

## DECOMPOSITION OF THE INVARIANT LAPLACIAN IN THE COMPLEX BALL

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ABSTRACT. We, in this note, decompose the invariant Laplacian of the unit complex ball of  $\mathbb{C}^n$  by the radial part and tangential part as

$$\tilde{\Delta} = \tilde{\Delta}_{rad} + \tilde{\Delta}_{tan}.$$

We give several properties and interpretations involved with this decomposition.

### 1. Introduction

Function theory on a subset of  $\mathbb{R}^{2n}$  often uses the Laplacian:

$$\Delta = \sum_{j=1}^n \left( \frac{\partial^2}{\partial^2 x_j} + \frac{\partial^2}{\partial^2 y_j} \right),$$

because it is one of the most convenient operators in measuring the local change of a function. Our interest in this note is on the decomposition of the Laplacian. Sometimes a decomposition of  $\Delta$  gives some benefits in measuring functional behaviors.

Throughout this note, terminologies are used as in [3, 4]. Let  $B = B_n$  be the open unit ball of the complex  $n$ -dimensional space  $\mathbb{C}^n$  topologically identified with  $\mathbb{R}^{2n}$ . Then  $\Delta$  can be expressed as

$$\Delta = 4 \sum_{j=1}^n D_j \bar{D}_j,$$

where  $D_j = \frac{\partial}{\partial z_j}$  and  $\bar{D}_j = \frac{\partial}{\partial \bar{z}_j}$ ,  $j = 1, 2, \dots, n$ . On  $B$ ,  $\Delta$  may be decomposed into the complex tangential Laplacian and the complex radial Laplacian as

$$\Delta = \Delta_{tan} + \Delta_{rad},$$

where  $\Delta_{rad}$  is defined, for  $f \in C^2(B)$  and  $z = r\zeta \in B - \{0\}$ , to be the Laplacian of the function  $\lambda \rightarrow f(z + \lambda\zeta)$  at the origin of  $\mathbb{C}$  (see [3, 17.3.2]). And lots of

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Received May 18, 2018; Accepted August 22, 2018.

2010 *Mathematics Subject Classification.* 35J05, 31C05, 31C10.

*Key words and phrases.* invariant Laplacian, tangential and radial decomposition.

properties of a function on  $B$  can be observed in terms of  $\Delta_{tan}$  and  $\Delta_{rad}$ . See for example [1].

We, in this note, decompose another more geometric Laplacian of  $B$ . Let  $\mathcal{M}$  denote the group of all automorphisms, that is, one to one biholomorphic onto map, of  $B$ .  $\mathcal{M}$  consists of all maps of the form  $U\varphi_a$ , where  $U$  is a unitary operator of  $\mathbb{C}^n$  and  $\varphi_a$  is defined by

$$(1.1) \quad \varphi_a(z) = \begin{cases} \frac{a - P_a z - \sqrt{1 - |a|^2} Q_a z}{1 - \langle z, a \rangle}, & \text{if } a \neq 0, \\ 0, & \text{if } a = 0. \end{cases}$$

Here  $\langle, \rangle$  is the Hermitian inner product of  $\mathbb{C}^n$ :  $\langle z, w \rangle = \sum_{j=1}^n z_j \bar{w}_j$ ,  $z, w \in \mathbb{C}^n$ ,  $P_a z$  is the projection of  $\mathbb{C}^n$  onto the subspace generated by  $B$ :

$$P_a z = \frac{\langle z, a \rangle}{\langle a, a \rangle} a, \quad \text{if } a \neq 0 \quad \text{and} \quad P_0 z = 0,$$

and  $Q_a(z) = z - P_a z$ . Then the ( $\mathcal{M}$ -)invariant Laplacian  $\tilde{\Delta}$  is defined, for  $f \in C^2(B)$ , as

$$\tilde{\Delta}f(a) = \Delta(f \circ \varphi_a)(0), \quad a \in B.$$

It is  $\mathcal{M}$ -invariant in the sense that

$$(\tilde{\Delta}f)(\psi(a)) = \tilde{\Delta}(f \circ \psi)(a), \quad a \in B,$$

for all  $\psi \in \mathcal{M}$ , and a straightforward calculation shows that

$$(1.2) \quad \tilde{\Delta}f(a) = 4(1 - |a|^2) \sum_{i,j=1}^n (\delta_{i,j} - \bar{a}_i a_j) (\bar{D}_i D_j f)(a), \quad a \in B.$$

See [3, 4.1.3]. Now, in this paper, we define:

**Definition 1.1.** Define  $\tilde{\Delta}_{rad}$ , for  $f \in C^2(B)$  and  $z = r\zeta \in B - \{0\}$ , as the (invariant) Laplacian of the function  $\lambda \rightarrow f \circ \varphi_z(\lambda\zeta)$  at the origin of  $\mathbb{C}$ . Define

$$\tilde{\Delta}_{tan} = \tilde{\Delta} - \tilde{\Delta}_{rad}.$$

Note that the round bracket in Definition 1.1 comes from the fact that  $\tilde{\Delta}f(0) = \Delta f(0)$  for any  $f \in C^2(B)$ .

We give several properties involved with this decomposition.

**Theorem 1.2.** Let  $f \in C^2(B)$  and  $a = r\zeta$ ,  $0 < r < 1$ ,  $\zeta \in S$ . Then we have the following.

$$\begin{aligned} (a) \quad \tilde{\Delta}_{rad}f(a) &= 4(1 - |a|^2)^2 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a) \\ &= (1 - |a|^2)^2 \Delta_{rad}f(a) \\ &= 4(1 - |a|^2)^2 \lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \int_0^{2\pi} \{f(a + \rho e^{i\theta} \zeta) - f(a)\} \frac{d\theta}{2\pi}. \\ (b) \quad \tilde{\Delta}_{tan}f(a) &= 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a) = (1 - |a|^2) \Delta_{tan}f(a). \end{aligned}$$

(c) If  $f$  is radial, then

$$\tilde{\Delta}_{rad}f = (1 - r^2)^2 \frac{\partial^2 f}{\partial r^2} + (1 - r^2) \frac{1}{r} \frac{\partial f}{\partial r}; \quad \tilde{\Delta}_{tan}f = 2(n - 1) \frac{(1 - r^2)^2}{r} \frac{\partial f}{\partial r}.$$

(d)  $\Delta, \Delta_{rad}, \Delta_{tan}, \tilde{\Delta}, \tilde{\Delta}_{rad}, \tilde{\Delta}_{tan}$  all commutes with the action of the unitary group.

(e)  $\tilde{\Delta} = \tilde{\Delta}_{tan} + \tilde{\Delta}_{rad} = (1 - r^2)\Delta_{tan} + (1 - r^2)^2\Delta_{rad}$  and

$$\Delta = \frac{\tilde{\Delta}}{1 - r^2} + r^2\Delta_{rad} = \frac{\tilde{\Delta}}{1 - r^2} + r^2(1 - r^2)^2\tilde{\Delta}_{rad}.$$

We next consider several consequences of Theorem 1.2. If we let  $f_a$  be defined by  $f_a(\lambda) = f(\lambda a)$ ,  $\lambda \in B_1$ , and  $\Delta_1$  denote the 1-variable Laplacian:

$$\Delta_1 = 4 \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}},$$

then

$$\Delta_1 f_a(1) = 4 \sum_{i,j=1}^n \bar{a}_i a_j (\bar{D}_i D_j f)(a),$$

so that (1.2) can be expressed as

$$(1.3) \quad \tilde{\Delta}f(a) = (1 - |a|^2) \{ \Delta f(a) - \Delta_1 f_a(1) \}$$

(See [3, 4.1.3]). About the formula (1.3), in [3, 19.3.16], W. Rudin asked “Is there a more intuitive way of seeing why  $\tilde{\Delta}$  is a difference of two ordinary Laplacian?” By Theorem 1.1, we can give an interpretation of (1.3) as follows.

Letting  $a = r\zeta$ ,  $0 < r < 1$ ,  $\zeta \in S$ ,  $S$  being the boundary of  $B$ , we see by a direct computation that

$$\Delta_1 f_a(1) = 4 \sum_{i,j=1}^n \bar{a}_i a_j (\bar{D}_i D_j f)(a) = 4r^2 \sum_{i,j=1}^n \bar{\zeta}_i \zeta_j (\bar{D}_i D_j f)(a),$$

so that  $\Delta_1 f_a(1) = r^2 \Delta_{rad} f(a)$ . Thus, (1.3) means

$$(1.4) \quad \tilde{\Delta} = (1 - r^2) (\Delta - r^2 \Delta_{rad}).$$

We have two observations of (1.4). First, since  $r^2 \Delta_{rad}$  represents radial directional change only,  $\tilde{\Delta} - (1 - r^2)\Delta$  has tangential component canceling property. That is, considering the quantities which equal  $-(1 - r^2)\Delta_1 f_a(1)$ , we have a kind of the tangential component canceling identity as in

**Corollary 1.3.**

$$\tilde{\Delta} - (1 - r^2)\Delta = \tilde{\Delta}_{rad} - (1 - r^2)\Delta_{rad} = -r^2(1 - r^2)\Delta_{rad}.$$

That is, for  $f \in C^2(B)$  and  $a = r\zeta$ ,  $0 < r < 1$ ,  $\zeta \in S$ , the quantity  $-(1 - r^2)\Delta_1 f_a(1)$  equals

$$\left[ \tilde{\Delta} - (1 - r^2)\Delta \right] f(a) = \left[ \tilde{\Delta}_{rad} - (1 - r^2)\Delta_{rad} \right] f(a) = -r^2(1 - r^2)\Delta_{rad} f(a).$$

Secondly, noting that  $\Delta = \Delta_{tan} + \Delta_{rad}$  and dividing tangential components and radial components, (1.4) matches the decomposition

$$\begin{aligned} \tilde{\Delta} &= (1 - r^2) (\Delta_{tan} + \Delta_{rad} - r^2 \Delta_{rad}) \\ &= (1 - r^2) \Delta_{tan} + (1 - r^2)^2 \Delta_{rad} = \tilde{\Delta}_{tan} + \tilde{\Delta}_{rad} \end{aligned}$$

by Theorem 1.2(a) and (b), which explicitly represents the tangential components and the radial components. And it is a natural and geometrical expression of  $\tilde{\Delta}$  in the sense that  $\tilde{\Delta}$  can be decomposed into two (orthogonal) directional Laplacians as  $\tilde{\Delta}_{tan} + \tilde{\Delta}_{rad}$  with the growth properly controlled, namely, twice better controlled in the radial direction (by powers of  $1 - r^2$ ) as

$$\tilde{\Delta}_{tan} = (1 - r^2) \Delta_{tan}; \quad \tilde{\Delta}_{rad} = (1 - r^2)^2 \Delta_{rad}.$$

The following gives an illustration of such a phenomenon (See [1] Theorem 3.1 for a proof).

**Theorem 1.4.** *If  $f \in C^2(B)$  satisfies  $\tilde{\Delta}f \equiv 0 \equiv \tilde{\Delta}f$ , then*

$$\sup \tilde{\Delta}|f|^2 < \infty \iff \sup \tilde{\Delta}_{tan}|f|^2 < \infty \iff \sup \tilde{\Delta}_{rad}|f|^2 < \infty.$$

### 2. Proof of Theorem 1.2

Theorem 1.2 follows from the following two lemmas.

**Lemma 2.1.** *Let  $f \in C^2(B)$  and  $a = r\zeta$ ,  $0 \leq r < 1$ ,  $\zeta \in S$ . Then we have the following.*

- (a)  $\Delta_{rad}f(a) = 4 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a)$   
 $= \lim_{\rho \rightarrow 0} \frac{4}{\rho^2} \int_0^{2\pi} \{f(a + \rho e^{i\theta} \zeta) - f(a)\} \frac{d\theta}{2\pi}.$
- (b)  $\Delta_{tan}f(a) = 4 \sum_{j,k=1}^n (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a).$
- (c) *If  $f$  is radial, then  $\Delta_{rad}f = \frac{\partial f^2}{\partial^2 r} + \frac{1}{r} \frac{\partial f}{\partial r}$  and  $\Delta_{tan}f = \frac{2(n-1)}{r} \frac{\partial f}{\partial r}.$*
- (d)  $\Delta, \Delta_{rad}, \Delta_{tan}, \tilde{\Delta}$  *all commutes with the action of the unitary group.*
- (e)  $\tilde{\Delta} = (1 - r^2) \Delta_{tan} + (1 - r^2)^2 \Delta_{rad}$  *and*  $\Delta = \frac{\tilde{\Delta}}{1-r^2} + r^2 \Delta_{rad}.$

**Lemma 2.2.** *Let  $f \in C^2(B)$  and  $a = r\zeta$ ,  $0 < r < 1$ ,  $\zeta \in S$ . Then we have*

- (a)  $\tilde{\Delta}_{rad}f(a) = 4(1 - |a|^2)^2 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a);$
- (b)  $\tilde{\Delta}_{tan}f(a) = 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a).$

*Proof of Lemma 2.1.* (a) Let  $F(\lambda) = f(a + \lambda\zeta)$ ,  $\lambda \in B_1$ . Then calculating  $\Delta F(0)$  with chain rule gives the first equality, and the Taylor expansion gives the second equality of (a).

- (b) follows from (a) and the fact  $\Delta_{tan} = \Delta - \Delta_{rad}.$
- (c) also follows from a direct calculation using (a) and (b).

(d) Using the unitary invariance of  $\sigma$  with the obvious equality

$$\Delta f(a) = \lim_{\rho \rightarrow 0} \frac{4n}{\rho^2} \int_S \{f(a + \rho\eta) - f(a)\} d\sigma(\eta),$$

that  $\Delta f(Ua) = \Delta(f \circ U)(a)$  follows immediately. Noting that  $a = r\zeta$  in the integral representation of  $\Delta_{rad}$  in (a),  $\Delta_{rad}f(Ua) = \Delta_{rad}(f \circ U)(a)$  follows. So is  $\Delta_{tan}$ . Applying the identity  $U\varphi_a U^{-1} = \varphi_{Ua}$ , which is obvious from (1.1), to the representation

$$\tilde{\Delta}f(a) = \lim_{\rho \rightarrow 0} \frac{4n}{\rho^2} \int_S \{f \circ \varphi_a(\rho\eta) - f(a)\} d\sigma(\eta)$$

(see [3, 4.1.3]), that  $\tilde{\Delta}f(Ua) = \tilde{\Delta}(f \circ U)(a)$  follows.

(e) Direct calculation gives (e). See [1] Theorem 2.1. □

*Proof of Lemma 2.2.* (a) Let  $a = r\zeta$ . Let  $f \in C^2(B)$  and let  $F(\lambda) = f \circ \varphi_a(\lambda\zeta)$ ,  $\lambda \in B_1$ . Let  $\Delta_1$  denote the one variable  $\Delta$ . Then by definition,

$$\tilde{\Delta}_{rad}f(a) = \Delta_1 F(0).$$

By the chain rule,

$$\begin{aligned} \Delta_1 F(\lambda) &= 4 \frac{\partial^2}{\partial \lambda \partial \bar{\lambda}} [f \circ \varphi_a(\lambda\zeta)] \\ &= 4 \sum_{j,k=1}^n (D_k \bar{D}_j f) \circ \varphi_a(\lambda\zeta) \frac{\partial}{\partial \lambda} [\varphi_a(\lambda\zeta)]_k \frac{\partial}{\partial \bar{\lambda}} [\bar{\varphi}_a(\lambda\zeta)]_j. \end{aligned}$$

Let  $s = \sqrt{1 - |a|^2}$ . Then by the definition of  $\varphi_a$ ,

$$\varphi_a(\lambda\zeta) = a - s\lambda\zeta + \frac{s}{1+s} \lambda|a|a + R(\lambda, a)$$

with  $\lambda$ -degree of  $R(\lambda, a) \geq 2$  (see [3, p. 48]). Hence

$$\begin{aligned} \frac{\partial}{\partial \lambda} [\varphi_a(\lambda\zeta)]_k \Big|_{\lambda=0} &= -s\zeta_k + \frac{s}{1+s} |a|^2 \zeta_k; \\ \frac{\partial}{\partial \bar{\lambda}} [\bar{\varphi}_a(\lambda\zeta)]_j \Big|_{\lambda=0} &= -s\bar{\zeta}_j + \frac{s}{1+s} |a|^2 \bar{\zeta}_j. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \Delta_1 F(0) &= 4 \sum_{j,k=1}^n (D_k \bar{D}_j f)(a) \left\{ -s\zeta_k + \frac{s}{1+s} |a|^2 \zeta_k \right\} \left\{ -s\bar{\zeta}_j + \frac{s}{1+s} |a|^2 \bar{\zeta}_j \right\} \\ &= 4s^4 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a), \end{aligned}$$

which gives the equality (a).

(b) By (a),

$$\tilde{\Delta}_{tan}f(a) = \tilde{\Delta}f(a) - \tilde{\Delta}_{rad}f(a)$$

$$\begin{aligned}
&= 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{k,j} - a_k \bar{a}_j) D_k \bar{D}_j f(a) \\
&\quad - 4(1 - |a|^2)^2 \sum_{j,k=1}^n \zeta_k \bar{\zeta}_j D_k \bar{D}_j f(a) \\
&= 4(1 - |a|^2) \sum_{j,k=1}^n (\delta_{k,j} - \zeta_k \bar{\zeta}_j) D_k \bar{D}_j f(a),
\end{aligned}$$

which gives (b).  $\square$

Now, Theorem 1.2 is straightforward from Lemma 2.1 and Lemma 2.2.

### 3. Remarks

Theorem 1.4 says that pluri-harmonic Bloch functions can be characterized in terms of the radial growth or tangential growth of the invariant Laplacian only. See [1, 2] for the definition of pluri-harmonic Bloch function.

This type of characterization can be expanded (possibly through Theorem 1.2(d)) also to some function classes involved with surface or volume integrals.

### References

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