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SOME STABILITY RESULTS FOR SEMILINEAR STOCHASTIC HEAT EQUATION DRIVEN BY A FRACTIONAL NOISE

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Abstract. In this paper, we consider a semilinear stochastic heat equation driven by an additive fractional white noise. Under the pathwise uniqueness property, we establish various strong stability results. As a consequence, we give an application to the convergence of the Picard successive approximation.

1. Introduction

During the last past years, some spectacular advances have been made in order to study the solutions to stochastic partial differential equations driven by general Brownian noises. Most of the research developed has been mainly focused on the analysis of heat and wave equations perturbed by Gaussian white noises (see, for instance, [5, 8, 23]). Recently, there has been a growing interest in studying stochastic partial differential equations driven by a Gaussian noise which has the covariance structure of the fractional Brownian motion (fBm) in time, combined with a white spatial covariance structure (see, for instance, [14, 20]). This interest comes from the large number of applications of the fBm in practice. To list only a few examples of the appearance of fractional noises in practical situations, we mention [16] for biophysics, [4] for financial time series, [12] for electrical engineering, and [10] for physics.

Let us now consider the quasi-linear stochastic partial differential equation (SPDE)

$$\frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial^2 x}u(t,x) + b(t,x,u(t,x)) + \frac{\partial^2}{\partial t\partial x}W^H(t,x), \tag{1}$$

with the initial condition $u(0,x) = u_0(x)$, $x \in [0,1]$ and Neumann boundary conditions

$$\frac{\partial}{\partial x}u(t,0) = \frac{\partial}{\partial x}u(t,1) = 0.$$

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We will assume that $u_0 \in C_0[0,1]$, where $C_0[0,1]$ denotes the set of continuous functions $v : [0,1] \to \mathbb{R}$ vanishing at 0. Here $b : [0,T] \times [0,1] \times \mathbb{R} \to \mathbb{R}$ is a measurable function, and $W^H = \{W^H(t,x),\ t \in [0,T],\ x \in [0,1]\}$ is a zero mean Gaussian process with covariance

$$E(W^H(t,x)W^H(s,y)) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)(x \wedge y).$$

That is, W^H is a Brownian motion in the space variable and a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ in the time variable.

Existence and uniqueness of a solution to (1) when the Hurst parameter $H > \frac{1}{2}$ is already known under weaker conditions on the drift. More specifically, in [19] the authors show that pathwise uniqueness for (1) holds if the drift b is allowed to be bounded and satisfies the following Hölder continuity property

$$|b(t,x,z) - b(s,x,r)| \leq C(|t-s|^{\gamma} + |z-r|^{\alpha}), \tag{2}$$

where $\gamma > H - \frac{1}{2}$ and $\alpha > \frac{2H-1}{3H-1}$. Moreover, u_0 is also Hölder continuous of order γ. We mention that (2) will not be used in the calculus below, it will just ensure the pathwise uniqueness property for the SPDE (1).

The aim of the present paper is to establish some various stability results under pathwise uniqueness of solutions. Indeed, we first prove that the pathwise uniqueness implies the stability of the solutions with respect to the initial condition. Then, we treat the relation between the pathwise uniqueness and the convergence of the Picard successive approximation, where we give a necessary and sufficient conditions which ensure the convergence of this latter. Furthermore, the stability of the solution under Lipschitz condition on the drift coefficient in a Hölder space is also given. Notice that the last stability result could be useful for situations where the drift is highly irregular and needs to be approximated by smooth drifts. We point out that the investigation of such questions for stochastic differential equations is carried out in [3], where the authors established some strong stability properties of the solutions using Skorokhod's selection theorem. The same result is investigated for stochastic differential equations driven by fractional Brownian motion in [13], also a considerable result in this direction has been established in [2] for an SPDE driven by a Gaussian noise white in time and colored in space.

The paper is organized as follows. In Section 2, we introduce some properties, definitions, and preliminary results. Section 3 is devoted to prove the stability with respect to the initial condition. In Section 4, we study the convergence of the Picard successive approximation. In Section 5, the stability of the solution under Lipschitz condition on the drift coefficient is investigated. Finally, an appendix gathers some technical results that will be needed in the work.

2. Preliminaries

In this section, we give some properties of the fractional white noise, definitions and some tools used in the proofs.

For any $H > \frac{1}{2}$, let us first introduce the formal definition of the fractional white noise $\frac{\partial^2}{\partial t \partial x} W^H(t,x)$ which appears in (1). Consider a centered Gaussian family of random variables $W^H = \{W^H(t,A) : t \in [0,T],\ A \in \mathbb{B}[0,1]\}$ defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with the covariance function

$$E(W^H(t,A)W^H(s,B)) = \frac{1}{2}\lambda(A \cap B)\left(t^{2H} + s^{2H} - |t-s|^{2H}\right),$$

$s,t \in [0,T],\ A,B \in \mathbb{B}[0,1]$, where λ denotes the Lebesgue measure. Notice that for a fixed t, $W^H(t,\cdot)$ is a Brownian measure with intensity t^{2H}, and for a fixed $A \in \mathbb{B}[0,1]$ with $\lambda(A) \neq 0$, the process $\frac{1}{\sqrt{\lambda(A)}} W^H(\cdot, A)$ is a fractional Brownian motion with Hurst parameter H. We will say that W^H is a fractional noise with Hurst parameter H.

For each $t \in [0,T]$, we denote by F_t^H the σ-field generated by the random variables $\{W^H(t,A) : t \in [0,T],\ A \in \mathbb{B}[0,1]\}$ and the sets of probability zero. We denote by $\mathbb{P}$ the σ-field of progressively measurable subsets of $[0,T] \times \Omega$.

Let ζ be the set of step functions on $[0,T] \times [0,1]$ and $\mathcal{H}$ be the Hilbert space defined as the closure of ζ with respect to the scalar product

$$\langle \mathbf{1}_{[0,t]\times A}, \mathbf{1}_{[0,s]\times B} \rangle_{\mathcal{H}} = E(W^H(t,A)W^H(s,B)).$$

The mapping $\mathbf{1}_{[0,t]\times A} \to W^H(t,A)$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(W^H)$ associated with W^H, such isometry is denoted by

$$\varphi \to W^H(\varphi) := \int_{[0,t]\times A} \varphi(s,y) W^H(ds,dy).$$

Now we introduce the linear operator K_H^* from ζ to $L^2([0,T]\times[0,1])$ defined by

$$(K_H^*\varphi)(s,x) = K_H(T,s)\varphi(s,x) + \int_s^T (\varphi(t,x) - \varphi(s,x)) \frac{\partial K_H}{\partial t}(t,s)dt,$$

where K_H is the square integrable kernel given by

$$(3)\quad K_H(t,s) = c_H(t-s)^{H-\frac{1}{2}} + c_H\left[(\frac{1}{2} - H)\int_s^t (u-s)^{H-\frac{3}{2}}(1 - (\frac{s}{u})^{\frac{1}{2}-H})du\right],$$

and $c_H = \left[\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H}\right]^{1/2}$. From (3) we have

$$\frac{\partial K_H}{\partial t}(t,s) = c_H\left(H - \frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}.$$

Moreover, the kernel K_H satisfies the following property

$$\begin{aligned}\int_0^{s\wedge t} K_H(t,r)K_H(s,r)dr &= \frac{1}{2}\left(t^{2H}+s^{2H}-|t-s|^{2H}\right)\\ &= R_H(t,s),\end{aligned}$$

$R_H(t,s)$ being the covariance kernel of the fractional Brownian motion (see for instance [1, 11, 17]).

The operator K_H^* is an isometry between ζ and $L^2([0,T]\times[0,1])$ that can be extended to the Hilbert space $\mathcal{H}$, then for any pair of step functions φ and ψ in ζ we have

$$\langle K_H^*\varphi, K_H^*\psi\rangle_{L^2([0,T]\times[0,1])} = \langle \varphi,\psi\rangle_{\mathcal{H}},$$

because

$$(K_H^*\mathbf{1}_{[0,t]\times A})(s,x) = K_H(t,s)\mathbf{1}_{[0,t]\times A}(s,x).$$

Define the process $B=\{B(t,A), t\in[0,T], A\in\mathbb{B}[0,1]\}$ by

$$B(t,A) = W^H((K_H^*)^{-1}\mathbf{1}_{[0,t]\times A}).$$

Then, B is a spacetime white noise, moreover W^H has the integral representation

$$W^H(t,x) = \int_0^t\int_0^x K_H(t,s)B(ds,dy).$$

We say that a $\mathbb{P}\times\mathbb{B}[0,1]$ measurable and continuous random field $u=\{u(t,x), t\in[0,T], x\in[0,1]\}$ is a solution to (1) if

$$\int_0^T\int_0^1 |b(s,x,u(s,x))|dxds<\infty$$

a.s., and for any $\phi\in C^2([0,1])$ such that $\phi'(0)=\phi'(1)=0$,

$$\begin{aligned}\int_0^1 u(t,x)\phi(x)dx &= \int_0^1 u_0(x)\phi(x)dx\\ &+\int_0^t\int_0^1\left(u(t,x)\phi''(x)+b(s,x,u(s,x))\phi(x)\right)dxds\\ &+\int_0^t\int_0^1\phi(x)W^H(t,x),\end{aligned}\tag{4}$$

a.s., for all $t\in[0,T]$. Notice that

$$\int_0^t\int_0^1\phi(x)W^H(t,x) = \int_0^t\int_0^1 K_H(t,s)\phi(x)B(ds,dy)$$

is well defined. We refer to [19] for a detailed account of these notions.

We denote by $G(t,x,y)$ the fundamental solution of the heat equation on $\mathbb{R}_+\times[0,1]$ with Neumann boundary conditions.

It is well known that

$$G(t,x,y) = \frac{1}{\sqrt{2\pi t}}\sum_{n\in\mathbb{Z}}\left\{\exp\left\{-\frac{(y-x-2n)^2}{4t}\right\}+\exp\left\{-\frac{(y+x-2n)^2}{4t}\right\}\right\}.$$

Assume we are given a bounded and measurable function $b(t,x,r)$. By a solution to our main SPDE (1), we shall mean an adapted and continuous random field $u(t,x)$ such that

$$
\begin{aligned}
u(t,x) = &\int_0^1 G(t,x,y)u_0(y)dy \\
&+ \int_0^t \int_0^1 G(t-s,x,y)b(s,y,u(s,y))dyds \\
&+ \int_0^t \int_0^1 G(t-s,x,y)W^H(ds,dy),
\end{aligned} \tag{5}
$$

$t \in [0,T],\ x \in [0,1]$, where the last term is equal to

$$
W^H(\mathbf{1}_{[0,t]}(\cdot)G(t-\cdot,x,\cdot)) = \int_0^t \int_0^1 K_H^* G(t-s,x,y)B(ds,dy).
$$

Definition 2.1. We say that the pathwise uniqueness holds for equation (5) if whenever (u, W^H) and $(\widetilde{u}, W^H)$ are two solutions of equation (5) defined on the same probability space $\big(\Omega, \mathcal{F}, P\big)$, then u and $\widetilde{u}$ are indistinguishable.

The main tool used in the proofs is Skorokhod's selection theorem given by the following lemma.

Lemma 2.2 ([22])**.** *Let (S,ρ) be a complete separable metric space, and let P, P_n, $n = 1,2,\dots$ be probability measures on $\big(S, \mathbb{B}(S)\big)$ such that P_n converges weakly to P as $n \to \infty$. Then, on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, we can construct S-valued random variables u, u_n, $n = 1,2,\dots$ such that*:

- (i) *$P_n = \widetilde{P}^{u_n}$, $n = 1,2,\dots$ and $P = \widetilde{P}^u$, where $\widetilde{P}^{u_n}$ and $\widetilde{P}^u$ are respectively the laws of u_n and u;*
- (ii) *u_n converges to u, $\widetilde{P}$–a.s.*

The following lemma gives criteria which allow us to apply Lemma 2.2 to sequences of laws associated to continuous processes.

Lemma 2.3 ([22])**.** *Let $\{u_n(t,x), n \geq 1\}$, be a sequence of real valued continuous processes satisfying the following two conditions*:

- (i) *There exist positive constants M and γ such that $\sup_{n\geq 1} E[|u_n(0,0)|^\gamma] \leq M$;*
- (ii) *there exist positive constants α, β_1, β_2 and C, such that*

  $$\sup_{n\geq 1} E[|u_n(t,x) - u_n(s,y)|^\alpha] \leq C\left(|t-s|^{2+\beta_1} + |x-y|^{2+\beta_2}\right)$$

  for every $t,s \in [0,T]$, and $x,y \in [0,1]$.

Then, there exist a subsequence (n_k), a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and a real valued continuous processes $\widetilde{u}$, $\widetilde{u}_{n_k}$, $k = 1,2,\dots$, defined on $\widetilde{\Omega}$ such that

- (1) *The laws of $\widetilde{u}_{n_k}$ and u_{n_k} coincide*;

(2) $\widetilde{u}_{n_k}(t,x)$ *converges to* $\widetilde{u}(t,x)$ *uniformly on every compact subset on* $\mathbb{R}_+ \times \mathbb{R}$ $\widetilde{P}$*–a.s.*

We will need also to give sufficient conditions which ensure the tightness criterium on the plane that generalizes a well-known theorem of Billingsley.

Theorem 2.4 ([24, Proposition 2.3]). *Let* $\{u_n(t,x), n \geq 1\}$ *be a family of random variables taking values in* $\mathcal{C}([0,T] \times [0,1])$. *The family of the laws of* u_n *is tight, if the following two conditions are satisfied*:

(i) *There exists a positive constant* γ *such that* $\sup_{n\geq 1} E[|u_n(0,0)|^\gamma] < \infty$;

(ii) *there exist positive constants* α, β_1, β_2 *and* C *such that*

$$\sup_{n\geq 1} E[|u_n(t,x) - u_n(s,y)|^\alpha] \leq C\left(|t-s|^{2+\beta_1} + |x-y|^{2+\beta_2}\right)$$

for every $t, s \in [0,T]$, *and* $x, y \in [0,1]$.

We will also make use of the following lemma which gives information about the increments of the Green function that can be found in [6, Lemma B.1], and in [7, Lemma 2.1].

Lemma 2.5. (a) *For any* $\alpha > 0$

$$\int_0^1 |G(t,x,y)|^\alpha dy \leq Ct^{\frac{1-\alpha}{2}}.$$

(b) *Let* $\alpha \in (\frac{3}{2}, 3)$. *Then, for all* $t \in [0,T]$ *and* $x, y \in [0,1]$,

$$\int_0^t \int_0^1 |G(t-s,x,z) - G(t-s,y,z)|^\alpha dz ds \leq C|x-y|^{3-\alpha}.$$

(c) *Let* $\alpha \in (1,3)$. *Then, for all* $s, t \in [0,T]$ *such that* $s \leq t$ *and* $x \in [0,1]$,

$$\int_0^s \int_0^1 |G(t-r,x,y) - G(s-r,x,y)|^\alpha dy dr \leq C|t-s|^{\frac{3-\alpha}{2}}.$$

(d) *Under the same hypothesis as* (c),

$$\int_s^t \int_0^1 |G(t-r,x,y)|^\alpha dy dr \leq C|t-s|^{\frac{3-\alpha}{2}}.$$

3. Stability with respect to the initial condition

In the theory of ordinary differential equations with continuous coefficients, uniqueness of solutions is sufficient to ensure the continuous dependence of the solution with respect to the initial condition (see for instance [9]). The purpose of this section is to give an analogue of the above result in the stochastic case. Define the sequence

$$u_n(t,x) = \int_0^1 G(t,x,y) u_0^n(y) dy \tag{6}$$

$$+\int_0^t \int_0^1 G(t-s,x,y) b(s,y,u_n(s,y)) dyds$$
$$+\int_0^t \int_0^1 G(t-s,x,y) W^H(ds,dy),$$

with the initial condition $u_n(0,x) = u_0^n(x)$, $x \in [0,1]$.

Assume that $u_0^n(x)$ converges to $u_0(x)$ uniformly in $x \in [0,1]$. Then, we have the following:

Theorem 3.1. *Let b be a continuous and bounded function. Then under pathwise uniqueness for SPDE* (5)*, we get*

$$\lim_{n\to\infty} E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x) - u(t,x)|^2\right] = 0.$$

Before we proceed to the proof of Theorem 3.1, we first state the following technical lemma which gives a tightness criterium on the plane.

Lemma 3.2. *Let b be a bounded function. Suppose that $\{u_0^n\}_{n\geq 0}$ is a sequence of functions which converges uniformly to u_0. Then, the sequence $\{u_n\}_{n\geq 0}$ defined by* (6) *is tight in $\mathcal{C}([0,T]\times[0,1])$.*

Proof. Set $u_n(t,x) = u_n^1(t,x) + u_n^2(t,x) + u^3(t,x)$ where

$$u_n^1(t,x) = \int_0^1 G(t,x,y) u_0^n(y) dy,$$

$$u_n^2(t,x) = \int_0^t \int_0^1 G(t-s,x,y) b(s,y,u_n(s,y)) dyds,$$

$$u^3(t,x) = \int_0^t \int_0^1 G(t-s,x,y) W^H(ds,dy).$$

We first prove that, for any $p \geq 2$

$$\sup_n E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x)|^p\right] \leq C. \tag{7}$$

Indeed, we know that since u_0^n converges uniformly to u_0, we have

$$|u_n(t,x)|^p \leq C\left(1 + |u_n^2(t,x)|^p + |u^3(t,x)|^p\right). \tag{8}$$

For $u_n^2(t,x)$, by Hölder's inequality, we obtain

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n^2(t,x)|^p\right]$$
$$\leq C_p \left(\int_0^T \int_0^1 (G(t-s,x,y))^2 dyds\right)^{\frac{p}{2}} E\left(\int_0^T \int_0^1 b(s,y,u_n(s,y))^2 dyds\right)^{\frac{p}{2}}.$$

Due to Lemma 2.5, and the boundedness of b

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u_n^2(t,x)|^p\right]\leq C. \tag{9}$$

For $u^3(t,x)$, we know that

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u^3(t,x)|^p\right]\leq C_p\left(\int_0^T\int_0^1[K_H^*(G(t-s,x,y))]^2dyds\right)^{\frac{p}{2}}.$$

Using the continuous embedding established in [18]

$$L^{\frac{1}{H}}([0,T]\times[0,1])\subset\mathcal{H}, \tag{10}$$

together with the estimates in Lemma 2.5, it holds that

$$\begin{aligned}&\left(\int_0^T\int_0^1[K_H^*(G(t-s,x,y))]^2dyds\right)^{\frac{p}{2}}\\&\leq C_H\left(\int_0^T\int_0^1(G(t-s,x,y))^{\frac{1}{H}}dyds\right)^{pH}\\&\leq C.\end{aligned}$$

From this we deduce that

$$E\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u^3(t,x)|^p\right]\leq C. \tag{11}$$

Therefore, combining (8) together with (9)–(11) estimate (7) follows.

On the other hand, let $(t',x'),(t,x)\in[0,T]\times[0,1]$, and it is assumed, without lost of generality, that $t'>t$ and $x'>x$. Then, we have for any $m>8$

$$\begin{aligned}&E|u_n^2(t',x')-u_n^2(t,x)|^m\\&\leq C_mE\left(\left|\int_0^t\int_0^1(G(t'-s,x',y)-G(t-s,x',y))b(s,y,u_n(s,y))dyds\right|\right)^m\\&\quad+C_mE\left(\left|\int_0^t\int_0^1(G(t-s,x',y)-G(t-s,x,y))b(s,y,u_n(s,y))dyds\right|\right)^m\\&\quad+C_mE\left(\left|\int_t^{t'}\int_0^1(G(t'-s,x',y)b(s,y,u_n(s,y))dyds\right|\right)^m.\end{aligned}$$

By Hölder's inequality we obtain

$$\begin{aligned}&E|u_n^2(t',x')-u_n^2(t,x)|^m\\&\leq C_m\left(\int_0^t\int_0^1(G(t'-s,x',y)-G(t-s,x',y))^2dyds\right)^{\frac{m}{2}}\end{aligned}$$

$$\times E\left(\int_0^t\int_0^1 b(s,y,u_n(s,y))^2 dyds\right)^{\frac{m}{2}}$$
$$+ C_m\left(\int_0^t\int_0^1 (G(t-s,x',y)-G(t-s,x,y))^2 dyds\right)^{\frac{m}{2}}$$
$$\times E\left(\int_0^t\int_0^1 b(s,y,u_n(s,y))^2 dyds\right)^{\frac{m}{2}}$$
$$+ C_m\left(\int_t^{t'}\int_0^1 (G(t'-s,x',y))^2 dyds\right)^{\frac{m}{2}}$$
$$\times E\left(\int_t^{t'}\int_0^1 b(s,y,u_n(s,y))^2 dyds\right)^{\frac{m}{2}}.$$

In view of the Lemma 2.5, and the fact that b is bounded, we get that

$$E|u_n^2(t',x')-u_n^2(t,x)|^m \le C_m(|t'-t|^{\frac{m}{4}}+|x'-x|^{\frac{m}{2}}). \tag{12}$$

For $u^3(t,x)$, we have for any $m>8$

$$\begin{aligned} E|u^3(t',x')-u^3(t,x)|^m &\le C_m\left(\int_0^t\int_0^1 [K_H^*(G(t'-s,x',y)-G(t-s,x',y))]^2 dyds\right)^{\frac{m}{2}} \\ &\quad + C_m\left(\int_0^t\int_0^1 [K_H^*(G(t-s,x',y)-G(t-s,x,y))]^2 dyds\right)^{\frac{m}{2}} \\ &\quad + C_m\left(\int_t^{t'}\int_0^1 [K_H^*(G(t'-s,x',y))]^2 dyds\right)^{\frac{m}{2}} \\ &= C_m(I_1+I_2+I_3). \end{aligned} \tag{13}$$

The three terms of the right hand side of the previous inequality will be dealt with as in [19]. Indeed, using the continuous embedding (10) and the estimates in Lemma 2.5

$$I_1 \le \left(\int_0^t\int_0^1 |G(t'-s,x',y)-G(t-s,x',y)|^{\frac{1}{H}} dyds\right)^{mH} \le C\,|t'-t|^{\frac{m}{2}(3H-1)}. \tag{14}$$

The following estimates also hold

$$I_2 \le \left(\int_0^t\int_0^1 |G(t-s,x',y)-G(t-s,x,y)|^{\frac{1}{H}} dyds\right)^{mH} \le C\,|x'-x|^{m(3H-1)}, \tag{15}$$

and

$$I_3 \le \left(\int_t^{t'}\int_0^1 (G(t'-s,x',y))^{\frac{1}{H}} dyds\right)^{mH} \le C\,|t'-t|^{\frac{m}{2}(3H-1)}. \tag{16}$$

Rearranging (13) in view of (14)–(16) yields

$$E|u^3(t',x') - u^3(t,x)|^m \leq C\left(|t'-t|^{\frac{m}{2}(3H-1)} + |x'-x|^{m(3H-1)}\right). \tag{17}$$

Then, in view of estimates (12) and (17) we deduce that the family $\{u_n^2 + u^3;\ n \geq 0\}$ is tight in $\mathcal{C}([0,T]\times[0,1])$ and since

$$\left|u_n^1(t,x) - \int_0^1 G(t,x,y)u_0(y)dy\right| \leq \sup_{x\in[0,1]} |u_0^n(x) - u_0(x)| \to 0 \tag{18}$$

it follows that $u_n = u_n^1 + u_n^2 + u^3$ is tight in $\mathcal{C}([0,T]\times[0,1])$, which finishes the proof. □

Now we are able to tackle the proof of Theorem 3.1.

Proof. Suppose that the claim of our theorem is false. Then, there exists a constant $\delta > 0$ such that:

$$\inf_n E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x) - u(t,x)|^2\right] \geq \delta.$$

According to Lemma 3.2, the sequence $Z_n = (u_n, u, W^H)$ is tight in $(\mathcal{C}([0,T]\times[0,1]))^3$. Then, by Skorokhod's selection theorem, there exist a subsequence $\{n_k, k\geq 1\}$, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and stochastic processes $\widetilde{Z} = (\widetilde{u}, \widetilde{v}, \widetilde{W^H})$, $\widetilde{Z}_{n_k} = (\widetilde{u}_{n_k}, \widetilde{v}_{n_k}, \widetilde{W^H_{n_k}})$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that:

(α) For each $k \geq 1$, the laws of $\widetilde{Z}_{n_k}$ and Z_{n_k} coincide;
(β) $\widetilde{Z}_{n_k}$ converges $\widetilde{P}$–a.s in $(\mathcal{C}([0,T]\times[0,1]))^3$ to $\widetilde{Z}$.

Thanks to property (α), we have, for $k \geq 1$

$$E\bigg|\widetilde{u}_{n_k}(t,x) - \int_0^1 G(t,x,y)u_0^{n_k}(y)dy - \int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}_{n_k}(s,y))dyds - \int_0^t\int_0^1 G(t-s,x,y)\widetilde{W^H_{n_k}}(ds,dy)\bigg|^2 = 0.$$

In other words $\widetilde{u}_{n_k}(t,x)$ satisfies the following SPDE:

$$\begin{aligned}\widetilde{u}_{n_k}(t,x) = &\int_0^1 G(t,x,y)u_0^{n_k}(y)dy\\ &+ \int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}_{n_k}(s,y))dyds\\ &+ \int_0^t\int_0^1 G(t-s,x,y)\widetilde{W^H_{n_k}}(ds,dy).\end{aligned}$$

Similarly,

$$\widetilde{v}_{n_k}(t,x) = \int_0^1 G(t,x,y)u_0(y)dy$$

$$+\int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{v}_{n_k}(s,y))dyds$$
$$+\int_0^t\int_0^1 G(t-s,x,y)\widetilde{W^H_{n_k}}(ds,dy).$$

Using (β) and Lemma A.2 (see Appendix), we deduce that

$$\int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{v}_{n_k}(s,y))dyds$$

converges in probability (as $k\to\infty$) to

$$\int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{v}(s,y))dyds.$$

Similarly,

$$\int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}_{n_k}(s,y))dyds$$

converges in probability (as $k\to\infty$) to

$$\int_0^t\int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}(s,y))dyds.$$

Due to the fact that the sequence u_0^n converges to u_0 uniformly, it follows from (18) that

$$\int_0^1 G(t,x,y)u_0^{n_k}(y)dy \to \int_0^1 G(t,x,y)u_0(y)dy.$$

Hence, by the pathwise uniqueness, $\widetilde{u}$ and $\widetilde{v}$ are indistinguishable.

On the other hand, by uniform integrability, we have that:

$$\begin{aligned}\delta &\le \liminf_n E\left[\sup_{(t,x)\in[0,T]\times[0,1]}|u_n(t,x)-u(t,x)|^2\right]\\ &\le \liminf_k \widetilde{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|\widetilde{u}_{n_k}(t,x)-\widetilde{v}_{n_k}(t,x)|^2\right]\\ &= \widetilde{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]}|\widetilde{u}(t,x)-\widetilde{v}(t,x)|^2\right],\end{aligned}$$

this contradicts our assumption. Therefore u_n converges to the unique solution u. □

4. Pathwise uniqueness and Picard's successive approximation

Let us consider the sequence of the Picard successive approximation associated to SPDE (5) defined as follows

(19)

$$\begin{cases} u_{n+1}(t,x) = \displaystyle\int_0^1 G(t,x,y)u_0(y)dy + \int_0^t\int_0^1 G(t-s,x,y)b(s,y,u_n(s,y))dyds \\ \qquad\qquad\quad + \displaystyle\int_0^t\int_0^1 G(t-s,x,y)W^H(ds,dy) \\ u_0(t,x) = \displaystyle\int_0^1 G(t,x,y)u_0(y)dy. \end{cases}$$

It is well known that in the theory of ordinary differential equations with continuous and bounded coefficients, the uniqueness of solution is not enough for the convergence of the Picard successive approximation (see for instance [21, p. 55]). The purpose of this section is to give a necessary and sufficient conditions which ensure the convergence of the Picard successive approximation when equation (5) admits a pathwise unique solution. More precisely, we have the following:

Theorem 4.1. *Let b be a continuous bounded function. Then under pathwise uniqueness for SPDE* (5)*, u_n converges in $L^2(\Omega;\mathcal{C}([0,T]\times[0,1]))$ to the unique solution of* (5) *if and only if $u_{n+1}-u_n$ converges to* 0 *in $L^2(\Omega;\mathcal{C}([0,T]\times[0,1]))$.*

As we did in the last section, we need first to ensure a tightness criterion. This is the goal of the next lemma.

Lemma 4.2. *Let b be a bounded function. Suppose that u_0 is Hölder continuous of order $H-\frac{1}{2}<\gamma<\frac{1}{2}$. Then, the sequence $\{u_n\}_{n\geq 0}$ defined by* (19) *is tight in $\mathcal{C}([0,T]\times[0,1])$.*

Proof. The proof is similar to the proof of Lemma 3.2, the only difference is to verify that for $(t',x'),(t,x)\in[0,T]\times[0,1]$, there exist positive constants β_1, β_2 such that

$$\left|\int_0^1 G(t',x',y)u_0(y)dy - \int_0^1 G(t,x,y)u_0(y)dy\right|^m \\ \leq C\left(|t'-t|^{2+\beta_1}+|x'-x|^{2+\beta_2}\right).$$

Note that in [6], it is shown that if u_0 is Hölder continuous of order γ, then, for any $m>\frac{4}{\gamma}$:

$$\left|\int_0^1 G(t',x',y)u_0(y)dy - \int_0^1 G(t,x,y)u_0(y)dy\right|^m \leq C\left(|t'-t|^{\frac{\gamma}{2}}+|x'-x|^{\gamma}\right)^m.$$

Then the result follows. □

Now we are able to complete the proof of Theorem 4.1.

Proof. Suppose that $u_{n+1} - u_n$ converges to 0 in quadratic mean, and there is some $\delta > 0$ such that

$$\inf_n E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x) - u(t,x)|^2\right] \geq \delta.$$

Due to Lemma 4.2 the sequence (u_n, u_{n+1}, u, W^H) satisfies conditions (i) and (ii) of Lemma 2.3. Then, by Skorokhod's selection theorem, there exist a subsequence $\{n_k, k \geq 1\}$, a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ and stochastic processes $(\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{W^H})$, $(\widetilde{u}_{n_k}, \widetilde{v}_{n_k}, \widetilde{w}_{n_k}, \widetilde{W^H_{n_k}})$ defined on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ such that:

- (α') For each $k \geq 1$, the laws of $(\widetilde{u}_{n_k}, \widetilde{v}_{n_k}, \widetilde{w}_{n_k}, \widetilde{W^H}_{n_k})$ and $(u_{n_k}, u_{n_k+1}, u, W^H)$ coincide;
- (β') $(\widetilde{u}_{n_k}, \widetilde{v}_{n_k}, \widetilde{w}_{n_k}, \widetilde{W^H_{n_k}})$ converges to $(\widetilde{u}, \widetilde{v}, \widetilde{w}, \widetilde{W^H})$ uniformly on every compact subset on $\mathbb{R}_+ \times \mathbb{R}$ $\widetilde{P}$–a.s.

Since $u_{n+1} - u_n$ converges to 0 in quadratic mean, we deduce that $\widetilde{u} = \widetilde{v}$, $\widetilde{P}$–a.s. Thanks to property (α'), we have, for every $k \geq 1$

$$\begin{aligned}\widetilde{v}_{n_k}(t,x) = & \int_0^1 G(t,x,y)u_0(y)dy \\ & + \int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}_{n_k}(s,y))dyds \\ & + \int_0^t \int_0^1 G(t-s,x,y)\widetilde{W^H_{n_k}}(ds,dy).\end{aligned}$$

Similarly

$$\begin{aligned}\widetilde{w}_{n_k}(t,x) = & \int_0^1 G(t,x,y)u_0(y)dy \\ & + \int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{w}_{n_k}(s,y))dyds \\ & + \int_0^t \int_0^1 G(t-s,x,y)\widetilde{W^H_{n_k}}(ds,dy).\end{aligned}$$

On the other hand, using (β') and Lemma A.2 (see Appendix)

$$\int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}_{n_k}(s,y))dyds$$

converges in probability (as $k \to \infty$) to

$$\int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{u}(s,y))dyds.$$

Similarly,

$$\int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{w}_{n_k}(s,y))dyds$$

converges in probability (as $k \to \infty$) to

$$\int_0^t \int_0^1 G(t-s,x,y)b(s,y,\widetilde{w}(s,y))dyds.$$

Thus, the processes $\widetilde{u}$ and $\widetilde{w}$ satisfy the same SPDE on $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$ with the same driving noise $\widetilde{W^H}$. Then, by pathwise uniqueness, we conclude that $\widetilde{u}$ and $\widetilde{w}$ are indistinguishable.

On the other hand, by uniform integrability, we have that:

$$\begin{aligned}
\delta &\leq \liminf_n E\left[\sup_{(t,x)\in[0,T]\times[0,1]} |u_n(t,x)-u(t,x)|^2\right] \\
&\leq \liminf_k \widetilde{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |\widetilde{u}_{n_k}(t,x)-\widetilde{w}_{n_k}(t,x)|^2\right] \\
&= \widetilde{E}\left[\sup_{(t,x)\in[0,T]\times[0,1]} |\widetilde{u}(t,x)-\widetilde{w}(t,x)|^2\right],
\end{aligned}$$

which is a contradiction. Then the desired result follows. □

5. Stability under Lipschitz condition in Hölder space

Let $\beta = (\beta_1, \beta_2)$ such that $\beta_1, \beta_2 > 0$, and denote by $C^\beta([0,T]\times[0,1];\mathbb{R})$ the set of β-Hölder continuous functions equipped with the norm

$$\|f\|_\beta = \sup_{(t,x)\in[0,T]\times[0,1]} |f(t,x)| + \sup_{s\neq t\in[0,T]} \sup_{x\neq y\in[0,1]} \frac{|f(t,x)-f(s,y)|}{|t-s|^{\beta_1}+|x-y|^{\beta_2}}.$$

Let g be a function defined on $[0,T]\times[0,1]\times\mathbb{R}$. For $T>0$, we set

$$\|g\|_\infty = \sup_{t\in[0,T]} \sup_{x\in[0,1]} \sup_{r\in\mathbb{R}} |g(t,x,r)|.$$

It follows from [19, Lemma 1], that the solution to the equation (5) is Hölder continuous in time, but using the same techniques as in [19, Lemma 1] and in Lemma 3.2, we can conclude also that u belongs to $C^\beta([0,T]\times[0,1];\mathbb{R})$ a.s.

Let $(b_n)_{n\geq 0}$ be a sequence of bounded functions on $[0,T]\times[0,1]\times\mathbb{R}$ which satisfy Lipschitz condition uniformly in n. Denote by $u_n(t,x)$ the unique solution of equation

$$\begin{aligned}
u_n(t,x) = &\int_0^1 G(t,x,y)u_0(y)dy \\
&+ \int_0^t \int_0^1 G(t-s,x,y)b_n(s,y,u_n(s,y))dyds \\
&+ \int_0^t \int_0^1 G(t-s,x,y)W^H(ds,dy).
\end{aligned}$$

Then, we have the following:

Theorem 5.1. *Assume that* $(b_n)_{n\geq 0}$ *converges to* b *uniformly on* $[0,T]\times[0,1]\times\mathbb{R}$*. Then, for any* $p\geq 2$

$$\lim_{n\to\infty} E\left(\|u_n-u\|_\beta^p\right)=0,$$

where u *is the unique solution of equation* (5)*.*

Before we proceed to the proof of Theorem 5.1, we now state a technical lemma which is crucial for our target.

Lemma 5.2. *Assume that there exists a real valued function* b *defined on* $[0,T]\times[0,1]\times\mathbb{R}$ *such that*

$$\lim_{n\to\infty}\|b_n-b\|_\infty=0.$$

Then, for any $p\geq 2$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\sup_{x\in[0,1]} E\left(|u_n(t,x)-u(t,x)|^p\right)=0,$$

where u *is the unique solution of equation* (5)*.*

Proof. For any $p\geq 2$, we know that

$$\begin{aligned}
&E|u_n(t,x)-u(t,x)|^p\\
&=E\left|\int_0^t\int_0^1 G(t-s,x,y)(b_n(s,y,u_n(s,y))-b(s,y,u(s,y))dyds\right|^p\\
&\leq C_pE\left|\int_0^t\int_0^1 G(t-s,x,y)(b_n(s,y,u_n(s,y))-b(s,y,u_n(s,y))dyds\right|^p\\
&\quad+C_pE\left|\int_0^t\int_0^1 G(t-s,x,y)(b(s,y,u_n(s,y))-b(s,y,u(s,y))dyds\right|^p.
\end{aligned}$$

We use Hölder inequality, Lipschitz condition on b_n and Lemma 2.5, to get

$$\begin{aligned}
&\sup_{x\in[0,1]} E\left|u_n(t,x)-u(t,x)\right|^p\\
&\leq C_p\left\|b_n-b\right\|_\infty^p(\phi(t))^{\frac{p}{2}}\\
&\quad+C_p(\phi(t))^{\frac{p}{2}}\int_0^t\sup_{y\in[0,1]}E\left|u_n(s,y)-u(s,y)\right|^p ds\\
&\leq C(p,T)\left(\|b_n-b\|_\infty^p+\int_0^t\sup_{y\in[0,1]}E\left|u_n(s,y)-u(s,y)\right|^p ds\right),
\end{aligned}$$

where

$$\phi(t)=\left(\int_0^t\int_0^1(G(t-s,x,y))^2dyds\right).$$

Therefore, using Gronwall's Lemma, we deduce that

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}E\left|u_n(t,x)-u(t,x)\right|^p\leq C(p,T)\left\|b_n-b\right\|_\infty^p,$$

which proves the Lemma. □

Let us now turn to the proof of Theorem 5.1

Proof. It is enough to prove that the sequence $(u_n - u)_{n\geq 1}$ satisfies the properties (1) and (2) of Lemma A.1 (see Appendix). The first property is easily checked by using the same arguments as in Lemma 3.2. The second property is given by Lemma 5.2. This achieves the proof. □

Appendix A

In this section we will recall some technical lemmas that have been used in the proofs.

Lemma A.1 ([6, Lemma A1]). *Let $\{Y_n(t,x)\}_{n\geq 1}$ be a sequence of processes indexed by $[0,T]\times[0,1]$ such that*

(1) *For any $p\geq 2$ there exist $C,\gamma_1,\gamma_2>0$ such that for any $t_1,t_2\in[0,T]$ and $x_1,x_2\in[0,1]$*

$$\sup_{n\geq 1} E[|Y_n(t_2,x_2)-Y_n(t_1,x_1)|^p] \leq C\left(|t_2-t_1|^{2+\gamma_1}+|x_2-x_1|^{2+\gamma_2}\right);$$

(2) *for every $(t,x)\in[0,T]\times\mathbb{R}$ and $p\geq 2$*

$$\lim_{n\to\infty} E[|Y_n(t,x)|^p]=0.$$

Then, for any $\beta_1\in(0,\frac{\gamma_1}{p})$ and $\beta_2\in(0,\frac{\gamma_2}{p})$

$$\lim_{n\to\infty} E[\|Y_n(t,x)\|_\beta^p]=0.$$

Lemma A.2 ([15, Lemma 4.3]). *For every integer $n\geq 0$, let $z_n=\{z_n(t,x): t\in[0,T],\ x\in[0,1]\}$ be a continuous adapted random field. Assume that for every $\epsilon>0$ and $T>0$*

$$\lim_{n\to\infty} P\left[\sup_{t\in[0,T]}\sup_{x\in[0,1]}|z_n(t,x)-z(t,x)|\geq\epsilon\right]=0.$$

Let $h(t,x,r)$ be a continuous bounded function of $(t,x,r)\in[0,T]\times[0,1]\times\mathbb{R}$. Then

$$\int_0^t\int_0^1 h(s,y,z_n(s,y))dyds \to \int_0^t\int_0^1 h(s,y,z(s,y))dyds$$

in probability for every $t\in[0,T]$ and $x\in[0,1]$.

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