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ON STRONGLY QUASI PRIMARY IDEALS

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Abstract. In this paper, we introduce strongly quasi primary ideals which is an intermediate class of primary ideals and quasi primary ideals. Let R be a commutative ring with nonzero identity and Q a proper ideal of R. Then Q is called strongly quasi primary if $ab \in Q$ for $a, b \in R$ implies either $a^2 \in Q$ or $b^n \in Q$ ($a^n \in Q$ or $b^2 \in Q$) for some $n \in \mathbb{N}$. We give many properties of strongly quasi primary ideals and investigate the relations between strongly quasi primary ideals and other classical ideals such as primary, 2-prime and quasi primary ideals. Among other results, we give a characterization of divided rings in terms of strongly quasi primary ideals. Also, we construct a subgraph of ideal based zero divisor graph $\Gamma_I(R)$ and denote it by $\Gamma_I^*(R)$, where I is an ideal of R. We investigate the relations between $\Gamma_I^*(R)$ and $\Gamma_I(R)$. Further, we use strongly quasi primary ideals and $\Gamma_I^*(R)$ to characterize von Neumann regular rings.

1. Introduction

In this article, all rings are assumed to be commutative with nonzero identity and all modules are unital. Let R always denote such a ring and $L(R)$ denote the lattice of all ideals of R. Assume that Q is an ideal of R. Then the radical of Q, written by $\sqrt{Q}$, is defined to be

$$\sqrt{Q} := \{r \in R : r^n \in Q,\ \exists n \in \mathbb{N}\}.$$

Also $(Q : a) = \{r \in R : ra \in Q\}$. Recall that a proper ideal P of R is said to be prime if $ab \in P$ implies either $a \in P$ or $b \in P$ for any $a, b \in R$ [6]. Note that a proper ideal P of R is prime if and only if $(P : a) = P$ for every $a \notin P$. A proper ideal Q of R is called primary if whenever $a, b \in R$ and $ab \in Q$, then either $a \in Q$ or $b \in \sqrt{Q}$ ($a \in \sqrt{Q}$ or $b \in Q$) [6]. In this case, $\sqrt{Q} = P$ is a prime ideal and Q is said to be P-primary. Also recall from [12] that a quasi primary ideal is a proper ideal Q of R whose radical is prime, i.e., $\sqrt{Q}$ is a prime ideal. Note that the class of quasi primary ideals properly contains the class of primary ideals.

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The notion of prime ideals and its generalizations have a distinguished place in commutative algebra and algebraic geometry. They are useful tools to determine the properties of commutative rings. Let R be an integral domain and F its quotient field. Then R is called a valuation domain if for any $x \in F$, either $x \in R$ or $x^{-1} \in R$ [16]. Note that an integral domain R is valuation domain if and only if $L(R)$ is totally ordered by inclusion if and only if for any $a, b \in R$, either $a|b$ or $b|a$, where $a|b$ stands for a divides b [16]. Beddani and Messirdi, in [10], introduced the concept of 2-prime ideals and they used it to characterize valuation domains. Recall that a proper ideal P of R is said to be a 2-prime ideal if $ab \in P$ for $a, b \in R$, then either $a^2 \in P$ or $b^2 \in P$. In [10, Theorem 2.1], it was shown that an integral domain R is valuation domain if and only if every ideal of R is 2-prime.

Our aim in this paper (especially in Section 2) is to introduce an intermediate class of ideals between primary ideals and quasi primary ideals. Let Q be a proper ideal of R. Then Q is said to be a strongly quasi primary ideal if $ab \in Q$ for $a, b \in R$ implies either $a^2 \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$ ($a^n \in Q$ or $b^2 \in Q$). In Section 2, we give many properties of strongly quasi primary ideals. Among many results in this paper, in Proposition 2.2, we give a characterization of strongly quasi primary ideals. In Proposition 2.5, we investigate the strongly quasi primary ideals in fractional ring $S^{-1}R$ of R at a multiplicatively closed set S. From Lemma 2.1 to Theorem 2.1, we determine all strongly quasi primary ideals of direct product of rings. From Theorem 2.3 to Corollary 2.4, we study the strongly quasi primary ideals of polynomial rings, formal power series rings and idealizations of modules. Recall that a ring R is a divided ring if any prime ideal is comparable with each principal ideal [7]. Note that every valuation domain is a divided ring, and also a ring R is divided if and only if for any $a, b \in R$ then either $a|b$ or $b|a^n$ for some $n \in \mathbb{N}$ [7, Proposition 2]. With Theorem 2.2, we give our main result in Section 2 characterizing divided integral domains in terms of strongly quasi primary ideals.

In Section 3, we study the application of strongly quasi primary ideals to graph theory. In this section, we construct a subgraph of ideal based zero divisor graph and denote this graph by $\Gamma_I^*(R)$, where I is an ideal of R. In Proposition 3.1, in terms of $\Gamma_I^*(R)$, we determine when a non prime ideal is a strongly quasi primary ideal. Recall that a ring R is a reduced ring if it has no nonzero nilpotent elements, i.e., $\sqrt{0} = 0$. Also a commutative ring R is called von Neumann regular if for any $a \in R$ there exists $x \in R$ such that $a = a^2x$ [19]. In this case the principal ideal generated by $a \in R$, $(a) = (e)$ for some idempotent $e \in R$. So far there has been a lot of studies on this issue and this kind of rings has some applications in other areas such as graph theory [15], [4]. Note that a ring R is von Neumann regular if and only if for any ideal I of R, $\sqrt{I} = I$ if and only if R is reduced and every prime ideal is maximal. In Theorem 3.1, we determine when R/I is reduced ring in terms of the connectivity of the graph $\Gamma_I^*(R)$. Finally, in Corollary 3.3, we characterize the von Neumann regular rings by using strongly quasi primary ideals and $\Gamma_I^*(R)$.

2. Characterization of strongly quasi primary ideals

Definition 2.1. Let Q be a proper ideal of R. Q is called a strongly quasi primary ideal if whenever $a, b \in R$ and $ab \in Q$, then either $a^2 \in Q$ or $b^n \in Q$ ($a^n \in Q$ or $b^2 \in Q$) for some $n \in \mathbb{N}$.

Proposition 2.1. *Let Q be a proper ideal of R. Then the following statements are satisfied*:

(i) *If Q is a primary ideal, then Q is a strongly quasi primary ideal.*

(ii) *If Q is a 2-prime ideal, then Q is a strongly quasi primary ideal.*

(iii) *If Q is a strongly quasi primary ideal, then Q is a quasi primary ideal.*

(iv) *Assume that $\sqrt{Q}^2 \subseteq Q$. Then Q is a 2-prime ideal if and only if Q is a strongly quasi primary ideal if and only if Q is a quasi primary ideal.*

Proof. (i), (ii): It is obvious.

(iii) Suppose that Q is a strongly quasi primary ideal. Now, we will show that$\sqrt{Q}$ is a prime ideal. Take $ab \in \sqrt{Q}$ for some $a, b \in R$. Then we have $(ab)^n = a^n b^n \in Q$ for some $n \in \mathbb{N}$. Since Q is a strongly quasi primary ideal, we get either $(a^n)^2 = a^{2n} \in Q$ or $(a^n)^m = a^{nm} \in Q$ for some $m \in \mathbb{N}$. Then we have either $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$, that is, Q is a quasi primary ideal.

(iv) Let Q be a quasi primary ideal and $ab \in Q \subseteq \sqrt{Q}$ for some $a, b \in R$. Since $\sqrt{Q}$ is a prime ideal, we have either $a \in \sqrt{Q}$ or $b \in \sqrt{Q}$. As $\sqrt{Q}^2 \subseteq Q$, we conclude that $a^2 \in \sqrt{Q}^2 \subseteq Q$ or $b^2 \in Q$. Thus Q is a 2-prime ideal. The rest follows from (ii) and (iii). $\square$

The following examples show the differences between strongly quasi primary ideals and other classical ideals such as 2-prime ideals, primary ideals and quasi primary ideals.

Example 2.1. Consider the subring $S = \{a_0 + a_1X + \cdots + a_nX^n : a_1$ is a multiple of $3\}$ of $\mathbb{Z}[X]$.

(i) Let $Q = (9X^2, X^3, X^4, X^5, X^6)$. Then note that $\sqrt{Q} = (3X, X^2, X^3)$.

Since $9X^2 \in Q$ but $X^2 \notin Q$ and $9^n \notin Q$ for all $n \in \mathbb{N}$, Q is not a primary ideal of S. Also note that Q is a quasi primary ideal and $\sqrt{Q}^2 = (9X^2, 3X^3, X^4, X^5, X^6) \subseteq Q$. Then by Proposition 2.1(iv), Q is a strongly quasi primary ideal.

(ii) Let $Q = (27X, 27X^2, X^3, X^4, X^5, X^6)$. Then note that $\sqrt{Q} = (3X, X^2, X^3)$ and so $S/\sqrt{Q}$ is isomorphic to $\mathbb{Z}$. Thus Q is a quasi primary ideal of S. Since $(3X)9 = 27X \in Q$ but $(3X)^2 = 9X^2 \notin Q$ and $9^n \notin Q$ for all $n \in \mathbb{N}$ and so Q is not a strongly quasi primary ideal of S.

Example 2.2. Let $R = F[X, Y]$, where F is a field and $Q = (X^3, XY, Y^3)$. Then Q is a primary ideal since $\sqrt{Q} = (X, Y)$ is a maximal ideal. By Proposition 2.1(i), Q is a strongly quasi primary ideal. Since $XY \in Q$ but $X^2 \notin Q$ and $Y^2 \notin Q$, Q is not a 2-prime ideal.

Example 2.3. Let R be a principal ideal domain. Thus every nonzero prime ideal is maximal. Then it is clear that primary ideals, strongly quasi primary ideals and quasi primary ideals are equal in any principal ideal domain.

From Proposition 2.1, we have the following diagram which clarifies the place of strongly quasi primary ideals in the lattice of all ideals $L(R)$.

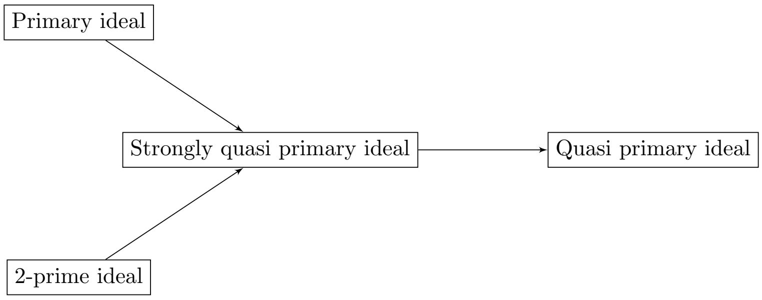

FIGURE 1. Relations between strongly quasi primary ideals and other classical ideals.

Example 2.4. Let R be a von Neumann regular ring. Then $Q^2 = Q = \sqrt{Q}$ for any ideal Q of R. Thus by Proposition 2.1, prime ideals, 2-prime ideals, primary ideals, strongly quasi primary ideals and quasi primary ideals coincide.

Let I be a proper ideal of R. Then the ideal generated by n th powers of elements of I is denoted by $I_n = (\{a^n : a \in I\})$ [2]. It is easy to note that $I_n \subseteq I^n \subseteq I$ and also the equality holds if $n = 1$. Further if $n!$ is a unit of R, then $I_n = I^n$ [2, Theorem 5].

Proposition 2.2. *Let Q be a proper ideal of R. Then the following statements are equivalent*:

(i) *Q is a strongly quasi primary ideal.*
(ii) *For every $a \in R$, either $(a) \subseteq (Q : a)$ or $(Q : a) \subseteq \sqrt{Q}$.*
(iii) *For any ideals J and K of R with $JK \subseteq Q$, either $J_2 \subseteq Q$ or $K \subseteq \sqrt{Q}$.*
(iv) *For every $a \in R$, either $a^n \in Q$ for some $n \in \mathbb{N}$ or $(Q : a)_2 \subseteq Q$.*

Proof. (i)$\Rightarrow$(ii) Suppose that Q is a strongly quasi primary ideal. Take an element $a \in R$. If $a^2 \in Q$, then $(a) \subseteq (Q : a)$. Now assume that $a^2 \notin Q$. Let $b \in (Q : a)$ for some $b \in R$. Then $ab \in Q$. Since Q is a strongly quasi primary ideal and $a^2 \notin Q$, we get $b \in \sqrt{Q}$ and this yields $(Q : a) \subseteq \sqrt{Q}$.

(ii)$\Rightarrow$(iii) Let $JK \subseteq Q$ for some ideals J and K of R. Assume that $K \nsubseteq \sqrt{Q}$. Then there exists $k \in K - \sqrt{Q}$. By assumption, for all $x \in J$, $xk \in Q$. Since

$k \in (Q : x) - \sqrt{Q}$, we have $(Q : x) \nsubseteq \sqrt{Q}$. Then by (ii), we get $(x) \subseteq (Q : x)$ and so $x^2 \in Q$. Thus $J_2 \subseteq Q$.

(iii)⇒(iv) Let $a \in R$. If $a \in \sqrt{Q}$, then we are done. Assume that $a \notin \sqrt{Q}$. Now put $J = (Q : a)$ and $K = (a)$. Then $JK = (Q : a)(a) \subseteq Q$. Since $K \nsubseteq \sqrt{Q}$, by (iii), we have $J_2 = (Q : a)_2 \subseteq Q$.

(iv)⇒(i) Let $ab \in Q$ with $b \notin \sqrt{Q}$. Then $a \in (Q : b)$ and so by (iv), $a^2 \in (Q : b)_2 \subseteq Q$. Hence, Q is a strongly quasi primary ideal. □

As a consequence of the previous proposition we give the following explicit result:

Corollary 2.1. *Let R be a ring and 2 be a unit of R. The following statements are equivalent for any proper ideal Q of R:*

(i) *Q is a strongly quasi primary ideal.*

(ii) *For every $a \in R$, either $(a) \subseteq (Q : a)$ or $(Q : a) \subseteq \sqrt{Q}$.*

(iii) *For any ideals J and K of R with $JK \subseteq Q$, either $J^2 \subseteq Q$ or $K \subseteq \sqrt{Q}$.*

(iv) *For every $a \in R$, either $a^n \in Q$ for some $n \in \mathbb{N}$ or $(Q : a)^2 \subseteq Q$.*

In the following proposition, we show that strongly quasi primary ideals have a similar property as that of primary ideals.

Proposition 2.3. *Let Q be a strongly quasi primary ideal of R and $a \in R$ such that $(a) = (a^2)$. If $a \notin Q$, then $(Q : a)$ is a strongly quasi primary ideal of R.*

Proof. Suppose that Q is a strongly quasi primary ideal of R. Since $a \notin (Q : a)$, by Proposition 2.2, $\sqrt{(Q : a)} = \sqrt{Q}$. Let $xy \in (Q : a)$ with $y^n \notin (Q : a)$ for all $n \in \mathbb{N}$. This implies that $(ax)y \in Q$ and $y^n \notin Q$ for all $n \in \mathbb{N}$. Since Q is a strongly quasi primary ideal of R, $(ax)^2 = x^2a^2 \in Q$ and so $x^2 \in (Q : a^2) = (Q : a)$. Thus, $(Q : a)$ is a strongly quasi primary ideal of R. □

Proposition 2.4. *Let $f : R \to S$ be a homomorphism of rings. Then the followings hold:*

(i) *If f is an epimorphism and Q is a strongly quasi primary ideal of R containing $Ker(f)$, then $f(Q)$ is a strongly quasi primary ideal of S.*

(ii) *If Q' is a strongly quasi primary ideal of S, then $f^{-1}(Q')$ is a strongly quasi primary ideal of R.*

Proof. (i) Let $ab \in f(Q)$, where $a, b \in S$. Then $a = f(x), b = f(y)$ for some $x, y \in R$ and so $f(xy) = f(x)f(y) \in f(Q)$. As $\mathrm{Ker}(f) \subseteq Q$, we get $xy \in Q$. Since Q is a strongly quasi primary ideal, we get either $x^2 \in Q$ or $y^n \in Q$ for some $n \in \mathbb{N}$ and this yields that $a^2 \in f(Q)$ or $b^n \in f(Q)$. Hence, $f(Q)$ is a strongly quasi primary ideal.

(ii) Let $xy \in f^{-1}(Q')$ for some $x, y \in R$. Then $f(xy) = f(x)f(y) \in Q'$. This implies that $f(x)^2 = f(x^2) \in Q'$ or $f(y)^n = f(y^n) \in Q'$ and so $x^2 \in f^{-1}(Q')$ or $y^n \in f^{-1}(Q')$. Hence, $f^{-1}(Q')$ is a strongly quasi primary ideal of R. □

Corollary 2.2. *Suppose that I is a proper ideal of R. Then the followings are satisfied*:

(i) *If Q is a strongly quasi primary ideal of R containing I, then Q/I is a strongly quasi primary ideal of R/I. Further, if Q/I is a strongly quasi primary ideal of R/I, then Q is a strongly quasi primary ideal of R.*

(ii) *If Q is a strongly quasi primary ideal of R and S is a subring of R with $S \nsubseteq Q$, then $S \cap Q$ is a strongly quasi primary ideal of S.*

Proof. (i) Consider the natural homomorphism $\pi : R \to R/I$, defined by $\pi(a) = a + I$ for each $a \in R$. Then (i) is obtained by Proposition 2.4(i).

(ii) Consider the injection $i : S \to R$, defined by $i(a) = a$ for each $a \in S$. Then the result follows from Proposition 2.4(ii). □

Corollary 2.3. *Let Q be a proper ideal of R. Then the following statements are equivalent*:

(i) *Q is a strongly quasi primary ideal of R.*

(ii) *(Q, X) is a strongly quasi primary ideal of $R[X]$.*

Proof. (i)⇔(ii) It follows from Corollary 2.2(i) and the isomorphism

$$(Q, X)/(X) \cong Q$$

in $R[X]/(X) \cong R$. □

Let I be an ideal of R. We denote the set of all elements $r \in R$ such that $rs \in I$ for some $s \notin I$ by $Zd_R(I)$.

Proposition 2.5. *Let S be a multiplicatively closed subset of R and Q a proper ideal of R. Then the following statements are satisfied*:

(i) *If Q is a strongly quasi primary ideal of R with $Q \cap S = \emptyset$, then $S^{-1}Q$ is a strongly quasi primary ideal of $S^{-1}R$.*

(ii) *If $S^{-1}Q$ is a strongly quasi primary ideal of $S^{-1}R$ with $S \cap Zd_R(Q) = \emptyset$, then Q is a strongly quasi primary ideal of R.*

Proof. (i) Let $\frac{a}{s}\frac{b}{t} \in S^{-1}Q$ for some $a, b \in R;\ s, t \in S$. Then there exists $u \in S$ such that $(ua)b \in Q$. Since Q is a strongly quasi primary ideal, we get either $(ua)^2 \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$. This implies that $(\frac{a}{s})^2 = \frac{a^2}{s^2} = \frac{u^2a^2}{u^2s^2} \in S^{-1}Q$ or $(\frac{b}{t})^n = \frac{b^n}{t^n} \in S^{-1}Q$. Hence, $S^{-1}Q$ is a strongly quasi primary ideal of $S^{-1}R$.

(ii) Let $ab \in Q$ for some $a, b \in R$. Then $\frac{a}{1}\frac{b}{1} \in S^{-1}Q$ so that $(\frac{a}{1})^2 = \frac{a^2}{1} \in S^{-1}Q$ or $(\frac{b}{1})^n = \frac{b^n}{1} \in S^{-1}Q$ for some $n \in \mathbb{N}$. This implies that $ua^2 \in Q$ or $tb^n \in Q$ for some $u, t \in S$. Since $S \cap Zd_R(Q) = \emptyset$, we get either $a^2 \in Q$ or $b^n \in Q$ which is required. □

Proposition 2.6. *Let $Q_1, Q_2, \ldots, Q_n$ be strongly quasi primary ideals with $\sqrt{Q_i} = P$ for each $i = 1, 2, \ldots, n$. Then $Q = \bigcap_{i=1}^{n} Q_i$ is a strongly quasi primary ideal of R.*

Proof. Let $ab \in Q$ for some $a, b \in R$. Assume that $b^k \notin Q$ for all $k \in \mathbb{N}$, that is, $b \notin P$. Since $ab \in Q_i$ and $b^k \notin Q_i$ for all $k \in \mathbb{N}$, we have $a^2 \in Q_i$ and so $a^2 \in Q$. Thus Q is a strongly quasi primary ideal of R. □

Lemma 2.1. *Let $R = R_1 \times R_2$ and $Q = Q_1 \times Q_2$, where Q_i's are ideals of R_i for $i = 1, 2$. Then the following statements are equivalent*:

(i) *Q is a strongly quasi primary ideal of R.*

(ii) *$Q_1 = R_1$ and Q_2 is a strongly quasi primary ideal of R_2 or $Q_2 = R_2$ and Q_1 is a strongly quasi primary ideal of R_1.*

Proof. (i)⇒(ii) Suppose that Q is a strongly quasi primary ideal of R. Then by Proposition 2.1, $\sqrt{Q} = \sqrt{Q_1} \times \sqrt{Q_2}$ is a prime ideal and so $Q_1 = R_1$ or $Q_2 = R_2$. Without loss of generality, we may assume that $Q_1 = R_1$. Now, we will show that Q_2 is a strongly quasi primary ideal of R_2. Let $ab \in Q_2$ for some $a, b \in R_2$. This implies that $(1, a)(1, b) \in Q$. Since Q is a strongly quasi primary ideal of R, $(1, a)^2 = (1, a^2) \in Q$ or $(1, b)^n = (1, b^n) \in Q$ for some $n \in \mathbb{N}$. Then we conclude that $a^2 \in Q_2$ or $b^n \in Q_2$. Consequently, Q_2 is a strongly quasi primary ideal of R_2.

(ii)⇒(i) Assume that $Q_1 = R_1$ and Q_2 is a strongly quasi primary ideal of R_2. Let $(a, b)(x, y) \in Q$ for some $a, x \in R_1$ and $b, y \in R_2$. Then we get $by \in Q_2$ and so either $b^2 \in Q_2$ or $y^n \in Q_2$ for some $n \in \mathbb{N}$. This implies that $(a, b)^2 \in Q$ or $(x, y)^n \in Q$ which is needed. In other case, one can similarly show that Q is a strongly quasi primary ideal of R. □

Theorem 2.1. *Let $R = R_1 \times R_2 \times \cdots \times R_n$ and $Q = Q_1 \times Q_2 \times \cdots \times Q_n$, where Q_i's are ideals of R_i and $n \in \mathbb{N}$. Then the following statements are equivalent*:

(i) *Q is a strongly quasi primary ideal of R.*

(ii) *Q_t is a strongly quasi primary ideal of R_t for some $t \in \{1, 2, \ldots, n\}$ and $Q_j = R_j$ for all $j \neq t$.*

Proof. We use induction on n. If $n = 1$, the result is valid. If $n = 2$, (i)⇔(ii) follows from Lemma 2.1. Assume that the claim is valid for all $k \in \mathbb{N}$ with $k < n$. Now suppose that $Q = Q_1 \times Q_2 \times \cdots \times Q_n$ and $R = R_1 \times R_2 \times \cdots \times R_n$. Put $Q' = Q_1 \times Q_2 \times \cdots \times Q_{n-1}$ and $R' = R_1 \times R_2 \times \cdots \times R_{n-1}$. Then by Lemma 2.1, $Q = Q' \times Q_n$ is a strongly quasi primary ideal of $R = R' \times R_n$ if and only if Q' is a strongly quasi primary ideal of R' and $Q_n = R_n$ or $Q' = R'$ and Q_n is a strongly quasi primary ideal of R_n. The rest follows by induction hypothesis. □

Theorem 2.2. *Let R be an integral domain. Then the following statements are equivalent*:

(i) *R is a divided ring.*

(ii) *Every proper principal ideal is a strongly quasi primary ideal.*

(iii) *Every proper ideal is a strongly quasi primary ideal.*

Proof. (i)⇒(ii) Let R be a divided ring and (x) be a proper ideal of R. Assume that $ab \in (x)$ for some $a, b \in R$. Since R is divided ring, by [7, Proposition

2], we have either $b|a$ or $a|b^n$ for some $n \in \mathbb{N}$. If $b|a$, then $a = kb$ and so $a^2 = k(ab) \in (x)$. Otherwise, we have $a|b^n$ and so $b^n = sa$ and this yields that $b^{n+1} = s(ab) \in (x)$. Thus, (x) is a strongly quasi primary ideal.

(ii)⇒(iii) Suppose that Q is a proper ideal and $ab \in Q$ for some $a, b \in R$. Since $ab \in (ab)$ and (ab) is a strongly quasi primary ideal, we have $a^2 \in (ab) \subseteq Q$ or $b^n \in (ab) \subseteq Q$ for some $n \in \mathbb{N}$.

(iii)⇒(i) Suppose that every proper ideal is a strongly quasi primary ideal. Let $a, b \in R$. Assume that a and b are not unit. Put $Q = (ab)$. Since Q is a strongly quasi primary ideal and $ab \in Q$, we have $a^2 \in (ab)$ or $b^n \in (ab)$ for some $n \in \mathbb{N}$. If $a^2 \in (ab)$, then $a^2 = abc$ for some $c \in R$ and so $a = bc$, i.e., $b|a$. If $b^n \in (ab)$, then $b^n = abs$ for some $s \in R$ and so $b^{n-1} = as$ so that $a|b^{n-1}$. Then by [7, Proposition 2], R is a divided ring. □

Let R be a ring and I a proper ideal of R. For any $f(X) = a_0 + a_1X + \cdots + a_nX^n \in R[X]$, the content $c(f)$ of f is defined as $c(f) = (a_0, a_1, \ldots, a_n)$. Also note that $I[X] = \{f \in R[X] : c(f) \subseteq I\}$ is an ideal of $R[X]$. Similarly, for any $f(X) = \sum_{i=0}^{\infty} a_iX^i \in R[[X]]$, the content $c(f)$ of f is defined by $c(f) = (\{a_i : i \in \mathbb{N}\})$. Furthermore, $I[[X]] = \{f = \sum_{i=0}^{\infty} a_iX^i \in R[[X]] : c(f) \subseteq I\}$ is an ideal of $R[[X]]$.

Theorem 2.3. *Let R be a ring and 2 be a unit of R. Then the following statements are equivalent*:

(i) *Q is a strongly quasi primary ideal of R.*

(ii) *$Q[X]$ is a strongly quasi primary ideal of $R[X]$.*

Proof. (i)⇒(ii) Suppose that Q is a strongly quasi primary ideal of R. Take $fg \in Q[X]$ with $g \notin \sqrt{Q[X]}$. First note that $\sqrt{Q[X]} = \sqrt{Q}[X]$. Since $g \notin \sqrt{Q[X]}$, we have $c(g) \nsubseteq \sqrt{Q}$. Also note that $c(fg) \subseteq Q$. Then by [13, Theorem 28.1], $c(f)c(g)^{n+1} = c(g)^n c(fg) \subseteq Q$, where $n = \deg(f)$. Since $c(g) \nsubseteq \sqrt{Q}$, $c(g)^{n+1} \nsubseteq \sqrt{Q}$. As Q is a strongly quasi primary ideal of R, by Corollary 2.1, we have $c(f)^2 \subseteq Q$. Since $c(f^2) \subseteq c(f)^2$, we get $c(f^2) \subseteq Q$ and so $f^2 \in Q[X]$. Hence, $Q[X]$ is a strongly quasi primary ideal of R.

(ii)⇒(i) Suppose that $Q[X]$ is a strongly quasi primary ideal of $R[X]$. Consider the injection $i : R \to R[X]$ defined by $i(a) = a$ for each $a \in R$. Then by Proposition 2.4(ii), $i^{-1}(Q[X]) = Q$ is a strongly quasi primary ideal of R. □

Theorem 2.4. *Let R be a Noetherian ring and 2 be unit of R. A proper ideal Q of R is a strongly quasi primary if and only if $Q[[X]]$ is a strongly quasi primary ideal of $R[[X]]$.*

Proof. (⇐) Since R is a subring of $R[[X]]$ and $Q = Q[[X]] \cap R$, then Q is a strongly quasi primary ideal of R by Corollary 2.2(ii).

(⇒) Let Q be a strongly quasi primary ideal of R. Let $fg \in Q[[X]]$ with $g \notin \sqrt{Q[[X]]}$. First note that $\sqrt{Q[[X]]} = \sqrt{Q}[[X]]$ by [1, Lemma 2.2]. Then we have $c(fg) \subseteq Q$ and $c(g) \nsubseteq \sqrt{Q}$. Since R is Noetherian, there exists a minimal number $\mu(c(f))$ of generators of $c(f)$. Thus we can choose $n \in \mathbb{N}$

as the maximum of the numbers $\mu(c(f)_m)$, taken over all maximal ideals m of R. Then by [11, Theorem 2.6], $c(f)c(g)^n = c(g)^{n-1}c(fg) \subseteq Q$. Since $c(g) \nsubseteq \sqrt{Q}$, $c(g)^n \nsubseteq Q$. Then $c(f)^2 \subseteq Q$ by Corollary 2.1 since Q is strongly quasi primary. As $c(f^2) \subseteq c(f)^2$, we get $c(f^2) \subseteq Q$ and so $f^2 \in Q[[X]]$. Hence, $Q[[X]]$ is a strongly quasi primary ideal of $R[[X]]$. □

Let M be an R-module. The idealization $R(+)M = \{(r, m) : r \in R, m \in M\}$ of M is a commutative ring with componentwise addition and multiplication

$$(a, m)(b, m') = (ab, am' + bm)$$

for each $a, b \in R$; $m, m' \in M$. Assume that I is an ideal of R and N is a submodule of M. Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$ [17] and [14]. In this case, $I(+)N$ is called a homogeneous ideal of $R(+)M$. In [3], radical of a homogeneous ideal is characterized as follows:

$$\sqrt{I(+)N} = \sqrt{I}(+)M.$$

Theorem 2.5. *Suppose that $Q(+)N$ is a homogeneous ideal of $R(+)M$.*

(i) *If $Q(+)N$ is a strongly quasi primary ideal of $R(+)M$, then Q is a strongly quasi primary ideal of R.*

(ii) *If Q is a strongly quasi primary ideal of R with $\sqrt{Q}M \subseteq N$, then $Q(+)N$ is a strongly quasi primary ideal of $R(+)M$.*

Proof. (i) Let $ab \in Q$ for some $a, b \in R$. Then note that $(a, 0)(b, 0) = (ab, 0) \in Q(+)N$. Since $Q(+)N$ is a strongly quasi primary ideal, $(a, 0)^2 = (a^2, 0) \in Q(+)N$ or $(b, 0)^n = (b^n, 0) \in Q(+)N$ for some $n \in \mathbb{N}$. This implies that $a^2 \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$. Hence, Q is a strongly quasi primary ideal of R.

(ii) Suppose that Q is a strongly quasi primary ideal of R with $\sqrt{Q}M \subseteq N$. Let $(a, m)(b, m') = (ab, am' + bm) \in Q(+)N$ for some $a, b \in R$; $m, m' \in M$. This implies that $ab \in Q$ and so $a^2 \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$. If $b^n \in Q$, then $b^n m' \in QM \subseteq N$ and so $(b, m')^{n+1} = (b^{n+1}, (n + 1)b^n m') \in Q(+)N$. Otherwise, we would have $a^2 \in Q$ and so $am \in \sqrt{Q}M \subseteq N$ and this yields $(a, m)^2 = (a^2, 2am) \in Q(+)N$ and thus $Q(+)N$ is a strongly quasi primary ideal of $R(+)M$. □

Corollary 2.4. *Let Q be a proper ideal of R and N a submodule of M with $(N : M) = \sqrt{(N : M)}$. Suppose that $Q(+)N$ is a homogeneous ideal of $R(+)M$. Then $Q(+)N$ is a strongly quasi primary ideal of $R(+)M$ if and only if Q is a strongly quasi primary ideal of R.*

3. Application to graph theory

The study of zero divisor graph of commutative ring R, $\Gamma(R)$ goes back to Beck [9], where it was first studied for coloring of a commutative ring R. Afterwards, Anderson and Livingston (in [5]) redefined and studied zero divisor graph by taking the set of vertices as just nonzero zero divisors $zd^*(R)$ of R. According to their definition, any two distinct vertices x and y are adjacent if

$xy = 0$. Then Redmond in [18] generalized this graph with respect to given an ideal I of R and called this graph: ideal based zero divisor graph. Let R be a ring and I be a proper ideal of R. The ideal based zero divisor graph $\Gamma_I(R)$ of R is a simple graph with a vertex set $V_I(R) = \{x \in R - I : xy \in I$ for some $y \in R - I\}$ such that any two distinct vertices x, y are adjacent if $xy \in I$ [18]. Note that if I is zero ideal, then ideal based zero divisor graph is the exactly zero divisor graph. The author in [18], gave many properties of ideal based zero divisor graph and investigated the relations between graph properties of $\Gamma_I(R)$ and algebraic properties of R. Our aim in this section is to construct a subgraph of ideal based zero divisor graph characterizing strongly quasi primary ideals and to find answers to many questions on this subgraph such as when two graphs are equal? Now, take the set of vertices $V_I^*(R) = V_I(R)$. We say that, for any distinct vertices x and y are adjacent if $xy \in I$ with $x^2 \notin I, y \notin \sqrt{I}$ or $x \notin \sqrt{I}, y^2 \notin I$. Then we denote this graph by $\Gamma_I^*(R)$. Note that $\Gamma_I^*(R)$ is a subgraph of ideal based zero divisor graph $\Gamma_I(R)$.

Definition 3.1. A graph G is called edgeless (n-empty) if it has some vertices (n vertices) but no edges. In particular, 0-empty graph is just called empty graph.

It was shown (in [18, Proposition 2.2]) that $\Gamma_I(R)$ is empty graph if and only if I is prime ideal, i.e., R/I is an integral domain. Now we characterize edgeless graph in terms of strongly quasi primary ideals.

Proposition 3.1. *Let R be a ring and I a proper ideal of R. Then*

(i) *I is a prime ideal if and only if $\Gamma_I^*(R)$ is an empty graph. In this case, $\Gamma_I^*(R) = \Gamma_I(R)$.*

(ii) *Let I be a non prime ideal. If $|V_I(R)| = 1$, then $\Gamma_I^*(R)$ is edgeless (1-empty) and $\Gamma_I^*(R) = \Gamma_I(R)$.*

(iii) *Let $|V_I(R)| \geq 2$. Then I is a strongly quasi primary ideal if and only if $\Gamma_I^*(R)$ is an edgeless graph.*

(iv) *If I is not a strongly quasi primary ideal, then $|V_I(R)| \geq 2$ and $\Gamma_I^*(R)$ can not be an edgeless graph.*

(v) *If I is a radical ideal, i.e., $\sqrt{I} = I$, then $\Gamma_I^*(R) = \Gamma_I(R)$. In particular, if R is a reduced ring then $\Gamma_0^*(R)$ and $\Gamma(R)$ are the same graph.*

Proof. (i), (ii): It is clear.

(iii) Assume that $|V_I(R)| \geq 2$. Suppose that I is a strongly quasi primary ideal. Now we will show that $\Gamma_I^*(R)$ is edgeless. Suppose to the contrary. Then there exist two distinct vertices x, y that are adjacent in $\Gamma_I^*(R)$. This implies that $xy \in I$ with $x^2 \notin I,\ y \notin \sqrt{I}$ or $x \notin \sqrt{I},\ y^2 \notin I$. Assume that $x^2 \notin I,\ y \notin \sqrt{I}$. Since I is a strongly quasi primary ideal and $xy \in I$. Then we get either $x^2 \in I$ or $y \in \sqrt{I}$, a contradiction. If $x \notin \sqrt{I},\ y^2 \notin I$, similarly we can get a contradiction since I is a strongly quasi primary ideal and $xy \in I$. Now, assume that $\Gamma_I^*(R)$ is an edgeless graph. Suppose that I is not a strongly quasi primary ideal. Then there exist $x, y \in R$ such that $xy \in I$ with $x^2 \notin I,\ y^n \notin I$

for all $n \in \mathbb{N}$. Since $x^2 \notin I$ and $xy \in I$, we have $x \neq y$ and $x, y \in V_I(R)$. Thus x, y are adjacent in $\Gamma_I^*(R)$, a contradiction. Hence, I is a strongly quasi primary ideal of R.

(iv) Follows from (iii).

(v) Suppose that $I = \sqrt{I}$. We may assume that $|V_I(R)| \geq 2$. If I is a strongly quasi primary ideal, then I is prime since $I = \sqrt{I}$ so that $|V_I(R)| = 0$, a contradiction. Let x, y be two distinct vertices that are adjacent in $\Gamma_I^*(R)$. Then it is clear that x, y are adjacent in $\Gamma_I(R)$. Now, assume that x, y are adjacent in $\Gamma_I(R)$ and not adjacent in $\Gamma_I^*(R)$. Then $xy \in I$ and also the condition $x^2 \notin I, y \notin \sqrt{I}$ or $x \notin \sqrt{I}, y^2 \notin I$ does not imply. Thus we have $x \in \sqrt{I}$ or $y \in \sqrt{I}$. Since $\sqrt{I} = I$, we have either $x \in I$ or $y \in I$, a contradiction. Hence, $\Gamma_I^*(R) = \Gamma_I(R)$. □

Let G be a graph and x, y be two distinct vertices. Then we denote $x - y$ if $x \neq y$ and x, y are adjacent. Also $x_1 - x_2 - x_3 - \cdots - x_n$ is called a path from x_1 to x_n of length $n-1$ if all $x_i \neq x_j$ for $i \neq j$. A graph G is connected if for any distinct vertices x and y, there exists a path from x to y. Otherwise, G is called disconnected.

Example 3.1. (i) Consider the ring of integers modulo 6, $\mathbb{Z}_6$. One can see that $\Gamma_0^*(\mathbb{Z}_6)$ is a line graph as in Figure 2.

(ii) Consider the ring of integers modulo 12, $\mathbb{Z}_{12}$. In Figure 3, compare $\Gamma_0^*(\mathbb{Z}_{12})$ and $\Gamma_0(\mathbb{Z}_{12})$. Note that $\Gamma_I^*(R)$ may be disconnected while $\Gamma_I(R)$ is always connected [18, Theorem 2.5].

FIGURE 2. $\Gamma_0^*(\mathbb{Z}_6)$

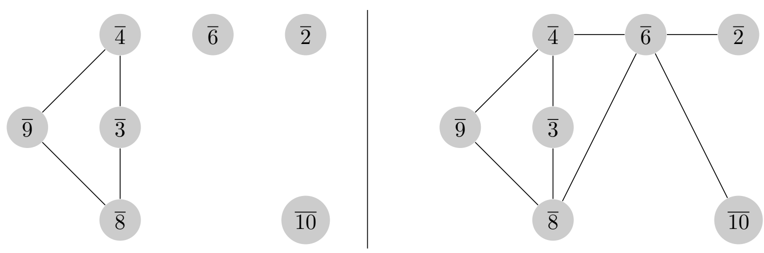

FIGURE 3. $\Gamma_0^*(\mathbb{Z}_{12})$ vs $\Gamma_0(\mathbb{Z}_{12})$

Let $a \in V_I(R)$. Then we denote the set of all vertices that are adjacent to a in $\Gamma_I^*(R)$ by $N_{\Gamma^*}(a)$. Similarly, $N_\Gamma(a) = \{b \in V_I(R) : b - a \text{ is an edge of } \Gamma_I(R)\}$.

Proposition 3.2. *Suppose that $a \in V_I(R)$ such that $N_{\Gamma^*}(ra) = N_\Gamma(ra)$ for all $r \notin (I : a)$. Then either $\sqrt{(I : a)} = (I : a)$ or $\sqrt{(I : a)} = \sqrt{I}$.*

Proof. The inclusion $(I : a) \cup \sqrt{I} \subseteq \sqrt{(I : a)}$ always holds. It is sufficient to show that the reverse inclusion also holds when the condition $N_{\Gamma^*}(ra) = N_\Gamma(ra)$ is satisfied for all $r \notin (I : a)$. Let $b \in \sqrt{(I : a)}$. Then $b^n a \in I$ for some $n \in \mathbb{N}$. Choose the smallest aforementioned integer $n \in \mathbb{N}$. If $n = 1$, then we are done. Assume that $n \geq 2$. Then $b^{n-1}a \notin I$. If $b = b^{n-1}a$, then $b(b^{n-1}a) = b^2 = b^n a \in I$ and so $b \in \sqrt{I}$. Now assume that $b \neq b^{n-1}a$. Since $b(b^{n-1}a) \in I$, b and $b^{n-1}a$ are adjacent in $\Gamma_I(R)$, that is, $b \in N_\Gamma(b^{n-1}a)$. Since $b^{n-1} \notin (I : a)$, by assumption, $b \in N_\Gamma(b^{n-1}a) = N_{\Gamma^*}(b^{n-1}a)$ so that b and $b^{n-1}a$ are adjacent in $\Gamma_I^*(R)$. This implies that $b^2 \notin I, (b^{n-1}a) \notin \sqrt{I}$ or $b \notin \sqrt{I}, (b^{n-1}a)^2 \notin I$. But note that $(b^{n-1}a)^2 = b^{2n-2}a^2 \in I$ since $2n - 2 \geq n$. Hence, we have $\sqrt{(I : a)} \subseteq (I : a) \cup \sqrt{I}$. Thus we have either $\sqrt{(I : a)} = (I : a)$ or $\sqrt{(I : a)} = \sqrt{I}$. □

An isolated point x of a graph G is a point that there is no edge between x and y for any point y of G. As one can see in Figure 3, in the graph $\Gamma_I^*(R)$ there may exist some isolated points. The following proposition detects the isolated points in the graph $\Gamma_I^*(R)$.

Proposition 3.3. *Let $x \in V_I(R)$.*

(i) *Assume that $x \in \sqrt{I}$. If $x^2 \in I$, then $N_{\Gamma^*}(x) = \emptyset$.*

(ii) *Assume that $x \in \sqrt{I}$ with $x^2 \notin I$. Then $N_{\Gamma^*}(x) = \emptyset$ if and only if $(I : x) \subseteq \sqrt{I}$.*

(iii) *Assume that $x \notin \sqrt{I}$. If $N_{\Gamma^*}(x) = \emptyset$ if and only if $(I : x)_2 \subseteq I$.*

Proof. (i) It is clear.

(ii) Suppose that $x^2 \notin I$ and $(I : x) \subseteq \sqrt{I}$. Let $y - x$ be an edge of $\Gamma_I^*(R)$. Then $yx \in I$ with $x^2 \notin I, y \notin \sqrt{I}$. Since $y \in (I : x) \subseteq \sqrt{I}$, we have a contradiction so that $N_{\Gamma^*}(x) = \emptyset$. Conversely, assume that $N_{\Gamma^*}(x) = \emptyset$. Let $y \in (I : x)$. Then $yx \in I$. If $y \in \sqrt{I}$, then we are done. Assume that $y \notin \sqrt{I}$. Since $x^2 \notin I$, we have $x \neq y$ and so $x - y$ is an edge of $\Gamma_I^*(R)$, a contradiction.

(iii) Let $N_{\Gamma^*}(x) = \emptyset$ for some $x \notin \sqrt{I}$. Now, we will show that $(I : x)_2 \subseteq I$. Suppose not. Then there exists $y \in (I : x)$ such that $y^2 \notin I$. Then $yx \in I$. As $y^2 \notin I$, note that $y \neq x$ and so $x - y$ is an edge in $\Gamma_I^*(R)$, a contradiction. Conversely, assume that $(I : x)_2 \subseteq I$. Let $x - y$ be an edge of $\Gamma_I^*(R)$. Then $yx \in I$ and $y^2 \notin I, x \notin \sqrt{I}$ or $y \notin \sqrt{I}, x^2 \notin I$. Since $y \in (I : x)$, by assumption, $y^2 \in I$, a contradiction. Hence, we have $N_{\Gamma^*}(x) = \emptyset$. □

As a consequence of the previous proposition, we give the following results.

Corollary 3.1. (i) *Let $x \in V_I(R)$ with $x^2 \notin I$. If $(I : x) \nsubseteq \sqrt{I}$, then $N_{\Gamma^*}(x) \neq \emptyset$.*

(ii) *Let R be a ring and I a proper ideal of R. Assume that for all $x \in V_I(R)$, $x^2 \notin I$ and $(I : x) \nsubseteq \sqrt{I}$, then $\Gamma_I^*(R)$ has no isolated point.*

As one can see in Figure 3, $\Gamma_I^*(R)$ and $\Gamma_I(R)$ are not the same graph since $\Gamma_I^*(R)$ may not be connected while $\Gamma_I(R)$ is always connected. The following theorem give an answer when $\Gamma_I^*(R)$ and $\Gamma_I(R)$ are coincide or when $\Gamma_I^*(R)$ is connected.

Theorem 3.1. *Let I be a proper ideal of R and $|V_I(R)| \geq 2$. Then the following statements are equivalent*:

(i) *$\Gamma_I^*(R)$ is a connected graph.*

(ii) *$I = \sqrt{I}$, i.e., R/I is a reduced ring.*

(iii) *$\Gamma_I^*(R) = \Gamma_I(R)$.*

Proof. (i)⇒(ii) Assume that $\Gamma_I^*(R)$ is a connected graph. Now, we will show that $I = \sqrt{I}$. Assume that $I \neq \sqrt{I}$. Then there exists $x \in \sqrt{I} - I$. This implies that $x^n \in I$ for some $n \in \mathbb{N}$. Let n be the smallest positive integer such that $x^n \in I$. Note that $x^{n-1} \notin I$ and $x^{n-1} \in V_I(R)$ since $xx^{n-1} = x^n \in I$ and $x \notin I$. As $\Gamma_I^*(R)$ is a connected graph, there exists $y \in V_I(R)$ such that x^{n-1} and y are adjacent in $\Gamma_I^*(R)$.This implies that $x^{n-1}y \in I$ with $(x^{n-1})^2 \notin I, y \notin \sqrt{I}$ or $x^{n-1} \notin \sqrt{I}, y^2 \notin I$. Since $2(n-1) \geq n$, we have $(x^{n-1})^2 = x^{2(n-1)} \in I$, a contradiction. Thus $\sqrt{I} = I$ which completes the proof.

(ii)⇒(iii) Follows from Proposition 3.1(v).

(iii)⇒(i) Follows from [18, Theorem 2.4]. □

Let G be a graph and x, y be two distinct vertices. Then $d(x, y)$ is the shortest length of a path from x to y. If there is no such a path, $d(x, y) = \infty$. Also, diameter of G, $diam(G) = \sup\{d(x, y) : x \neq y\}$.

Corollary 3.2. *Let R be a ring and I a proper ideal of R such that $|V_I(R)| \geq 2$.*

(i) *If $I \neq \sqrt{I}$, then $diam(\Gamma_I^*(R)) = \infty$.*

(ii) *If $I = \sqrt{I}$, then $diam(\Gamma_I^*(R)) \leq 3$.*

Proof. (i), (ii): Follows from previous Theorem and [18, Theorem 2.4]. □

As a consequence of Theorem 3.1, we determine when a ring R is von Neumann regular in terms of strongly quasi primary ideals and $\Gamma_I^*(R)$.

Corollary 3.3. *Let R be a ring. Then the following statements are equivalent*:

(i) *R is a von Neumann regular ring.*

(ii) *Every strongly quasi primary ideal is prime and for any non strongly quasi primary ideal I of R, $\Gamma_I^*(R) = \Gamma_I(R)$.*

Recall that a graph G is a bipartite if the vertex set is a union of two disjoint subvertex set, i.e., $V(G) = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ and also each edge of G joins a vertices of V_1 to a vertices V_2. A complete bipartite graph G is a bipartite graph such that all vertices $x_1 \in V_1$ and $x_2 \in V_2$ are adjacent.

Theorem 3.2. *Let I be a proper ideal of R. Then the following statements are equivalent*:

(i) *$\Gamma_I^*(R)$ is a complete bipartite graph.*

(ii) *$I = P_1 \cap P_2$ for some distinct prime ideals P_1 and P_2 of R that are minimal over I.*

Proof. (i)⇒(ii) Assume that $\Gamma_I^*(R)$ is a complete bipartite graph. Then $|V_I(R)| \geq 2$ and also $\Gamma_I^*(R)$ is connected. By Theorem 3.1, $I = \sqrt{I}$ and $\Gamma_I^*(R) = \Gamma_I(R)$. Now, we will show that I is a 2-absorbing ideal of R. Let $abc \in I$ for some $a, b, c \in R$ with $ab \notin I$ and $ac \notin I$. If $ab = ac$, then $abc = ac^2 \in I$ and so $ac \in \sqrt{I} = I$, a contradiction. So assume that $ab \neq ac$. Also it is clear that $ab - ac$ is an edge of $\Gamma_I(R)$. As $\Gamma_I^*(R) = \Gamma_I(R)$ is a complete bipartite graph, we can write $V_I(R) = V_1 \cup V_2$ such that $V_1 \cap V_2 = \emptyset$. Without loss of generality, we may assume that $ab \in V_1$ and $ac \in V_2$. Now, we will show that $a \in R$ is not a vertex. Suppose to the contrary. Then $a \in V_I(R) = V_1 \cup V_2$. Then either $a \in V_1$ or $a \in V_2$. Assume that $a \in V_1$. If $a = ab$, then $ac = abc \in I$, a contradiction. Thus $a \neq ab$. Since $\Gamma_I^*(R)$ is complete bipartite, $a - ac$ is an edge of $\Gamma_I(R)$ so that $a(ac) = a^2c \in I$ and this yields that $ac \in \sqrt{I} = I$, a contradiction. If $a \in V_2$, one can similarly get a contradiction. Then we have $a \notin V_I(R)$. Now, we will show that $(I : a) \subseteq I$. Assume that $x \in (I : a) - I$. Then $xa \in I$. As x and a are not in I, we have $a \in V_I(R)$, a contradiction. Thus $(I : a) \subseteq I$ and so $bc \in (I : a) \subseteq I$. Hence, I is a 2-absorbing ideal of R. Then by [8, Theorem 2.4], $I = \sqrt{I} = P$ for some prime ideal P of R or $I = \sqrt{I} = P_1 \cap P_2$ for some distinct prime ideals P_1 and P_2 of R that are minimal over I. If $I = \sqrt{I} = P$, then $\Gamma_I^*(R) = \Gamma_I(R)$ is empty, a contradiction so that $I = P_1 \cap P_2$ for some distinct prime ideals P_1 and P_2 of R that are minimal over I.

(ii)⇒(i) First, note that $V_I(R) = V_1 \cup V_2$, where $V_1 = \{x \in R - I : x \in P_1 - P_2\}$ and $V_2 = \{x \in R - I : x \in P_2 - P_1\}$. Also, it is clear that $V_1 \cap V_2 = \emptyset$. Now, we will show that $\Gamma_I^*(R)$ is complete bipartite. Let $x \in V_1$ and $y \in V_2$. Then $xy \in I = P_1 \cap P_2$ since $xy \in P_1$ and $xy \in P_2$. It is clear that $x^2 \notin I$, $y \notin \sqrt{I}$ and also $x \notin \sqrt{I}$, $y^2 \notin I$. □

References

[1] M. Achraf, H. Ahmed, and B. Ali, *2-absorbing ideals in formal power series rings*, Palestine J. Math. **6** (2017), no. 2, 502–506.

[2] D. D. Anderson, K. R. Knopp, and R. L. Lewin, *Ideals generated by powers of elements*, Bull. Australian Math. Soc. **49** (1994), no. 3, 373–376.

[3] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commutative Algebra **1** (2009), no. 1, 3–56.

[4] D. F. Anderson, R. Levy, and J. Shapiro, *Zero-divisor graphs, von Neumann regular rings, and Boolean algebras*, J. Pure Appl. Algebra **180** (2003), no. 3, 221–241.

[5] D. F. Anderson and P. S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), no. 2, 434–447.

[6] M. Atiyah, *Introduction to Commutative Algebra*, CRC Press, 2018.

[7] A. Badawi, *On divided commutative rings*, Comm. Algebra **27** (1999), no. 3, 1465–1474.

[8] ———, *On 2-absorbing ideals of commutative rings*, Bull. Austral. Math. Soc. **75** (2007), no. 3, 417–429.

[9] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988), no. 1, 208–226.

[10] C. Beddani and W. Messirdi, 2-*Prime ideals and their applications*, J. Algebra and Its Applications **15** (2016), no. 3, 1650051.
[11] N. Epstein and J. Shapiro, *A Dedekind-Mertens theorem for power series rings*, Proceedings of Amer. Math. Soc. **144** (2016), no. 3, 917–924.
[12] L. Fuchs, *On quasi-primary ideals*, Acta Univ. Szeged. Sect. Sci. Math. **11** (1947), 174–183.
[13] R. Gilmer, *Multiplicative Ideal Theory*, Marcel Dekker, Inc., New York, 1972.
[14] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Monographs and Textbooks in Pure and Applied Mathematics, **117**, Marcel Dekker, Inc., New York, 1988.
[15] C. Jayaram and Ü Tekir, *von Neumann regular modules*, Commun. Algebra **46** (2018), no. 5, 2205–2217.
[16] M. D. Larsen and P. J. McCarthy, *Multiplicative Theory of Ideals*, AcademicPress, New York, 1971.
[17] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers a division of John Wiley & Sons New York, 1962.
[18] S. P. Redmond, *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), no. 9, 4425–4443.
[19] J. von Neumann, *On regular rings*, Proceedings of the National Academy of Sci. **22** (1936), no. 12, 707–713.

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