

ON CONGRUENCES INVOLVING THE GENERALIZED CATALAN NUMBERS AND HARMONIC NUMBERS

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ABSTRACT. In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2 , one of which is

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d} \equiv 4(-1)^d \left\{ \frac{1}{3}d(2d^2+1)(4pH_d-1) - p \left(\frac{26}{9}d^3 + \frac{4}{3}d^2 + \frac{7}{9}d + \frac{1}{2} \right) \right\} \pmod{p^2},$$

where a prime number $p > 3$ and $1 \leq d \leq p$.

1. Introduction

The Catalan numbers have important applications in combinatorics, number theory and the analysis algorithms in [5, 6, 8, 10].

In [10], Shapiro gave the generalized Catalan numbers $B_{n,k}$ as follows:

$$B_{n,k} = \frac{k}{n} \binom{2n}{n-k}, \quad 0 \leq k \leq n.$$

The numbers $B_{n,k}$ are the entries of the Catalan triangles and satisfy the recurrence relation

$$B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad k \geq 2$$

with the initial conditions $B_{n,0} = 0 = B_{n,n+m}$, $m \geq 1$. The numbers $B_{n,k}$ are not known as Catalan numbers. However they have several applications [3, 7, 10].

Note that for $k = 1$, $B_{n,1}$ are the well known Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 1.$$

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Harmonic numbers are those rational numbers given by

$$H_0 = 0, \quad H_n = \sum_{k=1}^n \frac{1}{k}, \quad n \in \mathbb{Z}^+,$$

and for $m \in \mathbb{Z}^+$, harmonic numbers of order m are those rational numbers

$$H_{n,m} = \sum_{k=1}^n \frac{1}{k^m}, \quad n \in \mathbb{Z}^+.$$

Some interesting properties of harmonic numbers are given as follows [1,2,4,11]:

$$(1) \quad \sum_{k=1}^{n-1} \frac{k^m}{n-k} H_k = n^m (H_n (H_n - H_m) - H_{n,2} + H_{m,2}),$$

$$(2) \quad \sum_{k=1}^{n-1} k^m H_{n-k} = n^m \frac{n+1}{m+1} (H_{n+1} - H_{m+1}),$$

$$(3) \quad \sum_{k=1}^n k^m H_k = n^m \frac{n+1}{m+1} \left(H_{n+1} - \frac{1}{m+1} \right),$$

$$(4) \quad \sum_{k=1}^n \frac{H_k}{k} = \frac{1}{2} (H_n^2 + H_{n,2}),$$

where x^m stands for the falling factorial defined by

$$x^m = x(x-1)\cdots(x-m+1).$$

In [12], Sun showed that for any odd prime p and $0 \leq k < p$,

$$(5) \quad (-1)^k \binom{p-1}{k} = \prod_{0 < j \leq k} \left(1 - \frac{p}{j} \right) \equiv 1 - pH_k \pmod{p^2}.$$

In [6], Gutiérrez et al. gave some identities involving well-known Catalan numbers. For example,

$$\sum_{i=1}^m B_{n,i} B_{n,n+i-m} (n+2i-m) = (n+1) C_n \binom{2(n-1)}{m-1}, \quad m \leq n,$$

$$\sum_{i=1}^n i (B_{n,i})^2 = \frac{n(n+1)}{2} C_n C_{n-1}.$$

In [8], Miana and Romero showed the following identity that for $1 \leq m \leq n$,

$$\sum_{i=1}^m B_{n,i} B_{n,n+i-m} (n+2i-m)^3 = \binom{2n}{n} \binom{2(n-1)}{m-1} (n^2 + 4n - 2nm + m^2).$$

In [9], Ömür and Kopal obtained various congruences involving the numbers $B_{p,k-d}$. For example, for $1 \leq d \leq p-1$,

$$\sum_{k=1}^{p-1} B_{p,k} B_{p,k-d} \equiv 4(-1)^d \left(-2d + p \left(8d(H_d - 1) + \frac{2}{d} - 3 \right) \right) \pmod{p^2}$$

and for $1 \leq d \leq p$,

$$\sum_{k=1}^{p-1} k B_{p,k} B_{p,k-d} \equiv 2(-1)^d (-2d^2 + p(1 + 8d^2(H_d - 1) - 4d)) \pmod{p^2}.$$

In this paper, we prove some congruences involving the generalized Catalan numbers and harmonic numbers modulo p^2 , one of which is

$$\begin{aligned} \sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d} \equiv 4(-1)^d \left\{ \frac{1}{3}d(2d^2 + 1)(4pH_d - 1) \right. \\ \left. - p \left(\frac{26}{9}d^3 + \frac{4}{3}d^2 + \frac{7}{9}d + \frac{1}{2} \right) \right\} \pmod{p^2}, \end{aligned}$$

where a prime number $p > 3$ and $1 \leq d \leq p$.

2. Some congruences involving the generalized Catalan numbers and harmonic numbers

In this section, firstly, we will continue our work with the following lemmas.

Lemma 2.1. *Let $p > 3$ be a prime number. For $0 \leq m < p-1$ and $1 \leq d \leq p-1$, we have*

$$\begin{aligned} \sum_{k=d}^{p-1} k^m H_{k-d} \\ \equiv (p+d-1)^m \frac{p+d}{p(m+1)} - (d-1)^m \frac{d}{m+1} \left(\frac{1}{p} + H_d - H_{m+1} \right) \pmod{p}. \end{aligned}$$

Proof. Observed that

$$\begin{aligned} \sum_{k=1}^{p+d-2} k^m H_{p+d-1-k} &= \sum_{k=1}^{d-1} k^m H_{p+d-1-k} + \sum_{k=d}^{p-1} k^m H_{p+d-1-k} \\ &\quad + \sum_{k=p}^{p+d-2} k^m H_{p+d-1-k}. \end{aligned}$$

From the congruences for $1 \leq k, m < p$,

$$(6) \quad \begin{aligned} H_{p-k} &\equiv H_{k-1} \pmod{p}, \quad (p+k)^m \equiv k^m \pmod{p} \text{ and} \\ H_{p+k} &\equiv H_k + \frac{1}{p} \pmod{p}, \end{aligned}$$

we have

$$\begin{aligned} \sum_{k=1}^{p+d-2} k^m H_{p+d-1-k} &\equiv \sum_{k=1}^{d-1} k^m \left(H_{d-1-k} + \frac{1}{p} \right) + \sum_{k=d}^{p-1} k^m H_{k-d} + \sum_{k=0}^{d-2} k^m H_{d-1-k} \\ &\equiv 2 \sum_{k=1}^{d-2} k^m H_{d-1-k} + \frac{1}{p} \sum_{k=1}^{d-1} k^m + \sum_{k=d}^{p-1} k^m H_{k-d} \pmod{p}. \end{aligned}$$

Considering (2) and (6), we write

$$\begin{aligned} \sum_{k=d}^{p-1} k^m H_{k-d} &\equiv \sum_{k=1}^{p+d-2} k^m H_{p+d-1-k} - 2 \sum_{k=1}^{d-2} k^m H_{d-1-k} - \frac{1}{p} \sum_{k=1}^{d-1} k^m \\ &\equiv -(d-1)^m \frac{d}{m+1} (H_d - H_{m+1}) + (p+d-1)^m \frac{p+d}{p(m+1)} \\ &\quad - \frac{1}{p} \sum_{k=1}^{d-1} k^m \pmod{p}. \end{aligned}$$

It follows from the sum $\sum_{k=a}^n k^m = \frac{1}{m+1} ((n+1)n^{m+1} - a(a-1)^{m+1})$ that

$$\begin{aligned} \sum_{k=d}^{p-1} k^m H_{k-d} &\equiv (p+d-1)^m \frac{p+d}{p(m+1)} - (d-1)^m \frac{d}{m+1} \left(\frac{1}{p} + H_d - H_{m+1} \right) \pmod{p}. \end{aligned}$$

Thus, we complete the proof. □

Lemma 2.2. *Let p be an odd prime. For $1 < d \leq p-1$, we have*

$$\sum_{k=d}^{p-1} \frac{1}{k} H_{k-d} \equiv H_{p+d-1}^2 - H_{p+d-1,2} - 2H_{d-1}^2 + 2H_{d-1,2} - \frac{2}{p} H_{d-1} \pmod{p}.$$

Proof. We write

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} = \sum_{k=1}^{d-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=d}^{p-1} \frac{1}{k} H_{p+d-k-1} + \sum_{k=p}^{p+d-2} \frac{1}{k} H_{p+d-k-1}.$$

From the congruences $H_{p+k} \equiv H_k + \frac{1}{p} \pmod{p}$ and $H_{p-k} \equiv H_{k-1} \pmod{p}$ for $1 \leq k < p$, we get

$$\begin{aligned} \sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} &\equiv \sum_{k=1}^{d-1} \frac{1}{k} \left(H_{d-k-1} + \frac{1}{p} \right) + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} \\ &\quad + \frac{H_{d-1}}{p} + \sum_{k=1}^{d-2} \frac{1}{p+k} H_{d-k-1} \pmod{p}. \end{aligned}$$

By the congruence $\frac{1}{p+k} \equiv \frac{1}{k} \pmod{p}$ for $1 \leq k < p$, we have

$$\sum_{k=1}^{p+d-2} \frac{1}{k} H_{p+d-k-1} \equiv 2 \sum_{k=1}^{d-2} \frac{1}{k} H_{d-k-1} + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} + 2 \frac{H_{d-1}}{p} \pmod{p}.$$

Considering the sum

$$(7) \quad \sum_{k=1}^{n-1} \frac{1}{k} H_{n-k} = H_n^2 - H_{n,2}$$

from (1), we get

$$H_{p+d-1}^2 - H_{p+d-1,2} \equiv 2(H_{d-1}^2 - H_{d-1,2}) + \sum_{k=d+1}^{p-1} \frac{1}{k} H_{k-d} + 2 \frac{H_{d-1}}{p} \pmod{p}.$$

Thus the desired result is obtained. □

Lemma 2.3. For $n \in \mathbb{N}$, we have

$$\sum_{k=1}^n (-1)^k k^2 H_k = (-1)^n \binom{n+1}{2} H_n + (-1)^n \frac{n}{4} + \frac{1}{8} (1 - (-1)^n).$$

Proof. Observed that

$$\begin{aligned} \sum_{k=1}^n (-1)^k k^2 H_k &= \sum_{k=1}^n (-1)^k k^2 \sum_{j=1}^k \frac{1}{j} = \sum_{j=1}^n \frac{1}{j} \sum_{k=j}^n (-1)^k k^2 \\ &= \sum_{j=1}^n \frac{1}{j} \sum_{k=1}^n (-1)^k k^2 - \sum_{j=1}^n \frac{1}{j} \sum_{k=1}^{j-1} (-1)^k k^2. \end{aligned}$$

Applying $\sum_{k=1}^n (-1)^k k^2 = (-1)^n \binom{n+1}{2}$, we write

$$\sum_{k=1}^n (-1)^k k^2 H_k = (-1)^n \binom{n+1}{2} H_n + \frac{1}{2} \sum_{j=1}^n (-1)^j (j-1).$$

By the sums

$$\sum_{k=1}^n (-1)^k k = -\frac{1}{4} + \frac{(-1)^n}{2} \left(n + \frac{1}{2}\right) \quad \text{and} \quad \sum_{k=1}^n (-1)^k = \frac{(-1)^n - 1}{2},$$

we conclude the proof. □

Now we give some congruences involving harmonic numbers in the following theorems.

Theorem 2.4. Let p be an odd prime. For $1 < d < p$, we have

$$\sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \equiv 4(-1)^d \left\{ H_d + H_{d-1} + \frac{2}{p} + p(4H_d^2 + 4H_{d-1}^2) \right\}$$

$$-2H_{d,2} - 2H_{d-1,2} - H_{p+d}^2 - H_{p+d-1}^2) \} \pmod{p^2},$$

and for $3 < d \leq p$,

$$\begin{aligned} & \sum_{k=1}^{p-1} (-1)^k k^2 B_{p,k} B_{p,k-d} \\ \equiv & -4d + 4p \left\{ 1 + \frac{(-1)^d}{2} + \binom{d+1}{2} \left(H_{\lfloor \frac{d-1}{2} \rfloor} + H_{\lfloor \frac{p+d-1}{2} \rfloor} \right) \right. \\ & \left. - \binom{d}{2} \left(H_{\lfloor \frac{d-2}{2} \rfloor} + H_{\lfloor \frac{p+d-2}{2} \rfloor} \right) \right\} \pmod{p^2}. \end{aligned}$$

Proof. From $B_{p,k-d} = -B_{p,d-k}$ for $k < d$, we write

$$\begin{aligned} \sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} &= \sum_{k=1}^d \frac{B_{p,k} B_{p,k-d}}{k} + \sum_{k=d+1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \\ &= -\sum_{k=1}^{d-2} \frac{B_{p,k} B_{p,d-k}}{k} + \sum_{k=d+1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \\ &\quad - \frac{1}{p^2} \binom{2p}{p-1} \binom{2p}{p-d+1}. \end{aligned}$$

For $1 < k \leq p$ and $d+2 \leq k$, using the congruence

$$(8) \quad B_{p,k-d} \equiv 2(-1)^{d-k-1} (1 - p(H_{k-d} + H_{k-1-d})) \pmod{p^2},$$

in [9], we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \\ \equiv & 4(-1)^d \left\{ p \sum_{k=1}^{d-2} \frac{H_k + H_{k-1}}{k} + p \sum_{k=1}^{d-2} \frac{H_{d-k} + H_{d-k-1}}{k} \right. \\ & \left. + \sum_{k=d+1}^{p-1} \frac{1}{k} - \sum_{k=1}^{d-2} \frac{1}{k} - p \sum_{k=d+1}^{p-1} \frac{H_{k-d} + H_{k-d-1}}{k} - p \sum_{k=d+1}^{p-1} \frac{H_k + H_{k-1}}{k} \right\} \\ & - \frac{1}{p^2} \binom{2p}{p-1} \binom{2p}{p-d+1} \pmod{p^2}. \end{aligned}$$

Using (4), (7) and Lemma 2.2, we write

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \\ \equiv & 4(-1)^d \left\{ p H_{d-2}^2 + p \left(H_d^2 + H_{d-1}^2 - H_{d,2} - H_{d-1,2} - \frac{1}{d-1} \right) + H_{p-1} - H_d - H_{d-2} \right. \\ & \left. - p \left(H_{p+d-1}^2 + H_{p+d}^2 - H_{p+d-1,2} - H_{p+d,2} - 2 \left(H_{d-1}^2 + H_d^2 - H_{d-1,2} - H_{d,2} \right) \right) \right\} \end{aligned}$$

$$+2(H_{d-1} + H_d) - p(H_{p-1}^2 - H_d^2) \Big\} - \frac{1}{p^2} \binom{2p}{p-1} \binom{2p}{p-d+1} \pmod{p^2}.$$

Since the congruences $H_{p+k,2} \equiv H_{k,2} + \frac{1}{p^2} \pmod{p}$ for $1 \leq k \leq p-1$ and $H_{p-1} \equiv 0 \pmod{p^2}$, we have

$$\begin{aligned} & \sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-d}}{k} \\ & \equiv 4(-1)^d \left\{ \frac{2}{p} - H_{d-2} + 2H_{d-1} + H_d + p \left(H_{d-2}^2 + 4H_d^2 - \frac{1}{d-1} \right) \right. \\ & \quad \left. - p(H_{p+d-1}^2 - 3H_{d-1}^2 + 2H_{d-1,2} + H_{p+d}^2 + 2H_{d,2}) \right\} \\ & \quad - \frac{1}{p^2} \binom{2p}{p-1} \binom{2p}{p-d+1} \pmod{p^2}. \end{aligned}$$

For $1 < d \leq p$, taking the congruence

$$(9) \quad \begin{aligned} & \frac{d-1}{p^2} \binom{2p}{p-d+1} \binom{2p}{p-1} \\ & \equiv 4(-1)^d \left(1 + \frac{p}{d-1} + \frac{2p}{d} - p - 2pH_d \right) \pmod{p^2}, \end{aligned}$$

in [9], we obtain the desired result. With the help of Lemma 2.3, the proof of the other result is given. \square

For example, for $p \geq 5$ and $d = 4$ in Theorem 2.4, we obtain the congruences

$$\sum_{k=1}^{p-1} \frac{B_{p,k} B_{p,k-4}}{k} \equiv 4 \left(\frac{47}{12} + \frac{2}{p} + p \left(\frac{1817}{72} - H_{p+4}^2 - H_{p+3}^2 \right) \right) \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} (-1)^k k^2 B_{p,k} B_{p,k-4} \equiv -16 + 4p \left\{ \frac{11}{2} + 10H_{\lfloor \frac{p+3}{2} \rfloor} - 6H_{\lfloor \frac{p+2}{2} \rfloor} \right\} \pmod{p^2}.$$

Theorem 2.5. *Let $p > 3$ be a prime number. For $0 < d \leq p$ and $0 < m < p-1$, we have*

$$\begin{aligned} & \sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ & \equiv \frac{4(-1)^d}{m+1} \left\{ p(m-1)!(-1)^m - (p+d)(p+d-1)^m - (p+d+1)(p+d)^m \right. \\ & \quad \left. + 2p(2H_d - H_m) \left((d+1)d^m + d(d-1)^m \right) \right. \\ & \quad \left. - p \left(\frac{8d}{(m+1)}(d-1)^m + \frac{m+1}{m}(d^m + (d-1)^m) \right) \right\} \pmod{p^2}. \end{aligned}$$

Proof. Observed that

$$\sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} = \sum_{k=1}^d k^m B_{p,k} B_{p,k-d} + \sum_{k=d+1}^{p-1} k^m B_{p,k} B_{p,k-d}.$$

From $B_{p,k-d} = -B_{p,d-k}$ for $k < d$, we get

$$\begin{aligned} \sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} &= -\sum_{k=1}^{d-2} k^m B_{p,k} B_{p,d-k} + \sum_{k=d+1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ &\quad - \frac{d-1}{p^2} (d-1)^m \binom{2p}{p-d+1} \binom{2p}{p-1}. \end{aligned}$$

It follows from (8) that

$$\begin{aligned} &\sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\sum_{k=1}^{d-2} k^m + p \sum_{k=1}^{d-2} k^m (H_k + H_{k-1}) + p \sum_{k=1}^{d-2} k^m (H_{d-k} + H_{d-k-1}) \right. \\ &\quad \left. + \sum_{k=d+1}^{p-1} k^m - p \sum_{k=d+1}^{p-1} k^m (H_{k-d} + H_{k-d-1}) - p \sum_{k=d+1}^{p-1} k^m (H_k + H_{k-1}) \right\} \\ &\quad - \frac{d-1}{p^2} (d-1)^m \binom{2p}{p-d+1} \binom{2p}{p-1} \pmod{p^2}. \end{aligned}$$

According to (2), (3) and Lemma 2.1, using the sum

$$\sum_{k=a}^n k^m = \frac{1}{m+1} ((n+1)n^m - a(a-1)^m),$$

we obtain

$$\begin{aligned} &\sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ &\equiv 4(-1)^d \left\{ -\frac{(d-1)(d-2)^m}{m+1} + 2p \left(\frac{(d-1)(d-2)^m}{m+1} \left(H_{d-1} - \frac{1}{m+1} \right) \right) \right. \\ &\quad + p \left(\frac{2(d+1)d^m}{m+1} (H_{d+1} - H_{m+1}) + \frac{2d(d-1)^m}{m+1} (H_d - H_{m+1}) - (d-1)^m \right) \\ &\quad - p \left(\frac{(p+d)(p+d-1)^m}{p(m+1)} - \frac{d(d-1)^m}{p(m+1)} + \frac{(p+d+1)(p+d)^m}{p(m+1)} - \frac{(d+1)d^m}{p(m+1)} \right) \\ &\quad - p \left(\frac{2p(p-1)^m}{m+1} \left(H_p - \frac{1}{m+1} \right) - 2 \frac{d+1}{m+1} d^m \left(H_{d+1} - \frac{1}{m+1} \right) - \frac{(p-1)^m - d^m}{m} \right) \\ &\quad \left. + \left(\frac{1}{m+1} (p(p-1)^m - (d+1)d^m) \right) - \frac{p}{m} (d-2)^m \right\} \end{aligned}$$

$$-\frac{d-1}{p^2}(d-1)^m \binom{2p}{p-d+1} \binom{2p}{p-1} \pmod{p^2}.$$

From (5) and (9), we write

$$\begin{aligned} & \sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ \equiv & 4(-1)^d \left\{ -\frac{(d-1)(d-2)^m}{m+1} + 2p \left(\frac{(d-1)(d-2)^m}{m+1} \left(H_{d-1} - \frac{1}{m+1} \right) \right) \right. \\ & + p \left(\frac{2(d+1)d^m}{m+1} (H_{d+1} - H_{m+1}) + \frac{2d(d-1)^m}{m+1} (H_d - H_{m+1}) - (d-1)^m \right) \\ & - \frac{p+d}{m+1} (p+d-1)^m + \frac{d(d-1)^m}{m+1} - \frac{p+d+1}{m+1} (p+d)^m \\ & - \frac{p}{m+1} m! (-1)^m - p \left(-2 \frac{d+1}{m+1} d^m \left(H_{d+1} - \frac{1}{m+1} \right) - \frac{1}{m} ((-1)^m m! - d^m) \right) \\ & \left. - (d-1)^m \left(\frac{p}{d-1} + \frac{2p}{d} + 1 - p - 2pH_d \right) - \frac{p}{m} (d-2)^m \right\} \pmod{p^2}. \end{aligned}$$

Using the identities $\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}$ and $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$, we have

$$\begin{aligned} & \sum_{k=1}^{p-1} k^m B_{p,k} B_{p,k-d} \\ \equiv & \frac{4(-1)^d}{m+1} \left\{ p(m-1)! (-1)^m - (p+d)(p+d-1)^m - (p+d+1)(p+d)^m \right. \\ & \quad + 2p(2H_d - H_m) ((d+1)d^m + d(d-1)^m) \\ & \quad \left. - p \left(\frac{8d}{(m+1)} (d-1)^m + \frac{m+1}{m} (d^m + (d-1)^m) \right) \right\} \pmod{p^2}. \end{aligned}$$

This completes the proof. □

Corollary 2.6. *Let $p > 3$ be a prime number. For $0 < d \leq p$, we have*

$$\begin{aligned} \sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-d} \equiv & 4(-1)^d \left\{ \frac{1}{3} d(2d^2 + 1)(4pH_d - 1) \right. \\ & \left. - p \left(\frac{26}{9} d^3 + \frac{4}{3} d^2 + \frac{7}{9} d + \frac{1}{2} \right) \right\} \pmod{p^2}. \end{aligned}$$

Proof. When $m = 2$ in Theorem 2.5, using the congruence

$$\sum_{k=1}^{p-1} k B_{p,k} B_{p,k-d} \equiv 2(-1)^d (-2d^2 + p(1 + 8d^2(H_d - 1) - 4d)) \pmod{p^2}$$

in [9], the proof easily is obtained. □

For example, for $p > 3$, $d = 2$ in Corollary 2.6, we have

$$\sum_{k=1}^{p-1} k^2 B_{p,k} B_{p,k-2} \equiv 2(11p - 12) \pmod{p^2}.$$

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