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COEFFICIENT MULTIPLIERS ON DIRICHLET TYPE SPACES

Dongxing Li, Hasi Wulan, and Ruhan Zhao

Abstract. We characterize coefficient multipliers from certain Dirichlet type spaces to Hardy spaces and weighted Bergman spaces.

1. Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disk, $\mathbb{T} := \{\xi \in \mathbb{C} : |\xi| = 1\}$ be the unit circle and $\mathrm{Hol}(\mathbb{D})$ denote the space of all analytic functions on $\mathbb{D}$. If $f \in \mathrm{Hol}(\mathbb{D})$, its sequence of Taylor coefficients is denoted by $\{\widehat{f}(n)\}_{n=0}^{\infty}$.

If $0 < r < 1$ and f is an analytic function in $\mathbb{D}$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}}, \quad 0 < p < \infty, \tag{1.1}$$

$$M_\infty(r, f) = \max_{|z|=r} |f(z)|. \tag{1.2}$$

For $0 < p \leq \infty$, the Hardy space H^p consists of those functions f analytic in $\mathbb{D}$ for which

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r, f) < \infty. \tag{1.3}$$

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those f analytic on $\mathbb{D}$ such that

$$\|f\|^p_{A^p_\alpha} := (\alpha + 1)\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < \infty,$$

where $dA(z) = dx\,dy/\pi$ is the normalized area measure on $\mathbb{D}$. The unweighted Bergman space A^p_0 is simply denoted by A^p. By classical theorems, as in [16], the weighted Bergman integral is equivalent to the appropriate weighted integral of the derivative, which can be used to extend the concept of the weighted

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Bergman space to the more general case with $-\infty < \alpha < \infty$ as follows. For f analytic on $\mathbb{D}$, we say that $f \in A^p_\alpha$ if

$$\int_{\mathbb{D}} |f^{(k)}(z)|^p (1-|z|^2)^{kp+\alpha}\, dA(z) < \infty$$

for any nonnegative integer k such that $kp + \alpha > -1$.

For $\alpha > -1$, the weighted Dirichlet space $\mathcal{D}^p_\alpha$ consists of those f analytic on $\mathbb{D}$ such that

$$\|f\|^p_{\mathcal{D}^p_\alpha} := (\alpha+1) \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha dA(z) < \infty.$$

Note that weighted Dirichlet spaces are actually special cases of the above general weighted Bergman spaces; that is, for $\alpha > -1$, $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$.

When $\alpha = 0$, the unweighted Dirichlet space will be simply denoted by $\mathcal{D}^p$. For $p = 2$, $\mathcal{D}^2_\alpha$ is a Hilbert space, and we denote it simply by $\mathcal{D}_\alpha$. It is also well-known that $\mathcal{D}_1 = H^2$ (the Hardy space).

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ and $g = \sum_{n=0}^\infty b_n z^n$ be analytic functions on $\mathbb{D}$. The Hadamard product of f and g is defined by $f * g(z) = \sum_{n=0}^\infty a_n b_n z^n$. It is easy to check that $f * g$ has the following equivalent integral definition.

$$f * g(r^2 e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it}) g(re^{i(\theta-t)}) dt.$$

Note that, if we let $r = 1$ and simplify the tangential function of f and g as $f(t) := f(e^{it})$ and $g(\theta - t) := g(e^{i(\theta-t)})$, then the convolution of two boundary functions can be reformulated as

$$f * g(\theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) g(\theta - t) dt.$$

Their Fourier coefficients satisfy the relation $\widehat{(f * g)}(n) = \widehat{f}(n)\widehat{g}(n)$.

Although we are mainly interested in spaces of analytic functions, it is convenient to focus our attention on their Taylor coefficients. We will regard spaces of analytic functions on the unit disk as sequence spaces by identifying a function with its sequence of Taylor coefficients. Without causing confusion, we will use the same notation to denote this sequence space and its original function space.

Let A and B be two vector spaces of sequences. A sequences $\lambda = \{\lambda_n\}$ is said to be a coefficient multiplier from A to B if $\{\lambda_n \alpha_n\} \in B$ whenever $\{\alpha_n\} \in A$. The set of all multipliers from A to B will be denoted by (A, B).

Let X and Y be spaces of analytic functions on $\mathbb{D}$. A function $g \in \mathrm{Hol}(\mathbb{D})$ is said to be a (Hadamard) coefficient multiplier from X to Y if $f * g \in Y$ whenever $f \in X$. The set of all coefficient multipliers from X to Y is denoted by (X, Y). Since $f * g = g * f$, we know that $Y \subset (X, Z)$ is equivalent to $X \subset (Y, Z)$. For $X \subset Y \subset Z$, it is obvious that the inclusion relation $(Y, Z) \subset (X, Z)$ holds.

We may also identify a space X of analytic functions on $\mathbb{D}$ with the sequence space of Taylor coefficients of functions in X. In this way, we may talk about multipliers between sequence spaces and spaces of analytic functions on $\mathbb{D}$.

Coefficient multipliers were first studied in a series papers by Hardy, Littlewood and Paley (see, [4, 6, 9]), and later on have been extensively studied by many authors. Most of the work are for coefficient multipliers between Bergman spaces, Hardy spaces, or certain mixed normed spaces. See, for example, [1–3, 7, 8, 10–12, 14] and [15]. We refer to [8] and the references therein for a fairly complete account for this study. In this note we are interested in studying coefficient multipliers involve weighted Dirichlet spaces.

The rest of the paper is organized as follows. In Section 2, we characterize bounded fractional derivative operator from $\mathcal{D}^p$ to H^p. In Section 3, we characterize the multiplier from $\mathcal{D}^p_\alpha$ to A^q and H^q with mean growth of function.

2. Fractional derivative operators from $\mathcal{D}^p$ to H^p

It is known that, the general weighted Bergman space A^2_α $(\alpha > -1)$ is a Hilbert space with the orthonormal basis (see, p. 56 in [16])

$$\left\{ e_n = \sqrt{\frac{\Gamma(n+2+\alpha)}{n!\Gamma(\alpha+2)}} z^n, n \in \mathbb{N} \right\},$$

and the norm of f in A^2_α is given by

$$\|f\|_{A^2_\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+2+\alpha)}{n!\Gamma(\alpha+2)} |\widehat{f}(n)|^2.$$

By Stirling's formula, $\|f\|_{A^2_\alpha}$ is comparable to

$$\sum_{n=0}^{\infty} \frac{|\widehat{f}(n)|^2}{(n+1)^{1+\alpha}}.$$

It easily follows that if $\alpha, \beta > -1$, then $\{\lambda_n\} \in (A^2_\alpha, A^2_\beta)$ if and only if $\{n^{\frac{\alpha-\beta}{2}} \lambda_n\} \in l^\infty$. Since $\mathcal{D}_\alpha = A^2_{\alpha-2}$, we immediately get that $\{\lambda_n\} \in (\mathcal{D}_\alpha, \mathcal{D}_\beta)$ if and only if $\{n^{\frac{\alpha-\beta}{2}} \lambda_n\} \in l^\infty$.

For an analytic function $f(z) = \sum_{n=0}^\infty \widehat{f}(n) z^n$ on $\mathbb{D}$ and $t \in \mathbb{R}$, define the fractional derivative multiplier D^t as follows.

$$D^t f(z) = \sum_{n=0}^{\infty} n^t \widehat{f}(n) z^n.$$

We also denote by $D^t X = \{D^t f : f \in X\}$ for any function space X. Similarly, given a sequence $\{a_n\} \in \ell^p$ $(0 < p \le \infty)$, we define

$$D^t\{a_n\} = \{n^t a_n\}$$

and $D^t \ell^p = \{D^t\{a_n\} : \{a_n\} \in \ell^p\}$.

From the above discussions, we have the following results (note here we identify a space X of analytic functions with the sequence space of the Taylor coefficients of functions in X).

Proposition 2.1. *The following results holds.*

(i) *Let* $\alpha, \beta > -1$. *Then* $(A^2_\alpha, A^2_\beta) = (\mathcal{D}_\alpha, \mathcal{D}_\beta) = D^{\frac{\beta-\alpha}{2}} l^\infty$.

(ii) *Let* $-1 < \alpha \leq 1$. *Then* $(\mathcal{D}_\alpha, H^2) = (\mathcal{D}_\alpha, \mathcal{D}_1) = D^{\frac{1-\alpha}{2}} l^\infty$.

(iii) *Let* $\alpha > -1$. *Then* $(H^2, A^2_\alpha) = (D_1, \mathcal{D}_{\alpha+2}) = D^{\frac{1+\alpha}{2}} l^\infty$.

The following is the main result of this section.

Theorem 2.1. *Let* $1 \leq p \leq 2$. *The fractional derivative multiplier* $D^{1-\frac{1}{p}}$ *is a bounded operator from* $\mathcal{D}^p$ *to* H^p.

Proof. For each complex number ζ with $0 \leq \operatorname{Re}\zeta \leq 1$, we can define a linear operator T_ζ on holomorphic function space $\operatorname{Hol}(\mathbb{D})$ by

$$T_\zeta f(z) = \int_0^1 f'(rz) \left(\log \frac{1}{r}\right)^{-\frac{\zeta}{2}} dr, \qquad z \in \mathbb{D}.$$

For the case $p = 1$, by using Fubini's theorem, we can easily obtain that

$$\begin{aligned} \|T_\zeta f\|_{H^1} &= \frac{1}{2\pi} \int_0^{2\pi} |T_\zeta f(e^{it})|\, dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f'(re^{it})| \left(\log \frac{1}{r}\right)^{-\frac{\operatorname{Re}\zeta}{2}} dr\, dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f'(re^{it})|\, dr\, dt \qquad \text{for } \operatorname{Re}\zeta = 0. \end{aligned}$$

Recall that $dA(z) = \frac{r}{2\pi} drdt$. It is well known that (see, for example, Lemma 15 in [17]) there exists a constant $C > 0$ such that

$$\int_0^1 \int_0^{2\pi} |f'(re^{it})| dtdr \leq C \int_{\mathbb{D}} |f'(z)| dA(z).$$

That is, for each $\operatorname{Re}\zeta = 0$, the operator T_ζ maps $\mathcal{D}^1$ boundedly into H^1. Further more, for $\operatorname{Re}\zeta = 0$, there exists a constant $C > 0$ such that for any $f \in \mathcal{D}^1$,

$$\|T_\zeta f\|_{H^1} \leq C \|f\|_{\mathcal{D}^1}.$$

Next, we consider the case $p = 2$. For an analytic function $f(z) = \sum_{n=0}^\infty a_n z^n$, changing variables twice we get that

$$\begin{aligned} T_\zeta f(z) &= \int_0^1 f'(rz) \left(\log \frac{1}{r}\right)^{-\frac{\zeta}{2}} dr \\ &= \sum_{n=0}^\infty (n+1) a_{n+1} z^n \int_0^1 r^n \left(\log \frac{1}{r}\right)^{-\frac{\zeta}{2}} dr \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty}(n+1)a_{n+1}z^n \int_0^\infty e^{-(n+1)x}x^{-\frac{\zeta}{2}}\,dx \\
&= \sum_{n=0}^{\infty}(n+1)^{\frac{\zeta}{2}}a_{n+1}z^n \int_0^\infty e^{-y}y^{-\frac{\zeta}{2}}\,dy \\
&= \sum_{n=0}^{\infty}\Gamma\left(1-\frac{\zeta}{2}\right)(n+1)^{\frac{\zeta}{2}}a_{n+1}z^n.
\end{aligned}
$$

By using the norm expression of H^2 and $\mathcal{D}^2$ in terms of Taylor coefficients, we obtain that

$$
\begin{aligned}
\|T_\zeta f\|_{H^2}^2 &= \left|\Gamma\left(1-\frac{\zeta}{2}\right)\right|^2 \sum_{n=0}^{\infty}(n+1)^{\operatorname{Re}\zeta}|a_{n+1}|^2 \\
&\le C\sum_{n=0}^{\infty}(n+1)|a_{n+1}|^2 = C\|f\|_{\mathcal{D}^2}^2
\end{aligned}
$$

for all $f\in\mathcal{D}^2$, $\operatorname{Re}\zeta = 1$.

We prove remaining cases by an interpolation argument. Assume that $1 < p < 2, \frac{1}{p}+\frac{1}{q} = 1$ and $\theta = \frac{2}{q}$. We need to show that the operator T_θ maps $\mathcal{D}^p$ into H^p. It suffices to show that there exists a constant $C > 0$ such that $\|T_\theta f\|_{H^p} \le C\|f\|_{\mathcal{D}^p}$ for every analytic polynomials f. If we fix an analytic polynomial, then $T_\theta f$ is again another polynomial and

$$
\|T_\theta f\|_{H^p} = \sup\left\{\frac{1}{2\pi}\int_0^{2\pi} T_\theta f(e^{it})\overline{g(e^{it})}dt : \|g\|_{H^q} = 1\right\}.
$$

For $\theta\in[0,1]$ and Banach spaces X_0, X_1, we denote by $[X_0, X_1]_\theta$ the complex interpolation space between X_0 and X_1. Since $\mathcal{D}^1 = A^1_{-1}, \mathcal{D}^2 = A^2_{-2}, \mathcal{D}^p = A^p_{-p}$, and by Theorem 36 in [16], $[A^1_{-1}, A^2_{-2}]_\theta = A^p_{-p}$ for some $\theta\in[0,1]$ satisfying $\frac{1}{p} = \frac{1-\theta}{1}+\frac{\theta}{2}$. We get that $[\mathcal{D}^1,\mathcal{D}^2]_\theta = \mathcal{D}^p$ for some $\theta\in[0,1]$ satisfying $1-\frac{\theta}{2} = \frac{1}{p}$, or $\theta = \frac{2}{q}$. Thus there exist a family of functions $f_\zeta(z), (z\in\mathbb{D}, 0\le\operatorname{Re}\zeta\le 1)$ such that $f_\theta = f$, $f_\zeta\in\mathcal{D}^1$ for $\operatorname{Re}\zeta = 0$ and $f_\zeta\in\mathcal{D}^2$ for $\operatorname{Re}\zeta = 1$. What is more,

$$
\max\left\{\sup_{\operatorname{Re}\zeta=0}\|f_\zeta\|_{\mathcal{D}^1}, \sup_{\operatorname{Re}\zeta=1}\|f_\zeta\|_{\mathcal{D}^2}\right\} \le C_1\|f\|_{\mathcal{D}^p},
$$

where C_1 is a positive constant independent of f.

Let $g\in H^q$ such that $\|g\|_{H^q} = 1$. We define a function

$$
F(\zeta) = \frac{1}{2\pi}\int_0^{2\pi} T_\zeta f_\zeta(e^{it})\overline{g(e^{it})}|g(e^{it})|^{\frac{q\zeta-2}{2}}\,dt.
$$

It is clear that $F(\zeta)$ is analytic for $0 < \operatorname{Re}\zeta < 1$, continuous and bounded for $0\le\operatorname{Re}\zeta\le 1$.

For $\theta = \frac{2}{q}$, from previous calculation we know that

$$zT_\theta f(z) = \Gamma\left(1-\frac{\theta}{2}\right)\sum_{n=0}^{\infty}(n+1)^{\frac{\theta}{2}}a_{n+1}z^{n+1} = \Gamma(p^{-1})D^{1-\frac{1}{p}}f(z). \tag{2.1}$$

Since $f_\theta = f$, we have that

$$F(\theta) = \frac{1}{2\pi}\int_0^{2\pi} T_\theta f_\theta(e^{it})\overline{g(e^{it})}\,dt = \langle T_\theta f_\theta, g\rangle = \langle T_\theta f, g\rangle. \tag{2.2}$$

By the Hadamard three circle theorem, we obtain that

$$|F(\theta)| \le M_0^{1-\theta}M_1^{\theta},$$

where $M_0 = \sup\limits_{\mathrm{Re}\,\zeta=0}|F(\zeta)|$ and $M_1 = \sup\limits_{\mathrm{Re}\,\zeta=1}|F(\zeta)|$. Using Fubini's theorem and Hölder's inequality, there is a constant $C_1 > 0$ such that

$$\begin{aligned} M_0 &= \sup_{\mathrm{Re}\,\zeta=0}\left|\frac{1}{2\pi}\int_0^{2\pi} T_\zeta f_\zeta(e^{it})\overline{g(e^{it})}|g(e^{it})|^{\frac{q\zeta-2}{2}}\,dt\right| \\ &\le \sup_{\mathrm{Re}\,\zeta=0}\frac{1}{2\pi}\int_0^{2\pi}|T_\zeta f_\zeta(e^{it})|\,dt \\ &\le C_1\sup_{\mathrm{Re}\,\zeta=0}\|f_\zeta\|_{\mathcal{D}^1} \\ &\le C_1\|f\|_{\mathcal{D}^p}. \end{aligned}$$

Using the Cauchy-Schwartz inequality and Hölder's inequality, there is a constant $C_2 > 0$ such that

$$\begin{aligned} M_1 &\le \sup_{\mathrm{Re}\,\zeta=1}\frac{1}{2\pi}\int_0^{2\pi}|T_\zeta f_\zeta(e^{it})|\cdot|g(e^{it})|^{\frac{q}{2}}\,dt \\ &\le \sup_{\mathrm{Re}\,\zeta=1}\left(\frac{1}{2\pi}\int_0^{2\pi}|T_\zeta f_\zeta(e^{it})|^2\,dt\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\int_0^{2\pi}|g(e^{it})|^q\,dt\right)^{\frac{1}{2}} \\ &\le C_2\sup_{\mathrm{Re}\,\zeta=1}\|f_\zeta\|_{\mathcal{D}^2} \\ &\le C_2\|f\|_{\mathcal{D}^p}. \end{aligned}$$

Therefore there is a constant $C > 0$ such that

$$|F(\theta)| \le C\|f\|_{\mathcal{D}^p}.$$

It follows from (2.2) that

$$\|T_\theta f\|_{H^p} \le C\|f\|_{\mathcal{D}^p},$$

and so $T_\theta f \in H^p$. The boundedness of $D^{1-\frac{1}{p}}$ now follows from (2.1). □

It is well known that the dual space of H^p can be identified with H^q under the pairing $\langle f, g\rangle = \sum_{n=0}^{\infty} a_n\bar{b}_n$, and the dual space of $\mathcal{D}^p$ can be identified with $\mathcal{D}^q$ under the pairing $\langle f, g\rangle = \sum_{n=1}^{\infty} na_n\bar{b}_n$, where f and g are given by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$. From these duality results and Theorem 2.1, we easily obtain the following results.

Corollary 2.1. *For each $2 \leq p < \infty$, the operator $D^{\frac{1}{p}-1}$ is bounded from H^p to $\mathcal{D}^p$.*

The following results are consequence of Theorem 2.1 and Corollary 2.1.

Corollary 2.2. *The following result hold.*

(i) *If $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^p$, then $\sum_{n=1}^{\infty} n|a_n|^q < \infty$.*

(ii) *If $1 < p \leq 2$, $\frac{1}{p}+\frac{1}{q} = 1$ and $\sum_{n=1}^{\infty} n|a_n|^p < \infty$, then $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in \mathcal{D}^q$.*

(iii) *If $1 \leq p \leq 2$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^p$, then $\sum_{n=1}^{\infty} n^{2p-3}|a_n|^p < \infty$.*

(iv) *If $2 \leq p < \infty$ and $\sum_{n=1}^{\infty} n^{2p-3}|a_n|^p < \infty$, then $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^p$.*

Proof. (i) Let $1 < p \leq 2$. By Theorem 2.1, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^p,$$

then

$$D^{1-\frac{1}{p}} f(z) = \sum_{n=0}^{\infty} n^{\frac{1}{q}} a_n z^n \in H^p.$$

By Theorem 6.1 in [4], this implies that $\{n^{\frac{1}{q}} a_n\} \in l^q$, i.e.,

$$\sum_{n=1}^{\infty} n|a_n|^q < \infty.$$

(ii) Let $1 < p \leq 2$. By Theorem 6.1 in [4] we know that if $\{n^{\frac{1}{p}} a_n\} \in \ell^p$, then

$$f(z) = \sum_{n=0}^{\infty} n^{\frac{1}{p}} a_n z^n \in H^q.$$

By Corollary 2.1, we get that

$$D^{-\frac{1}{p}} f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^q.$$

(iii) Let $1 \leq p \leq 2$. By Theorem 2.1, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{D}^p,$$

then

$$D^{1-\frac{1}{p}} f(z) = \sum_{n=0}^{\infty} n^{1-\frac{1}{p}} a_n z^n \in H^p.$$

By Theorem 6.2 in [4] we get that

$$\sum_{n=0}^{\infty} n^{p-2}\left|n^{1-\frac{1}{p}}a_n\right|^p=\sum_{n=0}^{\infty} n^{2p-3}|a_n|^p<\infty.$$

(iv) Let $2\le p<\infty$. From the given condition we have

$$\sum_{n=0}^{\infty} n^{p-2}\left[n^{1-\frac{1}{p}}|a_n|\right]^p=\sum_{n=1}^{\infty} n^{2p-3}|a_n|^p<\infty.$$

By Theorem 6.3 in [4] we get that

$$f(z)=\sum_{n=0}^{\infty} n^{1-\frac{1}{p}}a_n z^n\in H^p.$$

By Corollary 2.1 we get that

$$D^{\frac{1}{p}-1}f(z)=\sum_{n=0}^{\infty} a_n z^n\in \mathcal{D}^p.$$

The proof is complete. □

3. Multipliers with mean growth property

In this section we use the mean growth property of functions to characterize the coefficient multipliers from D^p_α to A^q_β and H^q. We need some Lemmas.

Lemma 3.1 ([13, p. 73]). *If $f\in H^p$, then $M_q(r,f)\le(1-r)^{\frac{1}{q}-\frac{1}{p}}M_p(r,f)$ for $p\le q$.*

Lemma 3.2 ([4, Theorem 5.6]). *Let f be analytic on $\mathbb{D}$ and suppose that $\alpha>-1$, $0<q<\infty$ and $0<p\le\infty$. Then*

$$\int_0^1(1-r)^\alpha M_p^q(r,f)dr<\int_0^1(1-r)^{\alpha+q}M_p^q(r,f')dr<\infty.$$

Lemma 3.3 ([4, Chapter 4]). *For $\alpha>1$ and $0<r<1$ we have*

$$\int_0^{2\pi}|1-re^{it}|^{-\alpha}dt=O((1-r)^{-\alpha+1}).$$

Lemma 3.4 ([15, Theorem 2.1]). *If $0<p\le s<\infty$, $0<q\le\lambda<\infty$, $0<\alpha<\infty$, then*

$$\left(\int_0^1(1-r)^{\lambda(\alpha+\frac{1}{p}-\frac{1}{s})-1}M_s^\lambda(r,f)\,dr\right)^{\frac{1}{\lambda}}\le C\left(\int_0^1(1-r)^{q\alpha-1}M_p^q(r,f)\,dr\right)^{\frac{1}{q}}.$$

Lemma 3.5 ([15, Theorem 2.3]). *Suppose that $0<q\le\infty$, $s=\min\{1,q\}$, $0<r<1$, and f,g are analytic on $\mathbb{D}$ then $h=g*f$ satisfies that*

$$r^{2m}M_q(r^4,h^{(m)})\le C(1-r)^{\frac{s-1}{s}}M_s(r,f)M_q(r,g^{(m)}),$$

where m is a positive integer.

Here is the main result of this section.

Theorem 3.1. *Let* $0 < p \leq 1$, $p \leq q < \infty$, $\alpha, \beta > -1$, *and* $m = [\frac{\alpha+1}{p} - 1]$. *Then*

(i) $(\mathcal{D}^p_\alpha, A^q_\beta) := \left\{ g : M_q(r, g^{(m)}) = O\left((1-r)^{\frac{\alpha+2}{p} - \frac{1+\beta}{q} - (m+2)} \right) \right\}$.

(ii) $(\mathcal{D}^p_\alpha, H^q) := \left\{ g : M_q(r, g^{(m)}) = O\left((1-r)^{\frac{\alpha+2}{p} - (m+2)} \right) \right\}$.

Proof. (i) Let $s = \min\{1, q\}$. Suppose that

$$M_q(r, g^{(m)}) = O\left((1-r)^{\frac{\alpha+2}{p} - \frac{1+\beta}{q} - (m+2)} \right),$$

$f \in \mathcal{D}^p_\alpha$ and $h = g * f$. Then by Lemma 3.5, we have that

$$\begin{aligned} r^{2mq}(1-r)^{mq+\beta} M_q^q(r^4, h^{(m)}) &\leq C(1-r)^{(m+1-\frac{1}{s}+q)+\beta} M_s^q(r, f) M_q^q(r, g^{(m)}) \\ &\leq C(1-r)^{q(\frac{\alpha+1}{p} - 1 + \frac{1}{p} - \frac{1}{s}) - 1} M_s^q(r, f). \end{aligned}$$

Since $p \leq s \leq q$, it follows from Lemma 3.4 and then Lemma 3.2 that

$$\begin{aligned} \int_0^1 (1-r)^{mq+\beta} M_q^q(r, h^{(m)})\, dr &\leq C \int_0^1 (1-r)^{q(\frac{\alpha+1}{p} - 1 + \frac{1}{p} - \frac{1}{s}) - 1} M_s^q(r, f)\, dr \\ &\leq C \left(\int_0^1 (1-r)^{\alpha - p} M_p^p(r, f)\, dr \right)^{\frac{q}{p}} \\ &\leq C \left(\int_0^1 (1-r)^{\alpha} M_p^p(r, f')\, dr \right)^{\frac{q}{p}} < \infty. \end{aligned}$$

By successive application of Lemma 3.2, this implies that

$$\int_0^1 (1-r)^\beta M_q^q(r, h)\, dr < \infty,$$

i.e., $h \in A^q_\beta$.

Conversely, suppose that $g \in (\mathcal{D}^p_\alpha, A^q_\beta)$. By the closed graph theorem, the operator $H_g : f \mapsto h = g * f$ is a bounded linear operator from $\mathcal{D}^p_\alpha$ to A^q_β. Let

$$f(z) = m! \frac{z^m}{(1-z)^{m+1}} = \sum_{n=m}^\infty \frac{n!}{(n-m+1)!} z^n,$$

and $f_t(z) = f(tz)$ where $0 < t < 1$. Then

$$f'(z) = m! \frac{z^{m-1}(m+z)}{(1-z)^{m+2}}.$$

A simple calculation shows that $f_t \in \mathcal{D}^p_\alpha$ with

$$\|f_t\|_{\mathcal{D}^p_\alpha} \asymp (1-t^2)^{\frac{\alpha+2}{p} - (m+2)}.$$

Since $H_g : \mathcal{D}^p_\alpha \to A^q_\beta$ is a bounded operator, we have that

$$\|h_t\|_{A^q_\beta} \leq C \|f_t\|_{\mathcal{D}^p_\alpha} \leq C(1-t^2)^{\frac{\alpha+2}{p} - (m+2)}.$$

On the other hand we have that

$$\|h_t\|_{A^q_\beta} \geq \left(\int_t^1 (1-r^2)^\beta M_q^q(r,h_t)dr\right)^{1/q} \geq (1-t^2)^{\frac{1+\beta}{q}} M_q(t,h_t).$$

Hence, we get that

$$M_q(t,h_t) = O\left((1-t^2)^{\frac{\alpha+2}{p}-\frac{1+\beta}{q}-(m+2)}\right),$$

which implies that

$$M_q(t^2,h) = O\left((1-t^2)^{\frac{\alpha+2}{p}-\frac{1+\beta}{q}-(m+2)}\right).$$

Setting $t^2=r$, and combining with the fact that $h(z)=g*f(z)=z^m g^{(m)}(z)$, we have that

$$\begin{aligned} M_q(r,h) &= M_q(r,z^m g^{(m)}) = r^m M_q(r,g^{(m)}) \asymp M_q(r,g^{(m)}) \\ &= O\left((1-r)^{\frac{\alpha+2}{p}-\frac{1+\beta}{q}-(m+2)}\right). \end{aligned}$$

(ii) Let

$$M_q(r,g^{(m)}) = O\left((1-r)^{\frac{\alpha+2}{p}-(m+2)}\right),$$

$f\in\mathcal{D}^p_\alpha$ and $h=g*f$. Then, by Lemma 3.5, we have that

$$\begin{aligned} r^{2m}(1-r)^{ms-1}M_q^s(r^4,h^{(m)}) &\leq C(1-r)^{(m+1)s-2}M_s^s(r,f)M_q^s(r,g^{(m)}) \\ &\leq C(1-r)^{(\frac{\alpha+2}{p})s-2-s}M_s^s(r,f). \end{aligned}$$

Since $p\leq s$, it follows from Lemma 3.4 and then Lemma 3.2 that

$$\begin{aligned} \int_0^1 (1-r)^{ms-1}M_q^s(r,h^{(m)})\,dr &\leq \int_0^1 (1-r)^{(\frac{\alpha+1}{p}+\frac{1}{p}-\frac{1}{s})s-1}M_s^s(r,f)\,dr \\ &\leq \left(\int_0^1 (1-r)^{\alpha-p}M_p^p(r,f)\,dr\right)^{\frac{s}{p}} \\ &\leq \left(\int_0^1 (1-r)^{\alpha}M_p^p(r,f')\,dr\right)^{\frac{s}{p}} < \infty. \end{aligned}$$

By successive application of Lemma 3.2, this implies that

$$\int_0^1 (1-r)^{s-1}M_q^s(r,h')dr < \infty.$$

Since $0<s\leq 1$, it follows from Theorem 5 of [5] that $h\in H^q$.

Conversely, by taking the test function as $f(z)=\frac{m!z^m}{(1-z)^{m+1}}$ and using the bounded property of H_g, we get that

$$M_q(t^2,h) = M_q(t,h_t) \leq \|h_t\|_{H^q} \leq C\|f_t\|_{\mathcal{D}^p_\alpha} = O\left((1-t^2)^{\frac{\alpha+2}{p}-(m+2)}\right).$$

Setting $t^2=r$ and combining with $h(z)=g*f(z)=z^m g^{(m)}(z)$, we have that

$$M_q(r,g^{(m)}) = O\left((1-r)^{\frac{\alpha+2}{p}-(m+2)}\right).$$

The proof is complete. □

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Dongxing Li
Department of Mathematics
Shantou University
Shantou, Guangdong 515063, P. R. China
Email address: `14dxli@stu.edu.cn`

Hasi Wulan
Department of Mathematics
Shantou University
Shantou, Guangdong 515063, P. R. China
Email address: `wulan@stu.edu.cn`

Ruhan Zhao
Department of Mathematics
SUNY Brockport
Brockport, NY 14420, USA
Email address: rzhao@brockport.edu