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BIISOMETRIC OPERATORS AND BIORTHOGONAL SEQUENCES

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Abstract. It is shown that a pair of Hilbert space operators V and W such that $V^*W = I$ (called a biisometric pair) shares some common properties with unilateral shifts when orthonormal bases are replaced with biorthogonal sequences, and it is also shown how such a pair of biisometric operators yields a pair of biorthogonal sequences which are shifted by them. These are applied to a class of Laguerre operators on $L^2[0,\infty)$.

1. Introduction

Throughout this paper $\mathcal{H}$ and $\mathcal{K}$ stand for Hilbert spaces. We use the same symbol $\langle\cdot\,;\cdot\rangle$ and $\|\cdot\|$ for the inner product and norm in both of them, respectively. Let $T\colon \mathcal{H}\to\mathcal{K}$ be a bounded linear transformation (referred to as an operator if $\mathcal{K}=\mathcal{H}$ — i.e., an operator on $\mathcal{H}$ is a bounded linear transformation of $\mathcal{H}$ into itself). Let I stand for the identity operator (either on $\mathcal{H}$ or on $\mathcal{K}$). Recall that T is an isometry if $T^*T = I$, identity on $\mathcal{H}$. Every isometry is injective. A transformation T is unitary if it is a surjective isometry (i.e., an invertible isometry), which means T is an isometry and a coisometry (i.e., $TT^* = I$, identity on $\mathcal{K}$; and $T^*T = I$, identity on $\mathcal{H}$). By a subspace of $\mathcal{H}$ we mean a *closed* linear manifold of $\mathcal{H}$. Let $\mathcal{M}^-$ and $\mathcal{M}^\perp$ denote closure and orthogonal complement, respectively, of a linear manifold $\mathcal{M}$ of $\mathcal{H}$ (both are subspaces of $\mathcal{H}$). The kernel and range of a bounded linear transformation T will be denoted by $\mathcal{N}(T)$ (a subspace of $\mathcal{H}$) and $\mathcal{R}(T)$ (a linear manifold of $\mathcal{K}$), respectively. The adjoint of T (which is a bounded linear transformation of $\mathcal{K}$ into $\mathcal{H}$) will be denoted by T^*. Let span A denote the linear span of an arbitrary set $A\subseteq\mathcal{H}$ and let $\bigvee A$ denote the closure of span A.

An operator $S\colon\mathcal{H}\to\mathcal{H}$ is a *unilateral shift* if there exists an infinite sequence $\{\mathcal{H}_k\}_{k=0}^\infty$ of nonzero pairwise orthogonal subspaces of $\mathcal{H}$ (i.e., $\mathcal{H}_j\perp\mathcal{H}_k$) such that $\mathcal{H}=\bigoplus_{k=0}^\infty\mathcal{H}_k$ (i.e., $\{\mathcal{H}_k\}_{k=0}^\infty$ spans $\mathcal{H}$) and S maps each $\mathcal{H}_k$ isometrically

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onto $\mathcal{H}_{k+1}$ so that $S|_{\mathcal{H}_k}\colon \mathcal{H}_k \to \mathcal{H}_{k+1}$ is a unitary transformation (i.e., a surjective isometry). Thus $\dim \mathcal{H}_{k+1} = \dim \mathcal{H}_k$ for every $k \geq 0$. Such a common dimension is the *multiplicity* of S. The adjoint $S^*\colon \mathcal{H} \to \mathcal{H}$ of S is referred to as a *backward unilateral shift*. Every unilateral shift S is an isometry (i.e., $S^*S = I$) and so S is injective (but not surjective). Moreover, for each $k \geq 0$

$$\mathcal{H}_k = S^k\mathcal{H}_0 \quad \text{with} \quad \mathcal{H}_0 = \mathcal{N}(S^*),$$

where $\mathcal{N}(S^*)$ denotes the kernel of S^*. Therefore

$$S\mathcal{H}_k = \mathcal{H}_{k+1} \quad \text{and} \quad S^*\mathcal{H}_{k+1} = \mathcal{H}_k.$$

In this paper we show that there exist pairs of Hilbert space operators V and W that satisfy the above displayed shifting properties (although they may not be unilateral shifts themselves, not even isometries), where orthogonality is replaced by biorthogonality. The motivation behind such a program comes from the following result from [2,4]. *If S and R are unilateral shifts on a Hilbert space $\mathcal{H}$ such that $SS^* + RR^* = I$, then $\mathcal{H}$ admits the dual-shift decomposition*, namely,

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} S^k\mathcal{N}(S^*) \oplus \bigoplus_{k=1}^{\infty} R^k\mathcal{N}(R^*).$$

(The symbol $\oplus$ stands for orthogonal direct sum.) This will be approached here in light of biorthogonal sequences (which are not necessarily individually orthogonal sequences) and biisometric operators (which are not necessarily individually isometric operators). Next section discusses these notions.

2. Biorthogonal sequences

Biorthogonal sequences are germane to Banach spaces and were introduced in the context of basis for separable Banach spaces [8, Definition 1.4.1], [6, Definition 1.f.1]. Thus let $\mathcal{H}$ be a *separable* Hilbert space. In a Hilbert space setting (where dual pair boils down to inner product after the Riesz Representation Theorem) the notion of biorthogonality reads as follows.

Definition 2.1. Two sequences $\{f_n\}$ and $\{g_n\}$ of vectors in $\mathcal{H}$ are said to be *biorthogonal* (to each other) if $\langle f_m; g_n\rangle = \delta_{m,n}$, where δ stands for the Kronecker delta function. If $\{f_n\}$ is such that there exists a sequence $\{g_n\}$ for which $\{f_n\}$ and $\{g_n\}$ are biorthogonal, then it is said that $\{f_n\}$ *admits a biorthogonal sequence* (and so does $\{g_n\}$) and the pair $\{\{f_n\}, \{g_n\}\}$ is referred to as a *biorthonormal system*.

Let $\{f_n\}$ and $\{g_n\}$ be a pair of biorthogonal sequences. If they are such that $f_n = g_n$ for all n, then we get the definition of an orthonormal sequence, although in general neither $\{f_n\}$ nor $\{g_n\}$ are orthogonal (much less orthonormal) sequences.

A sequence $\{f_n\}$ that admits a biorthogonal sequence $\{g_n\}$ was called *minimal* in [6, Definition 1.f.1], where it was pointed out that (i) $\{f_n\}$ *admits a biorthogonal sequence if and only if $f_k \notin \bigvee\{f_k\}_{n\neq k}$ for every integer k* (i.e., if and only if each vector f_k from $\{f_n\}$ is not in the closure of $\operatorname{span}\{f_n\}/\{f_k\}$) [6, p. 42] and (ii) *every basic sequence is minimal* [6, p. 43] (a sequence is *basic* if it is a Schauder basis for its closed span). So *every orthonormal sequence admits a biorthonormal sequence*.

A sequence $\{f_n\}$ spanning the whole space $\mathcal{H}$ is sometimes called *total* (or *complete*, or *fundamental*). This means $\bigvee\{f_n\} = \mathcal{H}$ or, equivalently, $f \perp f_n$ for every n implies $f = 0$ (i.e., $\{f_n\}$ is total if and only if the only vector orthogonal to every f_n is the origin). It was pointed out in [9] that (iii) *if* $\{f_n\}$ *admits a biorthogonal sequence* $\{g_n\}$, *then* $\{g_n\}$ *is unique if and only if* $\{f_n\}$ *is total*.

If $\{f_n\}$ admits a biorthogonal sequence $\{g_n\}$, and if $\{f_n\}$ is total, then $\{g_n\}$ is not necessarily total (i.e., the property "$\{f_n\}$ spans $\mathcal{H}$" is not inherited by $\{g_n\}$ — as in the example below). A total sequence $\{f_n\}$ that admits a (unique) biorthogonal sequence $\{g_n\}$ was called *exact* in [9], where it was shown that (iv) *if the sequence* $\{f_n\} = \{e^{i\,\alpha_n}\}$ *of vectors in the Hilbert space* $L^2(-\pi, \pi)$ (so that $f_n(t) = e^{i\,\alpha_n t}$ for each n almost everywhere in $(-\pi, \pi)$ — i.e., for almost every t in $(-\pi, \pi)$ with respect to Lebesgue measure) *is exact* (i.e., $\{f_n\}$ is total and admits a (unique) biorthogonal sequence $\{g_n\}$), *then the biorthogonal sequence* $\{g_n\}$ *is also exact*.

Example 2.1. Let $\{e_n\}_{n\geq 1}$ be any orthonormal basis for $\mathcal{H}$ (any orthonormal sequences that spans $\mathcal{H}$, thus total).

(a) The sequence $\{f_n\}_{n\geq 1} = \{e_1 + e_{n+1}\}_{n\geq 1}$ is total (since if $f \perp f_n$ for all n, then the absolute value of the Fourier coefficients of f with respect to the orthonormal basis $\{e_n\}_{n\geq 1}$ are constant, and so $f = 0$). Moreover, $\{f_n\}_{n\geq 1}$ admits a biorthogonal sequence $\{g_n\}_{n\geq 1} = \{e_{n+1}\}_{n\geq 1}$ which is unique and not total.

(b) The sequences $\{f_n\}_{n\geq 1} = \{e_1 + e_2 + e_{n+2}\}_{n\geq 1}$, $\{g_n\}_{n\geq 1} = \{e_{n+2}\}_{n\geq 1}$, and $\{h_n\}_{n\geq 1} = \{e_1 - e_2 + e_{n+2}\}_{n\geq 1}$ are pairwise biorthogonal to each other, and therefore they are all not total.

Furthermore, every vector from $\{f_n\}_{n\geq 1}$ in (a) or in (b), and from $\{h_n\}_{n\geq 1}$ in (b), is not orthogonal to any other vector from the same sequence, and all vectors in $\{f_n\}_{n\geq 1}$ and in $\{h_n\}_{n\geq 1}$ have squared norm 2 or 3 while $\{g_n\}_{n\geq 1}$ in (a) or in (b) is an orthonormal sequence.

Indeed, *there is no distinct pair of biorthonormal sequences* as we show below.

Theorem 2.1. *Take a pair of biorthogonal sequences* $\{f_n\}$ *and* $\{g_n\}$. *If* $\|f_n\| = \|g_n\|^{-1}$, *then* $\{f_n\}$ *and* $\{g_n\}$ *are proportional, which means for each* n *there exists a nonnegative number* α_n *for which* $f_n = \alpha_n g_n$. *Moreover,* $\alpha_n = \|f_n\|^2$.

Proof. Suppose $\{f_n\}$ and $\{g_n\}$ are biorthogonal sequences. Hence $\langle f_n, g_n\rangle = 1$. Take an arbitrary n. If $\|f_n\| = \|g_n\|^{-1}$, then

$$\langle f_n, g_n\rangle = \|f_n\|\|g_n\|.$$

But this is equivalent to saying that (see, e.g., [1, Problem 5.2])

$$f_n = \alpha_n g_n$$

for some positive real number α_n. Therefore $\{f_n\}$ and $\{g_n\}$ are proportional (and so biorthogonal to each other). Moreover, since $\|f_n\| = \|g_n\|^{-1}$, it follows that $\|f_n\| = \alpha_n\|g_n\| = \alpha_n\|f_n\|^{-1}$ and so $\alpha_n = \|f_n\|^2$. □

Corollary 2.1. *There is no distinct pair of biorthonormal sequences. In other words, if two sequences $\{f_n\}$ and $\{g_n\}$ are biorthogonal and if $\|f_n\| = \|g_n\| = 1$ for all n, then $f_n = g_n$ for all n.*

Proof. This is a particular case of Theorem 2.1 for $\|f_n\| = \|g_n\| = 1$ for all n. □

3. Biisometric operators

Consider a pair of operators V and W on a Hilbert space $\mathcal{H}$.

Definition 3.1. Two operators V and W are said to be *biisometric* if $V^*W = I$, and in this case $\{V, W\}$ is referred to as a *biisometric pair* on $\mathcal{H}$.

It is clear that $V^*W = I$ if and only if $W^*V = I$. Thus $\{V, W\}$ is a biisometric pair if and only if

$$V^*W = I = W^*V.$$

Let V and W be a pair of biisometric operators. If they are such that $W = V$, then we get the definition of an isometry, viz., $V^*V = I$, although in general neither V nor W are assumed to be isometries themselves.

Theorem 3.1. *Let V and W be operators on $\mathcal{H}$. Take arbitrary nonzero vectors v and w in $\mathcal{H}$. For each nonnegative integer n consider the vectors*

$$\phi_n = V^n w \quad \textit{and} \quad \psi_n = W^n v$$

in $\mathcal{H}$. If $\{V, W\}$ is a biisometric pair on $\mathcal{H}$, then there exist

$$v \in \mathcal{N}(V^*) \quad \textit{and} \quad w \in \mathcal{N}(W^*)$$

(equivalently, $v \in \mathcal{R}(V)^\perp$ and $w \in \mathcal{R}(W)^\perp$) such that the sequences $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal. Moreover,

$$V\phi_n = \phi_{n+1} \quad \textit{and} \quad W\psi_n = \psi_{n+1},$$

and also

$$V^*\psi_{n+1} = \psi_n \quad \textit{and} \quad W^*\phi_{n+1} = \phi_n.$$

Proof. Take arbitrary nonzero vectors $v \in \mathcal{N}(V^*)$ and $w \in \mathcal{N}(W^*)$, arbitrary nonnegative integers m, n, and set $\phi_n = V^n w$ and $\psi_n = W^n v$. Suppose $n < m$. Since $V^*W = I$, a trivial induction leads to $V^{*n}W^n = I$ which ensures

$$\langle \phi_m \,;\psi_n\rangle = \langle V^m w \,; W^n v\rangle = \langle w \,; V^{*(m-n)}V^{*n}W^n v\rangle = \langle w \,; V^{*(m-n)}v\rangle = 0$$

for $n < m$ because $v \in \mathcal{N}(V^*)$ implies that $v \in \mathcal{N}(V^{*m-n})$. On the other hand, suppose $m < n$. Since $W^*V = I$ and $w \in \mathcal{N}(W^*)$, a similar argument ensures

$\langle\phi_m\,;\psi_n\rangle = 0$ for $m<n$. Moreover, since $W^*V = I$, it also follows that $\mathcal{N}(V^*)^\perp = \mathcal{R}(V)^- \not\subseteq \mathcal{N}(W^*)$ and hence

$$\mathcal{N}(V^*) \not\perp \mathcal{N}(W^*).$$

Thus there exist $v\in\mathcal{N}(V^*)$ and $w\in\mathcal{N}(W^*)$ such that $\langle w\,;v\rangle \neq 0$, and so we may take $v\in\mathcal{N}(V^*)$ and $w\in\mathcal{N}(W^*)$ for which $\langle\, w\,;v\rangle = 1$. Then

$$\langle\phi_n\,;\psi_n\rangle = \langle\, w\,;v\rangle = 1.$$

Therefore $\langle\phi_m\,;\psi_n\rangle = \delta_{mn}$. This means $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal sequences. Moreover, according to definition of ϕ_m and ψ_n we get

$$V\phi_n = V^{m+1}w = \phi_{n+1} \quad \text{and} \quad W\psi_n = W^{m+1}v = \psi_{n+1},$$

and since $V^*W = I = W^*V$ we also get

$$V^*\psi_{n+1} = V^*W^{m+1}v = W^n v = \phi_n \quad \text{and} \quad W^*\phi_{n+1} = W^*V^{n+1}v = V^n v = \psi_n$$

for every nonnegative integer n. □

Corollary 3.1. *Let V and W be a biisometric pair on $\mathcal{H}$, and consider the biorthogonal sequences $\{\phi_k\}$ and $\{\psi_k\}$ defined in Theorem 3.1 in terms of nonzero vectors $v\in\mathcal{R}(V)^\perp$ and $w\in\mathcal{R}(W)^\perp$. If in addition these biorthogonal sequences span $\mathcal{H}$, i.e., if they are total in $\mathcal{H}$, then every $f\in\mathcal{H}$ can be decomposed as*

$$f = \sum_{k=0}^{\infty}\langle f\,;\psi_k\rangle\phi_k = \sum_{k=0}^{\infty}\langle f\,;\phi_k\rangle\psi_k,$$

and therefore

$$Vf = \sum_{k=0}^{\infty}\langle f\,;\psi_k\rangle\phi_{k+1} \quad \textit{and} \quad Wf = \sum_{k=0}^{\infty}\langle f\,;\phi_k\rangle\psi_{k+1},$$

$$V^*f = \sum_{k=0}^{\infty}\langle f\,;\phi_{k+1}\rangle\psi_k \quad \textit{and} \quad W^*f = \sum_{k=0}^{\infty}\langle f\,;\psi_{k+1}\rangle\phi_k.$$

Proof. Take an arbitrary $f\in\mathcal{H}$. If the biorthogonal sequences $\{\phi_k\}$ and $\{\psi_k\}$ span $\mathcal{H}$ (i.e., if $\bigvee\{\phi_k\} = \bigvee\{\psi_k\} = \mathcal{H}$), then

$$f = \sum_{k=0}^{\infty}\alpha_k\phi_k = \sum_{k=0}^{\infty}\beta_k\psi_k$$

for some pair of sequences of scalars $\{\alpha_k\}$ and $\{\beta_k\}$. To verify the identities

$$\alpha_n = \langle f\,;\psi_n\rangle \quad \text{and} \quad \beta_n = \langle f\,;\phi_n\rangle$$

for every $n\geq 1$ observe (by the continuity of the inner product and recalling that $\{\phi_k\}$ and $\{\psi_k\}$ are biorthogonal) that

$$\langle f\,;\psi_n\rangle = \sum_{k=0}^{\infty}\alpha_k\langle\phi_k\,;\psi_n\rangle = \alpha_n \quad \text{and} \quad \langle f\,;\phi_n\rangle = \sum_{k=0}^{\infty}\alpha_k\langle\psi_k\,;\phi_n\rangle = \beta_n$$

for every $n \geq 1$. To complete the proof recall from Theorem 3.1 that

$$V\phi_n = \phi_{n+1},\ W\psi_n = \psi_{n+1},\ V^*\psi_{n+1} = \psi_n, \text{ and } W^*\phi_{n+1} = \phi_n$$

for each nonnegative integer n, and expand Vf and W^*f in terms of $\{\phi_k\}$ and V^*f and Wf in terms of $\{\phi_k\}$ (using again the continuity of the inner product). □

4. Laguerre shifts

We now apply the results of Section 3 to a class of Laguerre operators. Recall that the Laguerre functions consist of an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for the concrete Hilbert space $L^2[0,\infty)$ (see, e.g., [1, Example 5.L(d)]) defined a.e. for $t \geq 0$ by

$$e_n(t) = e^{-\frac{1}{2}t}\, L_n(t)$$

for each integer $n \geq 0$, where $L_n(t)$ are the Laguerre polynomials of degree $n \geq 0$. Consider the operator $S\colon L^2[0,\infty) \to L^2[0,\infty)$ defined by $Sf = g$, where (for almost all $t \geq 0$ with respect to Lebesgue measure; i.e., almost everywhere (a.e.) on $[0,\infty)$)

$$(Sf)(t) = g(t) \quad \text{with} \quad g(t) = f(t) - \int_0^t e^{-\frac{1}{2}(t-\tau)}\, f(\tau)\, d\tau,$$

which is an isometry having the shift property, viz. (with $e_n(t) = e^{-\frac{1}{2}t}\, L_n(t)$),

$$(Se_n)(t) = e^{-\frac{1}{2}t} L_{n+1}(t) = e_{n+1}(t)$$

for every $t \geq 0$ and each integer $n \geq 0$. This is referred to as the Laguerre shift (of multiplicity 1) generating the Laguerre functions. Let

$$D_{2\alpha}\colon L^2[0,\infty) \to L^2[0,\infty)$$

be the dilation-by-2α-operator defined by $D_{2\alpha}f = g$ where for every $t \geq 0$

$$(D_{2\alpha}f)(t) = g(t) \quad \text{with} \quad g(t) = \sqrt{2\alpha}\, f(2\alpha t)$$

for each $\alpha \geq \frac{1}{2}$. The α-Laguerre functions are then defined for each $n \geq 0$ (again with $e_n(t) = e^{-\frac{1}{2}t}\, L_n(t)$) by

$$(D_{2\alpha}\, e_n)(t) = \sqrt{2\alpha}\, e_n(2\alpha t) = \sqrt{2\alpha}\, e^{-\alpha t} L_n(2\alpha t)$$

for every $t \geq 0$. Similarly, the α-Laguerre shift S_α — generating the α-Laguerre functions — is defined by $S_\alpha = D_{2\alpha}S$ so that for every $t \geq 0$

$$(S_\alpha f)(t) = (D_{2\alpha}S)f(t) = g(t) \ \text{ with } \ g(t) = f(t) - 2\alpha \int_0^t e^{-\alpha(t-\tau)} f(\tau)\, d\tau$$

and (with $(D_{2\alpha}\, e_n)(t) = \sqrt{2\alpha}\, e^{-\alpha t} L_n(2\alpha t)$) for each $n \geq 0$

$$(S_\alpha\, e_n)(t) = (D_{2\alpha}Se_n)(t) = (D_{2\alpha}e_{n+1})(t) = \sqrt{2\alpha}\, e^{-\alpha t}\, L_{n+1}(2\alpha t).$$

Now consider a class of 2-parameter Laguerre functions as follows. The $(\alpha+\beta)$-Laguerre functions are defined for every $t \geq 0$ by (recall: $e_n(t) = e^{-\frac{1}{2}t} L_n(t)$)

$$(D_{\alpha+\beta}\, e_n)(t) = \sqrt{\alpha+\beta}\, e^{-\frac{\alpha+\beta}{2}t} L_n\big((\alpha+\beta)t\big)$$

for each $n \geq 0$ where $\alpha, \beta \geq \frac{1}{2}$. (Compare with [3, Section 5].) From now on we proceed formally.

Lemma 4.1. *The sequences $\{\phi_n\}$ and $\{\psi_n\}$ given by*

$$\phi_n(t) = \sqrt{\alpha+\beta}\, e^{-\alpha t} L_n\big((\alpha+\beta)t\big) \quad \textit{and} \quad \psi_n(t) = \sqrt{\alpha+\beta}\; e^{-\beta t}\, L_n\big((\alpha+\beta)t\big)$$

for each $n \geq 0$ are biorthogonal and span $L^2[0,\infty)$.

Proof. For each $k, \ell \geq 0$

$$\begin{aligned} \langle \phi_k\,;\psi_\ell\rangle &= \int_0^\infty \sqrt{\alpha+\beta}\, e^{-\alpha t} L_k\big((\alpha+\beta)t\big) \sqrt{\alpha+\beta}\, e^{-\beta t} L_\ell\big((\alpha+\beta)t\big)\, dt \\ &= \int_0^\infty \sqrt{\alpha+\beta}\, e^{-\frac{\alpha+\beta}{2}t} L_k\big((\alpha+\beta)t\big) \sqrt{\alpha+\beta}\, e^{-\frac{\alpha+\beta}{2}t} L_\ell\big((\alpha+\beta)t\big)\, dt \\ &= \delta_{k\ell} \end{aligned}$$

since the above-defined $(\alpha+\beta)$-Laguerre functions, namely, $D_{\alpha+\beta}\big(e^{-\frac{1}{2}t} L_n(t)\big)$ $= \sqrt{\alpha+\beta}\, e^{-\frac{\alpha+\beta}{2}t} L_n\big((\alpha+\beta)t\big)$, are orthonormal. It remains to show that $\{\phi_n\}$ and $\{\psi_n\}$ span $L^2[0,\infty)$. Suppose there is a nonzero $h \in L^2[0,\infty)$ such that

$$\int_0^\infty e^{-\alpha t} L_n\big((\alpha+\beta)t\big) h(t)\, dt = 0$$

for every $t \geq 0$ and each $n \geq 0$, which can be rewritten as

$$\int_0^\infty e^{-\frac{\alpha+\beta}{2}t} L_n\big((\alpha+\beta)t\big)\big(e^{-\frac{\alpha-\beta}{2}t} h(t)\big)\, dt = 0.$$

Therefore, since the $(\alpha+\beta)$-Laguerre functions $\sqrt{\alpha+\beta}\, e^{-\frac{\alpha+\beta}{2}t} L_n\big((\alpha+\beta)t\big)$ also span $L^2[0,\infty)$,

$$e^{-\frac{\alpha-\beta}{2}t} h(t) = 0 \;\text{ for every }\; t \geq 0 \quad \Longrightarrow \quad h(t) = 0 \;\text{ for every }\; t \geq 0,$$

which ensures that the sequence $\{\phi_n\}$ spans $L^2[0,\infty)$. Interchanging α and β, $\{\psi_n\}$ also spans $L^2[0,\infty)$. □

Lemma 4.2. *The Laplace transforms of each ϕ_n and ψ_n are given by*

$$\mathcal{L}_s[\phi_n](s) = \Big[\tfrac{s-\beta}{s+\alpha}\Big]^n \tfrac{\sqrt{\alpha+\beta}}{s+\alpha} \quad \textit{and} \quad \mathcal{L}_s[\psi_n](s) = \Big[\tfrac{s-\alpha}{s+\beta}\Big]^n \tfrac{\sqrt{\alpha+\beta}}{s+\beta}.$$

Proof. The functions ϕ_n and ψ_n were defined in Lemma 4.1 as follows.

$$\phi_n(t) = \sqrt{\alpha+\beta}\, e^{-\alpha t} L_n\big((\alpha+\beta)t\big) \quad \text{and} \quad \psi_n(t) = \sqrt{\alpha+\beta}\; e^{-\beta t}\, L_n\big((\alpha+\beta)t\big).$$

Recall that the Laplace transform of the Laguerre polynomial $L_n(t)$ is

$$\mathcal{L}_s[L_n](s) = \tfrac{(s-1)^n}{s^{n+1}}.$$

Thus (formally)

$$\mathcal{L}_s\big[\sqrt{\alpha+\beta}\,e^{-\alpha t}L_n([\alpha+\beta]t\big] \;=\; \sqrt{\alpha+\beta}\;\tfrac{(s+\alpha-[\alpha+\beta])^n}{(s+\alpha)^{n+1}}$$
$$=\; \sqrt{\alpha+\beta}\;\tfrac{(s-\beta)^n}{(s+\alpha)^{n+1}} \;=\; \Big[\tfrac{s-\beta}{s+\alpha}\Big]^n\;\tfrac{\sqrt{\alpha+\beta}}{s+\alpha},$$

and hence

$$\mathcal{L}_s\big[\sqrt{\alpha+\beta}\,e^{-\beta t}L_n([\alpha+\beta]t)\big] = \Big[\tfrac{s-\alpha}{s+\beta}\Big]^n\;\tfrac{\sqrt{\alpha+\beta}}{s+\beta}$$

by interchanging α and β. □

The shift operator S corresponds to the operator *multiplication* by the function $H(\cdot)$ in the Hardy space H^2 which is given by

$$H(s) = \tfrac{s-\frac{1}{2}}{s+\frac{1}{2}}$$

for $\mathrm{Re}(s) > -\frac{1}{2}$. The Laplace transforms of ϕ_n and ψ_n in Lemma 4.2 imply the existence of functions $H_{\alpha\beta}$ and $H_{\beta\alpha}$ in H^2 given by

$$H_{\alpha\beta}(s) = \tfrac{s-\alpha}{s+\beta} \quad\text{ and }\quad H_{\beta\alpha}(s) = \tfrac{s-\beta}{s+\alpha}.$$

Consequently we consider the $(\alpha\beta)$-Laguerre operator $S_{\alpha\beta}$ on $L^2[0,\infty)$ defined by $S_{\alpha\beta}f = g$ where, for every $t \geq 0$

$$(S_{\alpha\beta}f)(t) = g(t) \quad\text{ with }\quad g(t) = f(t) - (\alpha+\beta)\int_0^t e^{-\beta(t-\tau)}f(\tau)\,d\tau,$$

generating for each $n \geq 0$ the function ψ_n given by

$$\psi_m(t) = \sqrt{\alpha+\beta}\,e^{-\beta t}L_n\big([\alpha+\beta]t\big).$$

Interchanging α and β we have the $S_{\beta\alpha}$-Laguerre operator $S_{\beta\alpha}$ on $L^2[0,\infty)$ defined by $S_{\beta\alpha}f = g$ where, for every $t \geq 0$

$$(S_{\beta\alpha}f)(t) = g(t) \quad\text{ with }\quad g(t) = f(t) - (\alpha+\beta)\int_0^t e^{-\alpha(t-\tau)}f(\tau)\,d\tau,$$

generating the function ϕ_m given by

$$\phi_m = \sqrt{\alpha+\beta}\,e^{-\alpha t}L_n\big([\alpha+\beta]t\big).$$

Observe that $S_{\alpha\beta}$ is associated with the α-Laguerre shift S_α while $S_{\beta\alpha}$ is associated with the β-Laguerre shift S_β.

Theorem 4.1. *The $(\alpha\beta)$ and $(\beta\alpha)$-Laguerre operators $S_{\alpha\beta}$ and $S_{\beta\alpha}$ consist of a biisometric pair on $L^2[0,\infty)$ having the following properties.*

$$\mathcal{N}(S^*_{\alpha\beta}) = \mathrm{span}\{e^{-\alpha(\cdot)}\} = \mathcal{N}(S^*_\alpha),$$
$$\mathcal{N}(S^*_{\beta\alpha}) = \mathrm{span}\{e^{-\beta(\cdot)}\} = \mathcal{N}(S^*_\beta),$$
$$S^*_{\alpha\beta}S_{\beta\alpha} = I = S^*_{\beta\alpha}S_{\alpha\beta},$$
$$S_{\alpha\beta}S_{\beta\alpha} = S_\alpha S_\beta = S_{\beta\alpha}S_{\alpha\beta},$$

and so $S_{\alpha\beta}S_{\beta\alpha}$ is a shift of multiplicity 2.

Proof. For $t \geq 0$

$$S^*_{\alpha\beta} f = g \quad \text{with} \quad g(t) = f(t) - (\alpha+\beta)\int_t^\infty e^{-\beta(\sigma-t)} f(\tau)\,d\tau.$$

Therefore $f \in \mathcal{N}(S^*_{\alpha\beta})$ if and only if $S^*_{\alpha\beta} f = 0$, which implies, for every $t \geq 0$,

$$f(t) = (\alpha+\beta)\int_t^\infty e^{-\beta(\sigma-t)} f(\tau)\,d\tau.$$

Differentiating both sides we get, for $t \geq 0$,

$$f(t) = \beta f(t) - (\alpha+\beta) f(t) = -\alpha f(t).$$

Solving for f we get

$$f(t) = Ke^{-\alpha t}$$

for $t \geq 0$ and some constant K. The same argument leads to

$$S^*_{\alpha} e^{-\alpha t} = 0$$

for every $t \geq 0$. This proves the first property. Interchanging α and β we get the second one. The next two properties are derived by simple calculations. Finally, since $S_{\alpha\beta}\, S_{\beta\alpha}$ is the convolution of two commutable shifts of multiplicities 1, viz., $S_\alpha S_\beta$, it is therefore a shift of multiplicity 2. □

The final result follows from Corollary 3.1, Lemmas 4.1, 4.2, and Theorem 4.1.

Corollary 4.1. *Tor each $n \geq 0$ consider the functions ϕ_n and ψ_m in $L^2[0,\infty)$ as follows.*

$$\phi_n(t) = \sqrt{\alpha+\beta}\; e^{-\beta t} L_n\big((\alpha+\beta)t\big) = [S_{\alpha\beta}]^n \sqrt{\alpha+\beta}\, e^{-\beta t},$$
$$\psi_m(t) = \sqrt{\alpha+\beta}\, e^{-\alpha t} L_m\big((\alpha+\beta)t\big) = [S_{\beta\alpha}]^n \sqrt{\alpha+\beta}\, e^{-\alpha t}.$$

The sequences $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal and both span $L^2[0,\infty)$. Moreover, the biisometric operators $S_{\alpha\beta}$ and $S_{\beta\alpha}$ shift the biorthogonal total sequences $\{\phi_n\}$ and $\{\psi_n\}$. That is, for each $n \geq 0$

$$S_{\alpha\beta}\phi_n = \phi_{n+1},$$

and for every $f \in L^2[0,\infty)$

$$f = \sum_{k=0}^{\infty} \langle f\,;\psi_k\rangle \phi_k \quad \textit{and so} \quad S_{\alpha\beta} f = \sum_{k=0}^{\infty} \langle f\,;\psi_k\rangle \phi_{k+1}.$$

Similarly, for each $n \geq 0$

$$S_{\beta\alpha}\psi_k = \psi_{k+1},$$

and for every $f \in \mathcal{L}^2[0,\infty)$,

$$f = \sum_{k=0}^{\infty} \langle f\,;\phi_k\rangle \psi_k \quad \textit{and so} \quad S_{\beta\alpha} f = \sum_{k=0}^{\infty} \langle f\,;\phi_k\rangle \psi_{k+1}.$$

Proof. Apply Corollary 3.1, Lemmas 4.1, 4.2, and Theorem 4.1. □

Remark 4.1. (a) The Gram–Schmidt orthonormalization procedure can be extended to biorthonormalization in Hilbert space. Indeed, take a pair of sequences $\{f_n\}$ and $\{g_n\}$, and construct the sequences $\{\phi_n\}$ and $\{\psi_n\}$ so that $\langle\phi_m\,;\psi_n\rangle = \delta_{m,n}$ as follows. To begin with set

$$\phi_0 = \frac{f_0}{\langle f_0\,;g_0\rangle^{\frac{1}{2}}} \quad \text{and} \quad \psi_0 = \frac{g_0}{\langle f_0\,;g_0\rangle^{\frac{1}{2}}},$$

so that $\langle\phi_0\,;\psi_0\rangle = 1$. Next set

$$\phi_1 = \frac{p_1}{\langle p_1\,;q_1\rangle^{\frac{1}{2}}} \quad \text{and} \quad \psi_1 := \frac{q_1}{\langle p_1\,;q_1\rangle^{\frac{1}{2}}},$$

where

$$p_1 = f_1 - \langle f_1\,;\psi_0\rangle\phi_0 \quad \text{and} \quad q_1 = g_1 - \langle g_1\,;\phi_0\rangle\psi_0.$$

It is plain that $\langle\phi_1\,;\psi_1\rangle = 1$, $p_1 \perp \psi_0$, and $q_1 \perp \phi_0$. Then

$$r_1\phi_1 = p_1 = f_1 - \langle f_1\,;\psi_0\rangle\phi_0 \quad \text{where} \quad r_1 = \langle p_1, q_1\rangle^{\frac{1}{2}}.$$

Thus $\phi_1 \perp \psi_0$. Similarly,

$$r_1\psi_1 = q_1 = g_1 - \langle g_1\,;\phi_0\rangle\psi_0,$$

and so $\psi_1 \perp \phi_0$. In general we have

$$r_n\phi_n = p_n = f_n - \sum_{k=0}^{n-1}\langle f_n\,;\psi_k\rangle\phi_k \perp \psi_0,\dots,\psi_{n-1} \quad \text{where} \quad r_n = \langle p_n\,;q_n\rangle^{\frac{1}{2}}.$$

Similarly,

$$r_n\psi_n = q_n = g_n - \sum_{k=0}^{n-1}\langle g_n\,;\phi_k\rangle\psi_k \perp \phi_0,\dots,\phi_{n-1}.$$

If $f_n \perp h$, then $\phi_n \perp h$. Therefore, if $\{f_n\}$ is total (i.e., complete), then so is $\{\phi_n\}$. Similarly, if $\{g_n\}$ is total (i.e., complete), then so is $\{\psi_n\}$.

(b) It is also worth noticing on the following points.

(i) If $f_n = g_n$, then the Gram–Schmidt biorthonormalization becomes the usual Gram-Schmidt orthonormalization.

(ii) If $f_n(t) = e^{-\alpha t}[(\alpha+\beta)t]^n$ and $g_n(t) = e^{-\beta t}[(\alpha+\beta)t]^n$, then we get $\phi_n(t) = e^{\alpha t}L_n(\alpha+\beta)t$ and $\psi_n(t) = e^{\beta t}L_n(\alpha+\beta)t$. Moreover $\{f_n\}$ and $\{g_n\}$ are total (i.e., complete), and ϕ_n and ψ_n admit the biisometric description as well. Also $\{\phi_n\}$ and $\{\psi_n\}$ are in this case independently total (i.e., complete).

5. Conclusion and remarks

We have seen in Section 4 that the α-Laguerre shift S_α satisfied for each $n \geq 0$ the following properly.

$$S_\alpha[\sqrt{2\alpha}\,e^{-\alpha t}\,L_n(2\alpha t)] = \sqrt{2\alpha}\,e^{-\alpha t}L_{n+1}(2\alpha t).$$

Moreover,

$$[S_\alpha^*]\,e^{-\alpha t} = 0.$$

The same type of results can be obtained for $S_{\alpha\beta}$. Indeed, in the space H^2,

$$[H_{\alpha\beta}]^n \mathcal{L}_s\big[e^{-\alpha t}\big] = \Big[\tfrac{s-\alpha}{s+\beta}\Big]^n \tfrac{1}{s+\alpha},$$

and hence

$$[H_{\alpha\beta}]^{n+1} \tfrac{1}{s+\alpha} = \Big[\tfrac{s-\alpha}{s+\alpha}\Big] \Big[\tfrac{s-\alpha}{s+\beta}\Big]^n \tfrac{1}{s+\beta} = H_\alpha [H_{\alpha\beta}]^n \tfrac{1}{s+\beta}$$

for each $n \geq 0$, where $H_\alpha = \frac{s-\alpha}{s+\alpha}$. Therefore for each $n \geq 0$

$$[S_{\alpha\beta}]^n\, e^{-\alpha t} = S_\alpha\, [e^{-\beta t} L_n([\alpha+\beta]t)].$$

Interchanging α and β we get

$$[S_{\beta\alpha}]^n\, e^{-\beta t} = S_\beta\big[e^{-\alpha t} L_n\big((\alpha+\beta)t\big)\big].$$

These functions, however, are neither orthogonal nor biorthogonal. The α-Laguerre functions and α-Laguerre shift S_α have been widely applied in systems analysis and identification, see for instance [7] and the references therein. Applications of biorthogonal Laguerre functions and biisometric Laguerre shifts will be reported elsewhere.

Finally, the biisometric operators $S_{\alpha\beta}$ and $S_{\beta\alpha}$ discussed above can be regarded as “Laguerre-like” shifts of multiplicity 1. A class of Laguerre shifts and Laguerre shift semigroups of finite multiplicities have been developed in [5]. We expect that one can also construct “Laguerre-like” shifts of finite multiplicities.

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