Bull. Korean Math. Soc. **56** (2019), No. 3, pp. 681–689
https://doi.org/10.4134/BKMS.b180491
pISSN: 1015-8634 / eISSN: 2234-3016

A NOTE ON COMPACT MÖBIUS HOMOGENEOUS SUBMANIFOLDS IN $\mathbb{S}^{n+1}$

Xiu Ji and TongZhu Li

Abstract. The Möbius homogeneous submanifold in $\mathbb{S}^{n+1}$ is an orbit of a subgroup of the Möbius transformation group of $\mathbb{S}^{n+1}$. In this note, We prove that a compact Möbius homogeneous submanifold in $\mathbb{S}^{n+1}$ is the image of a Möbius transformation of the isometric homogeneous submanifold in $\mathbb{S}^{n+1}$.

1. Introduction

A notable class of submanifolds in Möbius differential geometry is the Möbius homogeneous submanifold in $\mathbb{S}^{n+1}$. A submanifold $f : M^m \to \mathbb{S}^{n+1}$ in $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ is called a Möbius homogeneous submanifold, if, for any two points $x, y \in M^m$, there exists a Möbius transformation ϕ of $\mathbb{S}^{n+1}$ such that $\phi \circ f(M^m) = f(M^m)$ and $\phi \circ f(x) = f(y)$. Let $M\ddot{o}b(\mathbb{S}^{n+1})$ denote the Möbius transformation group of $\mathbb{S}^{n+1}$, and $f : M^m \to \mathbb{S}^{n+1}$ a Möbius homogeneous submanifold, we define

$$G = \{\phi \in M\ddot{o}b(\mathbb{S}^{n+1}) \,|\, \phi \circ f(M^m) = f(M^m)\}.$$

Then G is a subgroup of $M\ddot{o}b(\mathbb{S}^{n+1})$, and the submanifold f is the orbit of the subgroup G, i.e., $f(M^m) = G \cdot p$ for some point $p \in f(M^m)$. The isometric homogeneous submanifold in $\mathbb{S}^{n+1}$ is an orbit of a subgroup of the isometric transformation group $O(n+2)$ of $\mathbb{S}^{n+1}$. It is well-known that, for $n \geq 2$, the Möbius transformation group of $\mathbb{S}^{n+1}$ coincides with the conformal transformation group of $\mathbb{S}^{n+1}$. Thus, $O(n+2) \subseteq M\ddot{o}b(\mathbb{S}^{n+1})$ is a subgroup.

Standard examples of the Möbius homogeneous submanifolds in $\mathbb{S}^{n+1}$ are the image of a Möbius transformation of the homogeneous submanifolds in $\mathbb{S}^{n+1}$. Due to Hsiang-Lawson ([3]) and Takagi-Takahashi ([11]), the homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ are classified. Every homogeneous hypersurface in $\mathbb{S}^{n+1}$ can be obtained as a principal orbit of a linear isotropy representation of

Received May 23, 2018; Accepted September 17, 2018.
2010 *Mathematics Subject Classification.* 53A30, 53C30.
Key words and phrases. Möbius transformation group, isometric transformation group, Möbius homogeneous hypersurfaces, homogeneous hypersurfaces.

©2019 Korean Mathematical Society

a compact Riemannian symmetric pair (U, K) of rank two. Hsiang also classified the minimal homogeneous submanifolds with low cohomogeneity. But for general homogeneous submanifolds there is no classification.

There exist some examples of Möbius homogeneous submanifolds which are not Möbius equivalent to the homogeneous submanifolds in $\mathbb{S}^{n+1}$. In next proposition we give a method to construct the example (see Sect. 3). The inverse of the stereographic projection $\sigma : \mathbb{R}^{n+1} \mapsto \mathbb{S}^{n+1}$ is defined by

$$\sigma(u) = (\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}).$$

Proposition 1.1. *Let $u : M^m \to \mathbb{S}^{m+1}$ be an immersed hypersurface. We define the cone f over u as*

$$f : M^m \times \mathbb{R}^+ \times \mathbb{R}^{n-m-1} \to \mathbb{R}^{n+1},\ f(p,t,y) = (tu(p), y),\ 1 \le m \le n-1.$$

If $u : M^m \to \mathbb{S}^{m+1}$ is a homogeneous hypersurface, then $\sigma \circ f$ is a Möbius homogeneous hypersurface in $\mathbb{S}^{n+1}$. Where $\mathbb{R}^+ = \{t \,|\, t > 0\}$.

These examples come from the homogeneous hypersurfaces in $\mathbb{S}^{n+1}$. But there are some examples of Möbius homogeneous hypersurfaces which can't be obtained in this way. In [10], Sulanke has constructed a Möbius homogeneous surface, which is a cylinder over a logarithmic spiral in $\mathbb{R}^2$, and classified the Möbius homogeneous surfaces in $\mathbb{R}^3$. In [5], authors have constructed a Möbius homogeneous hypersurface, a logarithmic spiral cylinder, which is a high dimensional version of Sulanke's example, and classified the Möbius homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ with two distinct principal curvatures. In addition, in [5], authors also have classified the Möbius homogeneous hypersurfaces in $\mathbb{S}^4$. Recently, Li and Wang classified the Möbius homogeneous hypersurfaces provided that the dimension of the hypersurface or the number of distinct principal curvatures is small (see [4], [6]). In [7], authors classified the Möbius homogeneous Willmore 2-spheres.

In this paper, we prove that compact Möbius homogeneous submanifolds in $\mathbb{S}^{n+1}$ are the homogeneous submanifolds in $\mathbb{S}^{n+1}$ up to a Möbius transformation.

Theorem 1.1. *Let $f : M^m \to \mathbb{S}^{n+1}$ be a compact Möbius homogeneous submanifold. Then f is Möbius equivalent to the homogeneous submanifold in $\mathbb{S}^{n+1}$.*

Remark 1.1. Two submanifolds $f, \tilde{f} : M^m \to \mathbb{S}^{n+1}$ are Möbius equivalent if there exists a Möbius transformation $\phi \in M\ddot{o}b(\mathbb{S}^{n+1})$ such that $\phi \circ f(M^m) = \tilde{f}(M^m)$.

Remark 1.2. By Hsiang-Lawson and Takagi-Takahashi's work about classification of homogeneous hypersurfaces in $\mathbb{S}^{n+1}$, the results of Theorem 1.1 implies that the compact Möbius homogeneous hypersurfaces are completely classified.

We organize the paper as follows. In Section 2, we give the elementary facts about the Möbius transformation group of $\mathbb{S}^{n+1}$. In Section 3, we prove Theorem 1.1 and give some example of noncompact Möbius homogeneous hypersurfaces by the orbit of a subgroup of the Möbius transformation group.

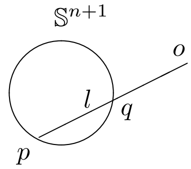
2. Möbius transformation group of $\mathbb{S}^{n+1}$

In this section, we recall some facts about the Möbius transformation group of $\mathbb{S}^{n+1}$. For details we refer to [1], [2], or [9].

Let $\mathbb{R}^{n+2}$ denote the $(n+2)$-dimensional Euclidean space, and $\langle\cdot,\cdot\rangle$ its inner product. The $(n+1)$-dimensional sphere $\mathbb{S}^{n+1} = \{x \in \mathbb{R}^{n+2} \mid \langle x, x\rangle = 1\}$. The hypersphere $S_p(\rho)$ in $\mathbb{S}^{n+1}$ with center $p \in \mathbb{S}^{n+1}$ and spherical radius $\rho, 0 < \rho < \pi$, is the intersection of $\mathbb{S}^{n+1}$ with the hyperplane in $\mathbb{R}^{n+2}$ given by

$$S_p(\rho) = \{y \in \mathbb{S}^{n+1} \mid \langle p, y\rangle = \cos\rho\},\ 0 < \rho < \pi.$$

A diffeomorphism $\phi : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1}$ is said to be a Möbius transformation, if ϕ takes the set of hyperspheres into the set of hyperspheres. All Möbius transformations form a transformation group, which is called the Möbius transformation group of $\mathbb{S}^{n+1}$ and denoted by $M\ddot{o}b(\mathbb{S}^{n+1})$. The isometric group $O(n+2)$ of $\mathbb{S}^{n+1}$ is a subgroup of $M\ddot{o}b(\mathbb{S}^{n+1})$.



Let D^{n+2} be the unit ball bounded by $\mathbb{S}^{n+1}$. Taking $o \in \mathbb{R}^{n+2}\backslash D^{n+2}$, a line l that passes through the point o intersects the sphere $\mathbb{S}^{n+1}$ in two points p, q (see above graph). Now we define the Möbius inversion Υ_o for the point $o \in \mathbb{R}^{n+2}\backslash D^{n+2}$ as follows,

$$\Upsilon_o : \mathbb{S}^{n+1} \to \mathbb{S}^{n+1},\ \Upsilon_o(p) = q.$$

Clearly, $\Upsilon_o \in M\ddot{o}b(\mathbb{S}^{n+1})$. When the point o is at infinity, the Möbius inversion Υ_o is an isometric transformation of $\mathbb{S}^{n+1}$, i.e., $\Upsilon_o \in O(n+2)$. The following results is well known.

Proposition 2.1 ([1])**.** *The Möbius transformation group $M\ddot{o}b(\mathbb{S}^{n+1})$ is generated by Möbius inversions Υ_o.*

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with the scalar product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+2}y_{n+2}$$

for $x = (x_0, x_1, \ldots, x_{n+2}), y = (y_0, y_1, \ldots, y_{n+2}) \in \mathbb{R}^{n+3}$.

Let $O(n+2,1)$ be the Lorentz orthogonal group of $\mathbb{R}_1^{n+3}$ defined by

$$O(n+2,1)=\{T\in GL(\mathbb{R}^{n+3})\,|\,TI_1T^t=I_1\},$$

where T^t denotes the transpose of T and $I_1=\begin{pmatrix}-1&0\\0&I\end{pmatrix}$.

Let

$$C_+^{n+2}=\{y=(y_0,y_1)\in\mathbb{R}\times\mathbb{R}^{n+2}=\mathbb{R}_1^{n+3}\,|\,\langle y,y\rangle=0,\ y_0>0\},$$

and $O^+(n+2,1)$ denote the subgroup of $O(n+2,1)$ defined by

$$O^+(n+2,1)=\{T\in O(n+2,1)\,|\,T(C_+^{n+2})=C_+^{n+2}\}.$$

Proposition 2.2 ([9])**.** *Let $T=\begin{pmatrix}w&u\\v&B\end{pmatrix}\in O(n+2,1)$. Then $T\in O^+(n+2,1)$ if and only if $w>0$.*

It is well-known that the subgroup $O^+(n+2,1)$ is isomorphic to the Möbius transformation group $M\ddot{o}b(\mathbb{S}^{n+1})$. In fact, for any

$$T=\begin{pmatrix}w&u\\v&B\end{pmatrix}\in O^+(n+2,1),$$

we can define the Möbius transformation $\varphi(T):\mathbb{S}^{n+1}\mapsto\mathbb{S}^{n+1}$ by

$$\varphi(T)(x)=\frac{Bx+v}{ux+w},\ x=(x_1,\ldots,x_{n+2})^t\in\mathbb{S}^{n+1}.$$

Then the map $\varphi:O^+(n+2,1)\mapsto M\ddot{o}b(\mathbb{S}^{n+1})$ is a group isomorphism.

Let $A\in O(n+2)$ be an isometric transformation of $\mathbb{S}^{n+1}$, then $A\in M\ddot{o}b(\mathbb{S}^{n+1})$ and

$$\varphi^{-1}(A)=\begin{pmatrix}1&0\\0&A\end{pmatrix}.$$

Thus $\varphi^{-1}(O(n+2))\subset O^+(n+2,1)$ is a subgroup.

Let

$$\Lambda_+^{n+3}=\{y=(y_0,y_1)\in\mathbb{R}\times\mathbb{R}^{n+2}=\mathbb{R}_1^{n+3}\,|\,\langle y,y\rangle<0,\ y_0>0\}$$

be the convex cone. Then $O^+(n+2,1)$ acts on Λ_+^{n+3} transitively. Since Λ_+^{n+3} is noncompact, thus the Lie group $O^+(n+2,1)$ is noncompact, but $O(n+2)$ is compact. The following Theorem 2.1 implies that a compact subgroup in $O^+(n+2,1)$ is conjugate to a compact subgroup in $\varphi^{-1}(O(n+2))$. To prove Theorem 2.1, we need the following lemma.

Lemma 2.1. *Let $H\subset O^+(n+2,1)$ be a subgroup. If there is a timelike vector $v_0\in\mathbb{R}_1^{n+3}$ such that*

$$h(v_0)=v_0\quad\textit{for all } h\in H,$$

then there exists $T\in O^+(n+2,1)$ such that THT^{-1} be a subgroup of

$$\varphi^{-1}(O(n+2)).$$

Proof. We call assume that $v_0 \in \Lambda_+^{n+3}$, otherwise we take $-v_0$ instead of v_0. Since $O^+(n+2,1)$ acts on Λ_+^{n+3} transitively, there exists $T \in O^+(n+2,1)$ such that

$$T(v_0) = (a, 0, \ldots, 0)^t \in \Lambda_+^{n+3}.$$

Thus

$$T(h)T^{-1} \cdot (a, 0, \ldots, 0)^t = Th \cdot v_0 = T \cdot v_0 = (a, 0, \ldots, 0)^t \text{ for all } h \in H.$$

Setting $T(h)T^{-1} = \begin{pmatrix} w & u \\ v & A \end{pmatrix}$. We have $w = 1, u = v = 0$ and $A \in O(n+2)$, that is $THT^{-1} \subset \varphi^{-1}(O(n+2))$. □

Theorem 2.1. *Let $H \subset O^+(n+2,1)$ be a compact subgroup. Then there exists $T \in O^+(n+2,1)$ such that THT^{-1} be a subgroup of $\varphi^{-1}(O(n+2))$.*

Proof. Since H is compact, there exists a bi-invariant measure $dvol_h$ in H. Now we take a timelike vector v_0 in Λ_+^{n+3}, define

$$v = \int_H (h \cdot v_0) dvol_h.$$

For any $h_1 \in H$,

$$h_1(v) = h_1(\int_H (h \cdot v_0) dvol_h) = \int_H (h_1 h \cdot v_0) dvol_h = \int_H (h_1 h \cdot v_0) dvol_{h_1 h} = v.$$

Since Λ_+^{n+3} is convex and $v_0 \in \Lambda_+^{n+3}$, thus $v = \int_H (h \cdot v_0) dvol_h \in \Lambda_+^{n+3}$ is a timelike vector such that $h(v) = v$ for all $h \in H$. By Lemma 2.1 we know that there exists $T \in O^+(n+2,1)$ such that $THT^{-1} \subset \varphi^{-1}(O(n+2))$. □

3. Proof of Theorem 1.1 and examples of Möbius homogeneous hypersurfaces in $\mathbb{S}^{n+1}$

Let $f : M^m \to \mathbb{S}^{n+1}$ be an immersed submanifold without umbilical points and $\{e_i \,|\, i = 1, \ldots, m\}$ be an orthonormal basis with respect to the induced metric $I = df \cdot df$ with the dual basis $\{\theta_i\}$. Let $\{e_\alpha \,|\, \alpha = m+1, \ldots, n+1\}$ be an orthonormal basis for the normal bundle, and $II = \sum_{ij\alpha} h_{ij}^\alpha \theta_i \otimes \theta_j e_\alpha$ the second fundamental form, $\overrightarrow{H} = \sum_{i,\alpha} \frac{h_{ii}^\alpha}{m} e_\alpha$ the mean curvature vector of f. To study the Möbius geometry of f, as in [12], one considers the Möbius position vector

$$Y = \rho(f)\,(1, f) : M^m \to C_+^{n+2} \subset \mathbb{R}_1^{n+3}$$

and the Möbius metric

$$g = \langle dY, dY \rangle = (\rho(f))^2 df \cdot df,$$

where $(\rho(f))^2 = \frac{m}{m-1}(|II|^2 - m|\overrightarrow{H}|^2)$. One basic fact for this approach is:

Lemma 3.1 ([12]). *Suppose that $f : M^m \to \mathbb{S}^{n+1}$ is an immersed submanifold without umbilical points and*

$$Y = \rho(f)\,(1, f) : M^m \to C_+^{n+2} \subset \mathbb{S}_1^{n+3}$$

is the Möbius position vector of f. Then, for any $T \in O^+(n+2,1)$, we have

$$T(Y) = \rho(\varphi(T)f)\,(1, \varphi(T)(f)) : M^m \to C_+^{n+2} \subset \mathbb{S}_1^{n+3}$$

and therefore the Möbius metric g stays invariant.

Theorem 3.1 ([12]). *Two submanifolds $f, \tilde{f} : M^m \to \mathbb{S}^{n+1}$ are Möbius equivalent if and only if there exists $T \in O^+(n+2,1)$ such that $\tilde{Y} = T(Y)$.*

Next we prove Theorem 1.1. Let $f : M^m \to \mathbb{S}^{n+1}$ be a compact Möbius homogeneous submanifold. If there are umbilical points, then the submanfold f is totally umbilical. Thus f is a homogeneous submanifold in $\mathbb{S}^{n+1}$.

Now let $f : M^m \to \mathbb{S}^{n+1}$ be a compact Möbius homogeneous submanifold without umbilical points. Thus $Y : M^m \to C_+^{n+2}$ is compact and the Möbius metric g is a Riemannian metric. Let

$$G = \{\phi \in M\ddot{o}b(\mathbb{S}^{n+1}) \,|\, \phi \circ f(M^m) = f(M^m)\}$$

and $G_p = \{\phi \in G \,|\, \phi \circ f(p) = f(p)\}$ be a stabilizer subgroup for some $p \in M^m$. Then

$$Y(M^m) = \frac{\varphi^{-1}(G)}{\varphi^{-1}(G_p)}.$$

Since the Riemannian metric g is invariant under $\varphi^{-1}(G)$, The stabilizer subgroup $\varphi^{-1}(G_p)$ is a compact subgroup. Since the quotient map

$$\pi : \varphi^{-1}(G) \to Y(M^m) = \frac{\varphi^{-1}(G)}{\varphi^{-1}(G_p)}$$

is an open map and $Y(M^m)$ is compact, $\varphi^{-1}(G)$ is compact. Since the subgroup $\varphi^{-1}(G) \subset O^+(n+2,1)$ is compact, by Theorem 2.1, we know that there exists $T \in O^+(n+2,1)$ such that $\varphi(T(\varphi^{-1}(G))T^{-1}) \subset O(n+2)$ is a subgroup of isometric transformation group $O(n+2)$. Thus f is Möbius equivalent to the homogeneous submanifold in $\mathbb{S}^{n+1}$ and we finish the proof of Theorem 1.1.

Next, we construct some examples of Möbius homogeneous hypersurfaces.

Example 3.1. Let $u : M^k \to \mathbb{S}^{k+1}$ be a homogeneous hypersurface in $(k+1)$-dimensional sphere. Then there exists a subgroup $H \subset O(k+2)$ which acts on $u(M^k)$ transitively.

We define the cone over u as

$$f : M^k \times \mathbb{R}^+ \times \mathbb{R}^{n-k-1} \to \mathbb{R}^{n+1},\ f(x,t,y) = (tu(x), y),\ 1 \le k \le n-1,$$

then the hypersurface $\sigma \circ f : M^k \times R^+ \times \mathbb{R}^{n-k-1} \to \mathbb{S}^{n+1}$ is a Möbius homogeneous hypersurface in $\mathbb{S}^{n+1}$.

Let

$$G = \left\{ \begin{pmatrix} O^+(n-k,1) & \\ & H \end{pmatrix} \right\} \subset O^+(n+2,1).$$

Then G is a subgroup of $O^+(n+2,1)$.

In fact, the hypersurface $\sigma \circ f : M^k \times R^+ \times \mathbb{R}^{n-k-1} \to \mathbb{S}^{n+1}$ is the orbit of the subgroup $\varphi(G) \subset M\ddot{o}b(\mathbb{S}^{n+1})$ acting on the point

$$p = (\underbrace{0,,0,\dots,\frac{1}{\sqrt{1+r^2}}}_{n-k+1},\underbrace{\frac{r}{\sqrt{1+r^2}},0,\dots,0}_{k+1}) \in S^{n+1}.$$

Example 3.2. Let $\gamma : I \to \mathbb{R}^2$ be the logarithmic spiral in the Euclidean plane $\mathbb{R}^2$ given by

$$\gamma(s) = (e^{cs}\cos s, e^{cs}\sin s), \;\; c > 0.$$

The cylinder in $\mathbb{R}^{n+1}$ over $\gamma(s)$ is defined by

$$f = (\gamma, id) : I \times \mathbb{R}^{n-1} \mapsto \mathbb{R}^{n+1},$$
$$f(s,y) = f(s,y_1,\dots,y_{n-1}) = (e^{cs}\cos s, e^{cs}\sin s, y_1,\dots,y_{n-1}) \in \mathbb{R}^{n+1}.$$

We call the hypersurface f a logarithmic spiral cylinder. The logarithmic spiral cylinder $f = (\gamma, id)$ is a Möbius homogeneous hypersurface in $\mathbb{R}^{n+1}$. The hypersurface $\sigma \circ f$ is a Möbius homogeneous hypersurface in $\mathbb{S}^{n+1}$.

g	(U,K)	(m_1,m_2)	$M^n = K/K_0$
1	$(S^1 \times SO(n+2), SO(n+1))$	n	S^n
2	$(SO(p+2)\times SO(n+2-p),$ $SO(p+1)\times SO(n+1-p))$ $1 \le p \le n-1$	$(p, n-p)$	$M^n = S^p \times S^{n-p}$
3	$(SU(3), SO(3))$	$(1,1)$	$M^3 = \frac{SO(3)}{Z_2+Z_2}$
3	$(SU(3)\times SU(3), SU(3))$	$(2,2)$	$M^6 = \frac{SU(3)}{T^2}$
3	$(SU(6), Sp(3))$	$(4,4)$	$M^{12} = \frac{Sp(3)}{Sp(1)^3}$
3	(E_6, F_4)	$(8,8)$	$M^{24} = \frac{F_4}{Spin(8)}$
4	$(SO(5)\times SO(5), SO(5))$	$(2,2)$	$M^8 = \frac{SO(5)}{T^2}$
4	$(SU(m+2),$ $S(U(m)\times U(2))), m \ge 2$	$(2, 2m-3)$	$M^{4m-2} = \frac{S(U(m)\times U(2))}{SU(m-2)\times T^2}$
4	$(SO(m+2),$ $SO(m)\times SO(2)), m \ge 3$	$(1, m-2)$	$M^{2m-2} = \frac{SO(m)\times SO(2))}{SO(m-2)\times Z_2}$
4	$(Sp(m+2),$ $Sp(m)\times Sp(2)), m \ge 2$	$(4, 4m-5)$	$M^{8m-2} = \frac{Sp(m)\times Sp(2))}{Sp(m-2)\times Sp(1)^2}$
4	$(SO(10), U(5))$	$(4,5)$	$M^{18} = \frac{U(5)}{SU(2)\times SU(2)\times T^1}$
4	$(E_6, Spin(10)\cdot T)$	$(6,9)$	$M^{30} = \frac{Spin(10)\cdot T}{SU(4)\cdot T}$
6	$(G_2\times G_2, G_2)$	$(2,2)$	$M^{12} = \frac{G_2}{T^2}$
6	$(G_2, SO(4))$	$(1,1)$	$M^6 = \frac{SO(4)}{Z_2+Z_2}$

Let

$$G(s,y)=\left\{\begin{pmatrix} \frac{1+|y|^2+e^{2cs}}{2e^{cs}} & \frac{1+|y|^2-e^{2cs}}{2e^{cs}} & 0 & 0 & y_1 & \cdots & y_{n-1} \\ \frac{1-|y|^2-e^{2cs}}{2e^{cs}} & \frac{1-|y|^2+e^{2cs}}{2e^{cs}} & 0 & 0 & -y_1 & \cdots & -y_{n-1} \\ 0 & 0 & \cos s & -\sin s & 0 & \cdots & 0 \\ 0 & 0 & \sin s & \cos s & 0 & \ddots & 0 \\ \frac{y_1}{e^{cs}} & \frac{y_1}{e^{cs}} & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{y_{n-1}}{e^{cs}} & \frac{y_{n-1}}{e^{cs}} & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}\right\},$$

then G is a subgroup of $O^+(n+2,1)$.

The logarithmic spiral cylinder $\sigma\circ f$ is the orbit of the subgroup $\varphi(G)\subset M\ddot{o}b(\mathbb{S}^{n+1})$ acting on the point $p=(1,0,\dots,0)\in\mathbb{S}^{n+1}$.

The above two examples of Möbius homogeneous hypersurfaces are noncompact. By Theorem 1.1, the compact Möbius homogeneous hypersurfaces are Möbius equivalent to the homogeneous submanifolds in $\mathbb{S}^{n+1}$.

The homogeneous hypersurfaces in $\mathbb{S}^{n+1}$ are the isoparametric hypersurfaces. By Müzner's result ([8]), the number g of distinct principal curvatures must be $1,2,3,4$ or 6, and the distinct principal curvatures have the multiplicities $m_1=m_3=\cdots$, $m_2=m_4=\cdots$. Next we list all homogeneous hypersurfaces in $\mathbb{S}^{n+1}$.

Acknowledgements. Authors would like to thank Prof. Ma Xiang for some useful discussions. Authors are partially supported by the grant No.11471021 and No. 11571037 of NSFC.

References

[1] T. E. Cecil, *Lie Sphere Geometry*, Universitext, Springer-Verlag, New York, 1992.

[2] U. Hertrich-Jeromin, *Introduction to Möbius Differential Geometry*, London Mathematical Society Lecture Note Series, **300**, Cambridge University Press, Cambridge, 2003.

[3] W. Hsiang and H. B. Lawson, Jr., *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry **5** (1971), 1–38.

[4] T. Li, *Möbius homogeneous hypersurfaces with three distinct principal curvatures in* $\mathbb{S}^{n+1}$, Chin. Ann. Math. Ser. B **38** (2017), no. 5, 1131–1144.

[5] T. Li, X. Ma, and C. Wang, *Möbius homogeneous hypersurfaces with two distinct principal curvatures in* S^{n+1}, Ark. Mat. **51** (2013), no. 2, 315–328.

[6] T. Li and C. Wang, *Classification of Möbius homogeneous hypersurfaces in a 5-dimensional sphere*, Houston J. Math. **40** (2014), no. 4, 1127–1146.

[7] X. Ma, F. Pedit, and P. Wang, *Möbius homogeneous Willmore* 2*-spheres*, Bull. London Math. Soc., 2018; doi:10.1112/blms.12155.

[8] H. F. Münzner, *Isoparametrische Hyperflächen in Sphären*, Math. Ann. **251** (1980), no. 1, 57–71.

[9] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.

[10] R. Sulanke, *Möbius geometry. V. Homogeneous surfaces in the Möbius space* S^3, in Topics in differential geometry, Vol. I, II (Debrecen, 1984), 1141–1154, Colloq. Math. Soc. János Bolyai, 46, North-Holland, Amsterdam, 1988.

[11] R. Takagi and T. Takahashi, *On the principal curvatures of homogeneous hypersurfaces in a sphere*, in Differential geometry (in honor of Kentaro Yano), 469–481, Kinokuniya, Tokyo, 1972.

[12] C. Wang, *Moebius geometry of submanifolds in S^n*, Manuscripta Math. **96** (1998), no. 4, 517–534.

Xiu Ji
Department of Mathematics
Beijing Institute of Technology
Beijing, 100081, P. R. China
Email address: jixiu1106@163.com

TongZhu Li
Department of Mathematics
Beijing Institute of Technology
Beijing, 100081, P. R. China
Email address: litz@bit.edu.cn