p,t-n(m-1)}(z) \\
\approx &||L_r^s f||_{L^p(G_{m,\beta p,\delta}\times G_{m,\beta p,\delta})}.
\end{aligned}$$

Thus, by Proposition 2.5, we obtain

$$\|f - f(0)\|_{F^p_{m,\alpha,t}} \lesssim \|L_r^s f\|_{L^p(G_{m,\beta p,\delta}\times G_{m,\beta p,\delta})}.$$

The constant suppressed above is independent of f. We complete that (b) implies (a).

Now, we assume that (a) holds. Let $r_0 = 2r(1+r)^{m-1}$.

$$\begin{aligned}
&||L_r^s f||_{L^p(G_{m,\beta p,\delta}\times G_{m,\beta p,\delta})} \\
= &\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |f(w) - f(z)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \chi_{E_r(z)}(w)\, dG_{m,\beta p,\delta}(w)\, dG_{m,\beta p,\delta}(z) \\
\lesssim &\int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |f(z) - f(0)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \chi_{E_{r_0}(z)}(w)\, dG_{m,\beta p,\delta}(w)\, dG_{m,\beta p,\delta}(z) \\
&+ \int_{\mathbb{C}^n}\int_{\mathbb{C}^n} |f(w) - f(0)|^p |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \chi_{E_{r_0}(w)}(z)\, dG_{m,\beta p,\delta}(z)\, dG_{m,\beta p,\delta}(w)
\end{aligned}$$

$$\lesssim \int_{\mathbb{C}^n} |f(z)-f(0)|^p \int_{E_{r_0}(z)} |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \, dG_{m,\beta p,\delta}(w) \, dG_{m,\beta p,\delta}(z).$$

For the first inequality, we used $E_r(z) \subset E_{r_0}(z)$, $E_r(z) \subset E_{r_0}(w)$ for $w \in E_r(z)$ and Fubini's theorem. And for $w \in E_{r_0}(z)$, we have from (2.3)

$$\begin{aligned} & ||L_r^s f||_{L^p(G_{m,\beta p,\delta} \times G_{m,\beta p,\delta})} \\ & \lesssim \int_{\mathbb{C}^n} |f(z)-f(0)|^p (1+|z|^{m-1})^n \\ & \quad \times \int_{E_{r_0}(z)} |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p \, dG_{m,\beta p,-n(m-1)}(w) \, dG_{m,\beta p,t}(z) \\ & = \int_{\mathbb{C}^n} |f(z)-f(0)|^p I_r(z) \, dG_{m,\alpha p,t}(z), \end{aligned}$$

where $I_r(z) := e^{\frac{(-s+\alpha)p}{2}|z|^m}(1+|z|^{m-1})^n \int_{E_{r_0}(z)} |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p dG_{m,\beta p,-n(m-1)}(w)$.

Now, we claim that $I_r(z) \lesssim 1$.

For $w \in E_{r_0}(z)$, we have Lemma 4.3, (2.2) and (2.3). It follows that

$$\begin{aligned} \int_{E_{r_0}(z)} |e^{s(\overline{z}\cdot w)^{\frac{m}{2}}}|^p dG_{m,\beta p,-n(m-1)}(w) & \lesssim \int_{E_{r_0}(z)} e^{sp|z|^m} dG_{m,\beta p,-n(m-1)}(w) \\ & \lesssim e^{\frac{(s-\alpha)p}{2}|z|^m}(1+|z|^{m-1})^n V[E_{r_0}(z)]. \end{aligned}$$

Since $V[E_{r_0}(z)] \approx (1+|z|^{m-1})^{-2n}$, we have $I_r(z) \lesssim 1$. Hence we complete that (a) implies (b). □

Acknowledgements. The authors would like to thank a reviewer for comments about restriction on positive integers for the index m. We would have some technical problems about binomial theorem if we have non-integer m. It seems to be possible that the research extends to Fock-type spaces with general weights, so the work will be our next research subject.

References

[1] B. R. Choe and K. Nam, *New characterizations for the weighted fock spaces*, Complex Anal. Oper. Theory, to appear.

[2] J. M. Ha, H. R. Cho, and H.-W. Lee, *A norm equivalence for the mixed norm of Fock type*, Complex Var. Elliptic Equ. **61** (2016), no. 12, 1644–1655.

[3] S. Li, H. Wulan, R. Zhao, and K. Zhu, *A characterisation of Bergman spaces on the unit ball of* $\mathbb{C}^n$, Glasg. Math. J. **51** (2009), no. 2, 315–330.

[4] S. Li, H. Wulan, and K. Zhu, *A characterization of Bergman spaces on the unit ball of* $\mathbb{C}^n$. *II*, Canad. Math. Bull. **55** (2012), no. 1, 146–152.

[5] H. Wulan and K. Zhu, *Lipschitz type characterizations for Bergman spaces*, Canad. Math. Bull. **52** (2009), no. 4, 613–626.

[6] K. Zhu, *Analysis on Fock Spaces*, Grad. Texts in Math. **263**, Springer, New York, 2012.

Hong Rae Cho
Department of Mathematics
Pusan National University
Pusan 46241, Korea
Email address: chohr@pusan.ac.kr

Jeong Min Ha
Department of Mathematics
Pusan National University
Pusan 46241, Korea
Email address: jm.ha@pusan.ac.kr

Kyesook Nam
Faculty of Liberal Education
Seoul National University
Seoul 08826, Korea
Email address: ksnam@snu.ac.kr