Bull. Korean Math. Soc. **56** (2019), No. 3, pp. 667–673
https://doi.org/10.4134/BKMS.b180477
pISSN: 1015-8634 / eISSN: 2234-3016

SOME REMARKS ON SUMSETS AND RESTRICTED SUMSETS

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Abstract. Let A be a finite set of integers. For any integer $h \geq 1$, let h-fold sumset hA be the set of all sums of h elements of A and let h-fold restricted sumset $h^\wedge A$ be the set of all sums of h distinct elements of A. In this paper, we give a survey of problems and results on sumsets and restricted sumsets of a finite integer set. In details, we give the best lower bound for the cardinality of restricted sumsets $2^\wedge A$ and $3^\wedge A$ and also discuss the cardinality of restricted sumset $h^\wedge A$.

1. Introduction

Let $\mathbb{N}$ denote the set of all nonnegative integers. Let A be a finite nonempty integer set and let $l(A)$ denote the difference of the largest and the smallest elements of A. For any finite set of integers A and any positive integer $h \geq 1$, define

$$hA = \{a_1 + \cdots + a_h : a_i \in A (1 \leq i \leq h)\},$$
$$h^\wedge A = \{a_1 + \cdots + a_h : a_i \in A (1 \leq i \leq h),\ a_i \neq a_j \text{ for all } i \neq j\}.$$

Here, $h^\wedge A = \emptyset$ if $|A| < h$. Let A, B be sets of integers, define

$$A + B = \{a + b : a \in A, b \in B\}.$$

Sumsets are one of the central objects of study in additive number theory. Nathanson [12] proved the following fundamental and important results:

Theorem A ([12], Theorem 1.3). *Let $h \geq 2$ be an integer and A a finite set of integers with $|A| = k$. Then*

$$|hA| \geq hk - h + 1.$$

Theorem B ([12], Theorem 1.6). *Let $h \geq 2$ be an integer and A a finite set of integers with $|A| = k$. Then*

$$|hA| = hk - h + 1$$

Received May 17, 2018; Revised October 18, 2018; Accepted January 8, 2019.
2010 *Mathematics Subject Classification.* 11B13.
Key words and phrases. restricted sumset, pigeonhole principle.
This work was supported by National Natural Science Foundation of China, Grant No. 11471017.

if and only if A is a k-term arithmetic progression.

In 1995, Nathanson [11] considered the set of all sums of distinct elements of A. He obtained a lower bound for $|h^\wedge A|$ and determined the structure of the finite sets A of integers for which $|h^\wedge A|$ is minimal.

Theorem C ([11], Theorem 1)**.** *Let A be a set of k integers and let $1 \leq h \leq k$. Then*

$$|h^\wedge A| \geq hk - h^2 + 1.$$

Theorem D ([11], Theorem 2)**.** *Let $k \geq 5$ and let $2 \leq h \leq k-2$. If A is a set of k integers such that*

$$|h^\wedge A| = hk - h^2 + 1,$$

then A is an arithmetic progression.

In 2014, Mistri and Pandey [9] generalized the above results. In 2015, Yang and Chen [15] generalized the results of Nathanson, and the results of Mistri and Pandey [9] which was actually proposed in [9], by Mistri and Pandey.

In 1959, Freiman [2] proved the following result:

Theorem E. *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. We have*

(i) *If $a_{k-1} \geq 2k-3$ and $\gcd(a_1, \ldots, a_{k-1}) = 1$, then $|2A| \geq 3k-3$.*

(ii) *If $a_{k-1} = k-1+r \leq 2k-3$ with $r \in [0, k-2]$, then $|2A| \geq 2k-1+r = k + a_{k-1}$.*

Theorem E shows that if $|A| = k$ and $|2A| \leq 3k-4$, then A is a subset of a short arithmetic progression. Moreover, Theorem E(ii) can be extend to $h \geq 2$ under the condition $a_{k-1} \leq 2k-3$ (see [12], Exercise 1.9.17).

Theorem F. *Let $h \geq 2$ and $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$. If $a_{k-1} = k-1+r \leq 2k-3$ with $r \in [0, k-2]$, then $|hA| \geq k + (h-1)a_{k-1}$.*

In 1962, Freiman [3] generalized Theorem E to the case of two sets.

Theorem G. *Let $A = \{a_1, \ldots, a_k\}$ and $B = \{b_1, \ldots, b_l\}$ be two sets of integers. If $a_k \leq k+l-3$, then $|A+B| \geq a_k + l$. If $a_k \geq k+l-2$ and $(a_1, \ldots, a_k, b_1, \ldots, b_l) = 1$, then $|A+B| \geq k+l+\min\{k,l\}-3$.*

There is a certain number of beautiful articles on this topic, see ([1], [4,5,7,8], [10], [13,14]).

In this paper, we give the best lower bound for the cardinality of restricted sumsets $2^\wedge A$ and $3^\wedge A$ under the condition $l(A) \leq 2|A|-5$. The paper is organized as follows. In Section 2, we focus on the cardinality of restricted sumset $2^\wedge A$. In Section 3, we focus on the cardinality of restricted sumset $3^\wedge A$. In Section 4, we give a remark on the cardinality of restricted sumset $h^\wedge A$.

2. The cardinality of restricted sumset $2^\wedge A$

The proof of Theorem 2.1 (which is actually an exercise in Nathanson's book [12]) have been already appeared in the article [6]. Here we give a simple combinatorial proof which was given by the anonymous referee in commenting on the first version of our manuscript.

Theorem 2.1. *Let A be a finite nonempty integer set with $|A| \geq 4$. If $l(A) \leq 2|A|-5$, then $|2^\wedge A| \geq |A| + l(A) - 2$.*

Proof. Without loss of generality, we may assume that $\{0, l(A)\} \subset A \subset [0, l(A)]$ and $l(A) \leq 2|A|-5$. Put $B := [0, l(A)] \setminus A$. We shall show that for each $b \in B$ one has $\{b, b+l(A)\} \cap 2^\wedge A \neq \emptyset$. Suppose that there exists an integer $b \in B$ such that neither b nor $b + l(A)$ lie in $2^\wedge A$, then by the pigeonhole principle, we have

$$\big|[0,b] \cap A\big| \leq \frac{b}{2} + 1 \text{ and } \big|[b, l(A)] \cap A\big| \leq \frac{l(A)-b}{2} + 1.$$

Thus

$$|A| \leq \frac{l(A)}{2} + 2,$$

which contradicts with the assumption $l(A) \leq 2|A| - 5$.

Since $\{a, a+l(A)\} \cup \{l(A)\} \subset 2^\wedge A$ whenever $a \in A \setminus \{0, l(A)\}$, this gives $|2^\wedge A| \geq 2|A|-3$. And when $b \in B$, we have $\{b, b+l(A)\} \cap 2^\wedge A \neq \emptyset$, hence

$$\begin{aligned} |2^\wedge A| &\geq 2|A| - 3 + |B| \\ &= 2|A| - 3 + l(A) + 1 - |A| \\ &= l(A) + |A| - 2. \end{aligned}$$

□

Remark 2.1. The lower bound in Theorem 2.1 is best possible. For example, let $A = \{0,2,3,4,5\}$, we have $2^\wedge A = \{2,3,4,5,6,7,8,9\}$ and $|2^\wedge A| = 8 = |A| + l(A) - 2$.

Remark 2.2. The assumption $l(A) \leq 2|A| - 5$ can not be relaxed in Theorem 2.1. For example, let $A = \{0, 1, l(A)-2, l(A)-1, l(A)\}$ with $l(A) \geq 2|A|-4 = 6$. Then

$$2^\wedge A = \{1, l(A)-2, l(A)-1, l(A), l(A)+1, 2l(A)-3, 2l(A)-2, 2l(A)-1\}$$

and $|2^\wedge A| = 8 < |A| + l(A) - 2$.

3. The cardinality of restricted sumset $3^\wedge A$

Theorem 3.1. *Let A be a finite nonempty integer set with $|A| \geq 5$. If $l(A) \leq 2|A|-5$, then $|3^\wedge A| \geq 2|A| + l(A) - 7$.*

Proof. Let $|A| = k$, we may assume that $A = \{a_0, a_1, \ldots, a_{k-1}\}$ with $0 = a_0 < a_1 < \cdots < a_{k-1} \leq 2|A| - 5$. Then $l(A) = a_{k-1}$. Define r by $a_{k-1} = k - 1 + r$, and let $B = [0, a_{k-1}] \setminus A$.

Consider the set

$$T = \{a_1 + a_i : i = 2, \ldots, k-2\} \cup \{a_i + a_{k-1} : i = 1, \ldots, k-2\}$$
$$\cup \{a_i + a_{k-2} + a_{k-1} : i = 1, \ldots, k-3\}.$$

Then $|T| = 3|A| - 8$. Since $B = [0, a_{k-1}] \setminus A$, we have $|B| = a_{k-1} + 1 - k = r$. By the proof of Theorem 2.1, we have $2^\wedge A \cap \{b, b + a_{k-1}\} \neq \emptyset$ for each $b \in B$.

If $b \in 2^\wedge A$, then $b = a_i + a_j$, where $a_i, a_j < a_{k-1}$ and $a_i \neq a_j$. Thus $b + a_{k-1} = a_i + a_j + a_{k-1} \in 3^\wedge A$.

If $b + a_{k-1} \in 2^\wedge A$, then $b + a_{k-1} = a_i + a_j$, where $a_i, a_j < a_{k-1}$. Thus $b + 2a_{k-1} = a_i + a_j + a_{k-1} \in 3^\wedge A$. Hence $3^\wedge A \cap \{b + a_{k-1}, b + 2a_{k-1}\} \neq \emptyset$ for each $b \in B$.

Next, we shall prove $|3^\wedge A \setminus T| \geq r$.

Case 1. $b + a_{k-1} \in 3^\wedge A$. If $b + a_{k-1} \notin T$, then $b + a_{k-1} \in 3^\wedge A \setminus T$. Noting that $b + a_{k-1} \neq a_i + a_{k-1} (i = 1, \ldots, k-2)$, we consider the following four cases.

Case 1.1. $b + a_{k-1} = a_1 + a_i (i = 2, \ldots, k-3)$. Then $b + 2a_{k-1} = a_1 + a_i + a_{k-1} (i = 2, \ldots, k-3)$. Since

$$a_{k-2} + a_{k-1} < b + 2a_{k-1} < a_1 + a_{k-2} + a_{k-1},$$

we have $b + 2a_{k-1} \in 3^\wedge A \setminus T$.

Case 1.2. $b + a_{k-1} = a_1 + a_{k-2}$. Then $b + 2a_{k-1} = a_1 + a_{k-2} + a_{k-1}$. We show that $2a_{k-1} \in 3^\wedge A$. Suppose that $2a_{k-1} \notin 3^\wedge A$, then except for $a_{k-1} = 2a_i$ for some $1 \leq i \leq k-2$, we have

$$\{2a_{k-1} - (a_j + a_{k-1}) : j = 1, \ldots, k-2\} \cap \{a_1, \ldots, a_{k-2}\} = \emptyset.$$

Write

$$A_1 = \{2a_{k-1} - (a_j + a_{k-1}) : j = 1, \ldots, k-2\}, A_2 = \{a_1, \ldots, a_{k-2}\}.$$

Then sets A_1, A_2 are pairwise disjoint except for at most one exception. Thus $|A_1 \cup A_2| \geq 2k-5$, which contradicts with the fact that $A_1, A_2 \subseteq \{1, \ldots, a_{k-1} - 1\} \subseteq \{1, \ldots, 2k-6\}$. Noting that $a_{k-2} + a_{k-1} < 2a_{k-1} < a_1 + a_{k-2} + a_{k-1}$, we have $2a_{k-1} \in 3^\wedge A \setminus T$.

Case 1.3. $b + a_{k-1} = a_1 + a_{k-2} + a_{k-1}$. Then $b = a_1 + a_{k-2}$. We show that $a_{k-1} \in 3^\wedge A$. Suppose that $a_{k-1} \notin 3^\wedge A$, then except for $a_{k-1} = 2a_i$ for some $1 \leq i \leq k-2$, we have

$$\{a_{k-1} - a_j : j = 1, \ldots, k-2\} \cap \{a_1, \ldots, a_{k-2}\} = \emptyset.$$

Write

$$B_1 = \{a_{k-1} - a_j : j = 1, \ldots, k-2\}, B_2 = \{a_1, \ldots, a_{k-2}\}.$$

Then sets B_1, B_2 are pairwise disjoint except for at most one exception. Thus $|B_1 \cup B_2| \geq 2k-5$, which contradicts with the fact that $B_1, B_2 \subseteq \{1, \ldots, a_{k-1} - 1\} \subseteq \{1, \ldots, 2k-6\}$. Noting that

$$a_1 + a_{k-2} = b < a_{k-1} < a_1 + a_{k-1},$$

then $a_{k-1} \in 3^\wedge A \setminus T$.

Case 1.4. $b + a_{k-1} = a_i + a_{k-2} + a_{k-1} (i = 2, \dots, k-3)$. Then $b = a_i + a_{k-2} (i = 2, \dots, k-3)$. Thus $b \in 3^\wedge A$. Moreover,

$$a_1 + a_{k-2} < b < a_1 + a_{k-1},$$

we have $b \in 3^\wedge A \setminus T$.

Case 2. $b + 2a_{k-1} \in 3^\wedge A$. If $b + 2a_{k-1} \notin T$, then $b + 2a_{k-1} \in 3^\wedge A \setminus T$. Noting that $b + 2a_{k-1} > a_{k-2} + a_{k-1}$, we consider the following two cases.

Case 2.1. $b + 2a_{k-1} = a_1 + a_{k-2} + a_{k-1}$. This is same as Case 1.2. We have $2a_{k-1} \in 3^\wedge A$. Moreover,

$$a_{k-2} + a_{k-1} < 2a_{k-1} < a_1 + a_{k-2} + a_{k-1},$$

we have $2a_{k-1} \in 3^\wedge A \setminus T$.

Case 2.2. $b + 2a_{k-1} = a_i + a_{k-2} + a_{k-1} (i = 2, \dots, k-3)$. Then $b + a_{k-1} = a_i + a_{k-2} (i = 2, \dots, k-3)$. Thus $b + a_{k-1} \in 3^\wedge A$. Moreover,

$$a_1 + a_{k-2} < b + a_{k-1} < a_{k-3} + a_{k-1}$$

and $b + a_{k-1} \neq a_j + a_{k-1}$ for each $a_j \in A$. We have $b + a_{k-1} \in 3^\wedge A \setminus T$.

By Case 1, Case 2 and the fact that

$$\bigcup_{b \in B} \{b\}, \{a_{k-1}\}, \{2a_{k-1}\}, \bigcup_{b \in B} \{b + a_{k-1}, b + 2a_{k-1}\}$$

are pairwise disjoint, we have $|3^\wedge A \setminus T| \geq r$. Hence, $|3^\wedge A| \geq 3k - 8 + r = 2|A| + l(A) - 7$.

This completes the proof of Theorem 3.1. □

Remark 3.1. The assumption $l(A) \leq 2|A| - 5$ can not be relaxed in Theorem 3.1. For example, let $A = \{0, 1, l(A)-2, l(A)-1, l(A)\}$ with $l(A) \geq 2|A|-4 = 6$. Then

$$3^\wedge A = \{l(A)-1, l(A), l(A)+1, 2l(A)-3, 2l(A)-2, 2l(A)-1, 2l(A), 3l(A)-3\}$$

and $|3^\wedge A| = 8 < 2|A| + l(A) - 7$.

Remark 3.2. The estimate for $|3^\wedge A|$ is sharp. For example, let $A = \{0, 2, 3, 4, 5, 6, 7\}$. We have $3^\wedge A = \{5, 6, \dots, 18\}$, and hence $|3^\wedge A| = 14 = 2|A| + l(A) - 7$.

4. Concluding remark

Remark 4.1. Let $h \geq 3$ and A be a finite nonempty integer set with $|A| \geq 5$. If $l(A) \leq 2|A| - 2h + 1$, then $|h^\wedge A| \geq (h-1)|A| + l(A) - h^2 + 2$.

Theorem 3.1 implies the result holds for $h = 3$. Now, let $h \geq 4$. Write $A = \{a_0, a_1, \dots, a_{k-1}\}$ with $0 = a_0 < a_1 < \cdots < a_{k-1} = l(A)$ and $k = |A|$. Assume that the result holds for $h - 1$, that is if $l(A) \leq 2|A| - 2h + 3$, then $|(h-1)^\wedge A| \geq (h-2)|A| + l(A) - (h-1)^2 + 2$. Now we shall prove the result holds for h. Write $B = A \setminus \{a_1\}$. Since $l(A) \leq 2|A| - 2h + 1$, we have

$$l(B) = l(A) \leq 2|A| - 2h + 1 = 2|B| - 2h + 3.$$

It follows from the induction hypothesis that

$$|(h-1)^\wedge B+a_1| \geq (h-2)|B|+l(B)-(h-1)^2+2 = (h-2)|A|+l(A)-(h-1)^2-h+4.$$

Notice also that $(h-1)^\wedge B + a_1 \subset h^\wedge A$ and $\max((h-1)^\wedge B + a_1) = a_1 + a_{k-h+1} + \cdots + a_{k-1}$. Consequently, the set $(h-1)^\wedge B + a_1$ is disjoint from the set

$$C = \{a_i + a_{k-h+1} + \cdots + a_{k-1} : 2 \leq i \leq k-h\} \subset h^\wedge A.$$

Therefore

$$\begin{aligned}|h^\wedge A| &\geq |(h-1)^\wedge B + a_1| + |C| \\ &\geq (h-2)|A| + l(A) - (h-1)^2 - h + 4 + (|A| - h - 1) \\ &= (h-1)|A| + l(A) - h^2 + 2.\end{aligned}$$

Hence, by induction the result holds for all $h \geq 3$.

Acknowledgment. We are grateful to the anonymous referees for their valuable comments on this work.

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