A NOTE ON MEROMORPHIC SOLUTIONS OF COMPLEX DIFFERENTIAL-DIFFERENCE EQUATIONS

Xiaoguang Qi and Lianzhong Yang

ABSTRACT. In this article, we consider properties of transcendental meromorphic solutions of the complex differential-difference equation

$$P_n(z)f^{(n)}(z + \eta_n) + \cdots + P_1(z)f'(z + \eta_1) + P_0(z)f(z + \eta_0) = 0,$$

and its non-homogeneous equation. Our results extend earlier results by Liu et al. [9].

1. Introduction

Although as early as thirty years ago, S. Shimomura [14] and N. Yanagihara [16, 17] et al. started a preliminary study of complex difference equations, the lack of vigorous research tools led to the slow development of subsequent research. Until 2006, Chiang-Feng [1] and Halburd-Korhonen [6] independently proved the difference analogue of the lemma on logarithmic derivative, Nevanlinna theory stared to be a powerful tool to investigate complex differences and difference equations, which results in the rapid development in this field. Furthermore, complex differences and difference equations have attracted more and more attentions in recent ten years. Many results on the complex difference equations are rapidly obtained until now, please refer to [2] for details.

The study of complex differential-difference equations can be traced back to Naftalevich's research. Using operator theory and iteration method, Naftalevich [12] ever considered the meromorphic solutions on complex differential-difference equations. However, there are few investigations on complex differential-difference field using Nevanlinna theory. Therefore, the relevant results are very limited, the reader is invited to see [3, 7, 8, 10, 13].

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevalinna theory of meromorphic functions (e.g. see [18]). We use $\sigma(f)$ to denote the order of a meromorphic

function $f(z)$, and $\lambda(f)$, $\lambda(\frac{1}{f})$ to denote the exponent of convergence of zeros and poles of $f(z)$, respectively.

Recently, Liu and Song [9] considered the exponent of convergence of zeros and the order for entire solutions of the linear differential-difference equation

$$(1.1) \qquad P_n(z)f^{(n)}(z+\eta_n) + \cdots + P_1(z)f'(z+\eta_1) + P_0(z)f(z+\eta_0) = 0$$

and its non-homogeneous equation

$$(1.2) \qquad P_n(z)f^{(n)}(z+\eta_n) + \cdots + P_1(z)f'(z+\eta_1) + P_0(z)f(z+\eta_0) = Q(z),$$

where $\eta_0, \ldots, \eta_n$ are distinct complex constants. They got the following results.

Theorem A. *Let $P_0(z)$, $P_1(z), \ldots, P_n(z)(\not\equiv 0)$ be polynomials. Suppose that $f(z)$ is a transcendental entire solution with finite order of (1.1). Then $\lambda(f - a) \geq \sigma(f) - 1$ for every finite value a.*

Theorem B. *Let $Q(z)$ and $P_j(z)$ $(j = 0, 1, 2, \ldots, n)$ be polynomials such that $P_n(z)Q(z) \not\equiv 0$ in (1.2). If $f(z)$ is a transcendental entire solution with finite order of (1.2), then $\lambda(f) = \sigma(f)$. If $Q(z) \not\equiv dP_0(z)$, then $\lambda(f - d) = \sigma(f)$, where d is a constant.*

Related to the theorems above, a natural question is: what happens if the coefficients of equations (1.1) and (1.2) are meromorphic functions other than polynomials? Concerning this question, we obtain following results.

Theorem 1.1. *Let $P_j(z)$ $(j = 0, 1, \ldots, n)$ be meromorphic functions, and set $\sigma = \max\{\sigma(P_j)\}$. If $f(z)(\not\equiv 0)$ is a finite order transcendental meromorphic solution of equation (1.1), then we have*
  (1) *if $\sigma \geq \lambda_f$, then $\sigma \geq \sigma(f) - 1$;*
  (2) *if $\sigma < \lambda_f$, then $\lambda_f \geq \sigma(f) - 1$, where $\lambda_f = \max\{\lambda(f), \lambda\left(\frac{1}{f}\right)\}$.*

Remark 1. (i) From [11], we know equation (1.1) has no properly meromorphic solutions when $P_j(z)$ are constants. However, equation (1.1) can admit properly meromorphic solutions when $P_j(z)$ are meromorphic functions. For example, $f(z) = \frac{e^z}{z}$ is a meromorphic solution of

$$f'(z + 2\pi i) - \frac{z(z + 2\pi i - 1)}{(z + 2\pi i)^2} f(z) = 0.$$

(ii) The equation

$$\frac{1}{4(z+2)^2+2} f''(z+2) + \frac{1}{2z+2}(e^z + e^{-z})f'(z+1) - (e^{4z+4} + e^{3z+1} + e^{z+1})f(z) = 0$$

has a solution $f(z) = e^{z^2}$. Here, $\sigma(f) = 2$, $\lambda_f = 0$ and $\sigma = 1$. The equation and its solution satisfy Theorem 1.1(1).

Moreover, the meromorphic function $f(z) = e^{z^2} - 1$ is a solution of equation

$$\frac{1}{4(z+2)^2+2} f''(z+2) - \frac{1}{2z+2}e^{2z+3}f'(z+1) = 0.$$

Here, $2 = \lambda_f > \sigma = 1$ and $\lambda_f > \sigma(f) - 1 = 1$. The equation and its solution satisfy Theorem 1.1(2) as well.

Theorem 1.2. *Let $P_j(z)$ $(j = 0, 1, \ldots, n)$, $Q(z)$ be meromorphic functions such that $Q(z) \not\equiv 0$. Suppose that $f(z)$ is an admissible finite order transcendental meromorphic solution of equation (1.2) in the sense that $T(r, P_j) = S(r, f)$ and $T(r, Q) = S(r, f)$. Then, we have $\lambda(f) = \sigma(f)$. Furthermore, if $Q(z) \not\equiv dP_0(z)$, then $\lambda(f - d) = \sigma(f)$, where d is a constant.*

Remark 2. (i) Checking the proof of Theorem 1.2, we see, whether or not $Q(z) \not\equiv 0$ is established, Theorem 1.2 holds, as long as the condition $Q(z) \not\equiv dP_0(z)$ holds.

(ii) The equation

$$\frac{1}{4(z+2)^2+2} f''(z+2) + \frac{1}{2z+2}(e^z + e^{-z})f'(z+1)$$
$$- (e^{4z+4} + e^{3z+1} + e^{z+1})f(z)$$
$$= \frac{1}{z}(e^{4z+4} + e^{3z+1} + e^{z+1}) + \frac{1}{2(z+1)^3}(e^z + e^{-z}) - \frac{2}{4(z+2)^5 + 2(z+2)^3}$$

has a solution $f(z) = e^{z^2} - \frac{1}{z}$. Here,

$$Q(z) = \frac{1}{z}(e^{4z+4} + e^{3z+1} + e^{z+1}) + \frac{1}{2(z+1)^3}(e^z + e^{-z}) - \frac{2}{4(z+2)^5 + 2(z+2)^3}$$

such that $T(r, Q) = S(r, f)$. Moreover, it is easy to verify that $P_0(z) = -(e^{4z+4} + e^{3z+1} + e^{z+1})$ satisfies $Q(z) \not\equiv dP_0(z)$. We see $\lambda(f) = \lambda(f - d) = \sigma(f) = 2$. The equation and its solution satisfy Theorem 1.2.

Theorem 1.3. *Let $P_j(z)$ $(j = 0, 1, \ldots, n)$, $Q(z)$ be meromorphic functions such that $Q(z) \not\equiv 0$. Suppose that $f(z)$ is an admissible finite order transcendental meromorphic solution of equation (1.2). If $\sigma(f) > \sigma(Q)$, then we have $\lambda_f = \sigma(f)$.*

In [15], Wu and Zheng obtained the following result.

Theorem C. *Let $A_{ij}(z)$ $(i = 0, 1, \ldots, n; j = 0, 1, \ldots, m)$ be meromorphic functions such that there exists an integer l $(0 \le l \le n)$ satisfying*

$$\max\{\sigma(A_{ij}), (i,j) \neq (l, 0)\} < \sigma(A_{l0}) < \infty, \quad \delta(\infty, A_{l0}) > 0.$$

If $f(z)$ $(\not\equiv 0)$ is a meromorphic solution of

$$\sum_{i=0}^{n} \sum_{j=0}^{m} A_{ij} f^{(j)}(z + c_i) = 0,$$

then we have $\sigma(f) \ge \sigma(A_{l0}) + 1$. Here, we define

$$\delta(a, f) = \liminf_{r \to \infty} \frac{m(r, \frac{1}{f-a})}{T(r, f)},$$

where $a(z)$ is a small function related to $f(z)$.

From Theorem C, we see:

Corollary 1.4. *If*

$$\max\{\sigma(P_j)\} < \sigma(P_0) < \infty, \quad \delta(\infty, P_0) > 0, \quad j = 1, 2, \ldots, n.$$

Then all meromorphic solutions of (1.1) *satisfy* $\sigma(f) \geq \sigma(P_0) + 1$.

Naftalevich [12], Liu-Song [9] discussed some properties of the following equation

$$(1.3) \qquad\qquad f(z+c) = e^{P(z)} f^{(k)}(z),$$

namely,

$$(1.4) \qquad\qquad f(z+c) - e^{P(z)} f^{(k)}(z) = 0,$$

where c is a non-zero constant and k is a positive integer. On one hand, we consider the non-homogeneous equation of (1.4)

$$(1.5) \qquad\qquad f(z+c) - e^{P(z)} f^{(k)}(z) = Q(z).$$

We obtain:

Theorem 1.5. *Let* $P(z)$, $Q(z) \not\equiv 0$ *be polynomials, and* $P(z)$ *be not a constant. If* $f(z)$ *is a finite order transcendental entire solution of* (1.5), *then we have* $\sigma(f) \geq \deg(P(z))$ *and* $\lambda(f) = \sigma(f)$.

On the other hand, we see, equation (1.3) can be rewritten as

$$(1.6) \qquad\qquad f^{(k)}(z) - e^{-P(z)} f(z+c) = 0.$$

Noting equation (1.6), a natural question is: what will happen if $e^{-P(z)}$ is replaced with a polynomial of exponential functions. Furthermore, we doubt if condition "$\max\{\sigma(P_j)\} < \sigma(P_0) < \infty$" in Corollary 1.4 is necessary. Indeed, from Remark 1(ii), we see $\max\{\sigma(P_j)\} = \sigma(P_0) = 1$, the conclusion of Corollary 1.4 holds as well. Due to above considerations, we investigate the following differential-difference equation:

$$(1.7) \qquad f^{(n)}(z + \eta_n) + \sum_{j=0}^{n-1} \{P_j(e^{A(z)}) + Q_j(e^{-A(z)})\} f^{(j)}(z + \eta_j) = 0,$$

where $P_j(z)$ and $Q_j(z)\, (j = 0, 1, \ldots, n-1)$ are polynomials in z, $A(z)$ is a polynomial of degree m. We obtain the following result.

Theorem 1.6. *Let* $P_j(z)$ *and* $Q_j(z)\, (j = 0, 1, \ldots, n-1)$ *be polynomials,* $A(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0$, $(a_m \neq 0)$ *be a non-constant polynomial. If*

$$\deg(P_0) > \deg(P_j) \quad or \quad \deg(Q_0) > \deg(Q_j), \quad j = 1, \ldots, n-1.$$

Then, each meromorphic solution $f(z)(\not\equiv 0)$ *with finite order of the equation* (1.7) *satisfies* $\sigma(f) \geq m+1$. *Moreover, we have* $\max\{\lambda(f-a), \lambda\left(\frac{1}{f}\right)\} = \sigma(f)$, *where* a *is non-zero constant.*

2. Some lemmas

Lemma 2.1 ([6, Corollary 2.2])**.** *Let $f(z)$ be a meromorphic function of finite order. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) + m\left(r, \frac{f(z)}{f(z+c)}\right) = S(r, f).$$

Lemma 2.2 ([1, Corollary 8.3])**.** *Let η_1, η_2 be two arbitrary complex numbers, and let $f(z)$ be a meromorphic function of finite order σ. Let $\varepsilon > 0$ be given constants, then there exits a subset $E \subset (1, \infty)$ of finite logarithmic measure, for all $|z| = r \notin [0, 1] \cup E$, we have*

$$\exp\{-r^{\sigma-1+\varepsilon}\} \leq \left|\frac{f(z+\eta_1)}{f(z+\eta_2)}\right| \leq \exp\{r^{\sigma-1+\varepsilon}\}.$$

Lemma 2.3 ([4])**.** *Let $f(z)$ be a transcendental meromorphic function of finite order σ, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin H \cup [0, 1]$, and for all k, j, $0 \leq j < k$, we have*

$$(2.1) \qquad \left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq r^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.4 ([5, Lemma 5])**.** *Let $g : (0, +\infty) \to R$, $h : (0, +\infty) \to R$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set E of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ hold for all $r > r_0$.*

Lemma 2.5 ([1, Lemma 5.1])**.** *Let $f(z)$ be a transcendental meromorphic function of finite order σ, and let $\varepsilon > 0$ be a given constant. Then*

$$T(r, f(z+c)) = T(r, f(z)) + O(r^{\sigma-1+\varepsilon}) + O(\log r).$$

Lemma 2.6 ([18, Theorem 1.51])**.** *Suppose that $f_j(z)$ $(j = 1, \ldots n)$ $(n \geq 2)$ are meromorphic functions and $g_j(z)$ $(j = 1, \ldots, n)$ are entire functions satisfying the following conditions.*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$.
(2) $1 \leq j < k \leq n$, $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$.
(3) *For $1 \leq j \leq n$, $1 \leq h < k \leq n$,*

$$T(r, f_j) = o\{T(r, e^{g_h-g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, \infty)$ is of finite linear measure.

Then $f_j(z) \equiv 0$.

Lemma 2.7 ([18, Theorem 1.24])**.** *Suppose $f(z)$ is a non-zero meromorphic function in the complex plane and k is a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

Lemma 2.8 ([6, Lemma 2.2]). *Let $T\colon (0, +\infty) \to (0, +\infty)$ be a non-decreasing continuous function, $s > 0$, $0 < \alpha < 1$, and let $F \subset R^+$ be the set of all r such that*

$$T(r) \leq \alpha T(r + s).$$

If the logarithmic measure of F is infinite, then

$$\limsup_{r \to \infty} \frac{\log T(r)}{\log r} = \infty.$$

Remark. By a simple geometric observation and Lemma 2.8, we conclude that: Let $f(z)$ be a meromorphic function of finite order, then

$$(2.2) \qquad N\left(r, \frac{1}{f(z+c)}\right) \leq N\left(r + |c|, \frac{1}{f(z)}\right) = N\left(r, \frac{1}{f}\right) + S(r, f).$$

3. Proof of Theorem 1.1

(1) Suppose that $\sigma \geq \lambda_f$, and that $\sigma < \sigma(f) - 1$. Then $\lambda_f < \sigma(f) - 1$ follows. From Hadamard's Theorem, then we may write $f(z)$ in the from

$$(3.1) \qquad f(z) = \frac{H_1(z)}{H_2(z)} e^{g(z)},$$

where $H_1(z)(H_2(z)) \not\equiv 0$ is the canonical product formed by zeros (poles) of $f(z)$ such that

$$(3.2) \qquad \begin{aligned} \lambda(H_1) &= \sigma(H_1) = \lambda(f) < \sigma(f) - 1, \\ \lambda(H_2) &= \sigma(H_2) = \lambda\left(\frac{1}{f}\right) < \sigma(f) - 1. \end{aligned}$$

In addition, we have

$$(3.3) \qquad g(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$$

is a polynomial with $\deg(g(z)) = m = \sigma(f)$, where $a_m, \ldots, a_0$ are constants and $a_m \neq 0$. Notice $\lambda_f < \sigma(f) - 1$, then we have $m \geq 2$. Furthermore, from (3.1), we can rewrite $f^{(k)}(z)(k = 1, 2, \ldots, n)$ as the following form

$$(3.4) \qquad f^{(k)}(z) = \left(\frac{H_1(z)}{H_2(z)} e^{g(z)}\right)^{(k)} = \Phi_k(z) e^{g(z)},$$

where $\Phi_k(z)$ is a polynomial formed by $\frac{H_1(z)}{H_2(z)}$, $g(z)$ and their derivatives. Substituting (3.4) into (1.1), we have

$$(3.5) \qquad \begin{aligned} & P_n(z)\Phi_n(z + \eta_n)e^{g(z+\eta_n)} + P_{n-1}(z)\Phi_{n-1}(z + \eta_{n-1})e^{g(z+\eta_{n-1})} \\ & + \cdots + P_1(z)\Phi_1(z + \eta_1)e^{g(z+\eta_1)} + P_0(z)\frac{H_1(z + \eta_0)}{H_2(z + \eta_0)}e^{g(z+\eta_0)} = 0. \end{aligned}$$

From (3.2) and Lemma 2.5, it follows that

$$\sigma\left(P_k\Phi_k(z+\eta_k)\right) \leq \max\{\sigma,\sigma(H_1),\sigma(H_2)\} \leq \max\{\sigma,\lambda_f\} < m-1,$$

(3.6)
$$\sigma\left(P_0\frac{H_1(z+\eta_0)}{H_2(z+\eta_0)}\right) \leq \max\{\sigma,\lambda_f\} < m-1.$$

On the other hand, since $\deg\left(g(z)\right) \geq 2$, we obtain for $i \neq j$ that

(3.7)
$$g(z+\eta_i) - g(z+\eta_j) = ka_k(\eta_i-\eta_j)z^{m-1}$$

is a non-constant polynomial with $\deg(g(z+\eta_i) - g(z+\eta_j)) = m-1$. Since $e^{g(z+\eta_i)-g(z+\eta_j)}$ is of regular growth, by (3.6), we get

$$T(r, P_k\Phi_k(z+\eta_k)) = o\left(T(r, e^{g(z+\eta_i)-g(z+\eta_j)})\right),$$

(3.8)
$$T\left(r, P_0\frac{H_1(z+\eta_0)}{H_2(z+\eta_0)}\right) = o\left(T(r, e^{g(z+\eta_i)-g(z+\eta_j)})\right).$$

Therefore, by Lemma 2.6, (3.5) and (3.8), we know

$$P_k\Phi_k(z+\eta_k) \equiv 0, \quad P_0\frac{H_1(z+\eta_0)}{H_2(z+\eta_0)} \equiv 0,$$

which is impossible. Hence, $\sigma \geq \sigma(f) - 1$.

(2) If $\sigma < \lambda_f$, the same method as (1), we get $\lambda_f \geq \sigma(f) - 1$.

4. Proof of Theorem 1.2

The main idea of this proof is from [6], while details are somewhat different. For the convenience of the reader, we give a complete proof.

By substituting $f(z) = g(z) + d$ into (1.2), we obtain that

(4.1)
$$P_n(z)g^{(n)}(z+\eta_n) + \cdots + P_1(z)g'(z+\eta_1)$$
$$+ P_0(z)g(z+\eta_0) + dP_0(z) - Q(z) = 0.$$

For simplicity, we set

$$H(z) = P_n(z)g^{(n)}(z+\eta_n) + \cdots + P_1(z)g'(z+\eta_1) + P_0(z)g(z+\eta_0).$$

Then, we have

(4.2)
$$m\left(r, \frac{1}{f-d}\right) = m\left(r, \frac{1}{g}\right),$$

and

$$\left|\frac{H(z)}{g(z)}\right| = \left|\frac{P_n(z)g^{(n)}(z+\eta_n) + \cdots + P_1(z)g'(z+\eta_1) + P_0(z)g(z+\eta_0)}{g(z)}\right|$$
$$\leq \left|\frac{P_n(z)g^{(n)}(z+\eta_n)}{g(z)}\right| + \cdots + \left|\frac{P_1(z)g'(z+\eta_1)}{g(z)}\right| + \left|\frac{P_0(z)g(z+\eta_0)}{g(z)}\right|.$$

Thus, Lemma 2.1 and the lemma on the logarithmic derivative yield

(4.3)
$$m\left(r, \frac{H(z)}{g(z)}\right) = S(r, g).$$

Now, noting $dP_0(z) - Q(z) \not\equiv 0$, then by (4.1)-(4.3), it follows that

$$m\left(r, \frac{1}{f-d}\right) = m\left(r, \frac{1}{g}\right) = m\left(r, \frac{dP_0 - Q}{g}\right) + m\left(r, \frac{1}{dP_0 - Q}\right)$$

$$= m\left(r, \frac{H}{g}\right) + S(r,g) = S(r,g) = S(r,f).$$

Hence, we have $\lambda(f-d) = \sigma(f)$.

5. Proof of Theorem 1.3

Suppose, contrary to the assertion, that $\lambda_f < \sigma(f)$. Then, using the same way as in the proof of Theorem 1.1, we obtain $f(z)$ is of regular growth. Since $\sigma(Q) < \sigma(f)$, $T(r,Q) = S(r,f)$ follows. From Theorem 1.2, we have $\lambda(f) = \sigma(f)$, which contradicts the assumption that $\lambda_f < \sigma(f)$. Hence, $\lambda_f = \sigma(f)$.

6. Proof of Theorem 1.5

Suppose that $f(z)$ is a finite order transcendental entire solution of (1.5). Then, equation (1.5) gives

$$\frac{f(z+c) - Q(z)}{f^{(k)}(z)} = e^{P(z)}.$$

By Lemma 2.5, it follows that

$$\deg\left(P(z)\right) \leq \max\{\sigma(f(z+c) - Q(z)), \sigma(f^{(k)}(z))\} = \sigma(f).$$

In the following, we will prove $\lambda(f) = \sigma(f)$. Otherwise, we assume $\lambda(f) < \sigma(f)$, then we know $f(z)$ is normal growth. Hence, we have

(6.1) $N\left(r, \frac{1}{f}\right) = S(r,f).$

By Lemma 2.7 and (2.2), we get that

(6.2) $N\left(r, \frac{1}{f(z+c)}\right) = S(r,f), \quad N\left(r, \frac{1}{f^{(k)}}\right) = S(r,f).$

Thus, by (1.5) and (6.2), it follows that

(6.3) $N\left(r, \frac{1}{f(z+c) - Q(z)}\right) = N\left(r, \frac{1}{e^{P(z)} f^{(k)}(z)}\right) = S(r,f).$

Noting $Q(z) \not\equiv 0$, then from (6.2) and (6.3), we obtain that

$$T(r, f(z+c)) \leq \overline{N}\left(r, \frac{1}{f(z+c)}\right) + \overline{N}\left(r, \frac{1}{f(z+c) - Q(z)}\right) + S(r,f) = S(r,f),$$

which is a contradiction. Hence, $\lambda(f) = \sigma(f)$.

7. Proof of Theorem 1.6

Suppose that $j = 0, 1, \ldots, n-1$ and

$$P_j(z) = a_{j\,p_j} z^{p_j} + a_{j\,p_{j-1}} z^{p_j-1} + \cdots + a_{j\,0},$$

$$Q_j(z) = b_{j\,q_j} z^{q_j} + b_{j\,q_{j-1}} z^{q_j-1} + \cdots + b_{j\,0}.$$

Assume that $f(z) \not\equiv 0$ is a solution of the equation (1.7) such that $\sigma(f) = \sigma < \infty$. From Lemma 2.2, Lemma 2.3 and Lemma 2.5, we get that, for any given $\varepsilon > 0$, there exits a subset $E \subset (1, \infty)$ with finite logarithmic measure such that for all $|z| = r \notin [0, 1] \cup E$,

$$(7.1) \quad \left| \frac{f^{(k)}(z + \eta_k)}{f(z + \eta_0)} \right| = \left| \frac{f^{(k)}(z + \eta_k)}{f(z + \eta_k)} \right| \left| \frac{f(z + \eta_k)}{f(z + \eta_0)} \right| \leq \exp\{r^{\sigma-1+\varepsilon}\} r^{k(\sigma-1+\varepsilon)},$$

where $k = 1, 2, \ldots, n$.

Case 1. If $\deg(P_0) > \deg(P_j)(j = 1, 2, \ldots, n-1)$, then we take a suitable z such that $a_m z^m = |a_m| r^m$. Combining (1.7) and (7.1), we have for all sufficiently large r and $r \notin [0, 1] \cup E$, that

$$(7.2)$$
$$\left| P_0(e^A) + Q_0(e^{-A}) \right|$$
$$= |a_{0p_0}| e^{p_0 r^m |a_m|} (1 + o(1))$$
$$\leq \left| \frac{f^{(n)}(z + \eta_n)}{f(z + \eta_0)} \right| + |P_{n-1}(e^A) + Q_{n-1}(e^{-A})| \left| \frac{f^{(n-1)}(z + \eta_{n-1})}{f(z + \eta_0)} \right| + \cdots$$
$$\quad + |P_1(e^A) + Q_1(e^{-A})| \left| \frac{f'(z + \eta_1)}{f(z + \eta_0)} \right|$$
$$\leq e^{r^{\sigma-1+\varepsilon}} r^{n(\sigma-1+\varepsilon)} + |a_{n-1p_{n-1}}| e^{p_{n-1} r^m |a_m|} e^{r^{\sigma-1+\varepsilon}} r^{(n-1)(\sigma-1+\varepsilon)} (1 + o(1))$$
$$\quad + \cdots + |a_{1p_1}| e^{p_1 r^m |a_m|} e^{r^{\sigma-1+\varepsilon}} r^{\sigma-1+\varepsilon} (1 + o(1)).$$

If $\sigma(f) < 1$, then equation (7.1) yields

$$(7.3) \qquad\qquad\qquad \left| \frac{f^{(k)}(z + \eta_k)}{f(z + \eta_0)} \right| = o(1)$$

for all sufficiently large r and $r \notin [0, 1] \cup E$. Hence, by (7.2) and (7.3), it follows that

$$|a_{0p_0}| e^{p_0 r^m |a_m|} (1 + o(1))$$
$$\leq \left| \frac{f^{(n)}(z + \eta_n)}{f(z + \eta_0)} \right| + |P_{n-1}(e^A) + Q_{n-1}(e^{-A})| \left| \frac{f^{(n-1)}(z + \eta_{n-1})}{f(z + \eta_0)} \right| + \cdots$$
$$\quad + |P_1(e^A) + Q_1(e^{-A})| \left| \frac{f'(z + \eta_1)}{f(z + \eta_0)} \right|$$
$$\leq 1 + |a_{n-1p_{n-1}}| e^{p_{n-1} r^m |a_m|} (1 + o(1)) + \cdots + |a_{1p_1}| e^{p_1 r^m |a_m|} (1 + o(1))$$
$$\leq nM e^{\max\{p_1, \ldots, p_{n-1}\} r^m |a_m|} (1 + o(1)),$$

and $M = \max\{|a_{n-1p_{n-1}}|, \ldots, |a_{1p_1}|, 1\}$. Since $p_0 > \max\{p_1, \ldots, p_{n-1}\} = N$, we have

$$\frac{|a_{0p_0}|}{nM} e^{(p_0-N)|a_m|r^m}(1 + o(1)) \leq 1,$$

which is impossible. Hence, we obtain that $\sigma(f) \geq 1$. From (7.2) again, we have

$$|a_{0p_0}| e^{p_0 r^m |a_m|}(1 + o(1)) \leq nM e^{r^{\sigma-1+\varepsilon}} e^{\max\{p_1,\ldots,p_{n-1}\}r^m |a_m|}(1 + o(1)),$$

which implies that

(7.4) $$\frac{|a_{0p_0}|}{nM} e^{(p_0-N)|a_m|r^m}(1 + o(1)) \leq e^{r^{\sigma-1+\varepsilon}}.$$

By Lemma 2.4 and (7.4), we have that $\sigma - 1 + \varepsilon \geq m$, which implies $\sigma(f) \geq m + 1$.

Case 2. If $\deg(Q_0) > \deg(Q_j)$, then taking a suitable z such that $a_m z^m = -|a_m|r^m$. Following a similar arguments as above, we also get $\sigma(f) \geq m + 1$.

In the following, we prove that $\max\{\lambda(f - a), \lambda\left(\frac{1}{f}\right)\} = \sigma(f)$, where $a \in \mathbb{C}\backslash\{0\}$. Conversely, suppose that $\max\{\lambda(f-a), \lambda\left(\frac{1}{f}\right)\} < \sigma(f)$. Then, similarly as Theorem 1.1, we obtain $f(z)$ is of regular growth. Noting $\sigma(f) \geq m + 1$, we have

$$T(r, \{P_j(e^{A(z)}) + Q_j(e^{-A(z)})\}) = S(r, f).$$

Clearly,

$$a[P_0(e^{A(z)}) + Q_0(e^{-A(z)})] \not\equiv 0.$$

From Theorem 1.2, we have $\lambda(f - a) = \sigma(f)$, which is a contradiction. Hence, $\max\{\lambda(f - a), \lambda\left(\frac{1}{f}\right)\} = \sigma(f)$.

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XIAOGUANG QI
UNIVERSITY OF JINAN
SCHOOL OF MATHEMATICS
JINAN, SHANDONG, 250022, P. R. CHINA
Email address: `xiaoguang.202@163.com, xiaogqi@mail.sdu.edu.cn`

LIANZHONG YANG
SHANDONG UNIVERSITY
SCHOOL OF MATHEMATICS
JINAN, SHANDONG, 250100, P. R. CHINA
Email address: `lzyang@sdu.edu.cn`