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ON CLIQUES AND LAGRANGIANS OF HYPERGRAPHS

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Abstract. Given a graph G, the Motzkin and Straus formulation of the maximum clique problem is the quadratic program (QP) formed from the adjacent matrix of the graph G over the standard simplex. It is well-known that the global optimum value of this QP (called Lagrangian) corresponds to the clique number of a graph. It is useful in practice if similar results hold for hypergraphs. In this paper, we attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques when the number of edges is in a certain range. Specifically, we obtain upper bounds for the Lagrangian of a hypergraph when the number of edges is in a certain range. These results further support a conjecture introduced by Y. Peng and C. Zhao (2012) and extend a result of J. Talbot (2002). We also establish an upper bound of the clique number in terms of Lagrangians for hypergraphs.

1. Introduction

Given a graph G, the Motzkin and Straus formulation of the maximum clique problem is the quadratic program (QP) formed from the adjacent matrix of the graph G over the standard simplex. It is well-known that the global optimum value of this QP (called Lagrangian) has applications in both combinatorics and optimization. Motzkin and Straus' result basically says that the Lagrangian of a graph corresponds to the clique number of this graph (the precise statement is given in Theorem 2.1). This result provides a solution to the optimization problem for a class of homogeneous quadratic multilinear functions over the standard simplex of an Euclidean plane. The Motzkin-Straus result and its extension were successfully employed in maximum clique problem (see [1–3, 7]). It has been also generalized to vertex-weighted graphs [7] and edge-weighted graphs with applications to pattern recognition in image analysis (see [1–3, 7, 12–14, 20]).

For hypergraphs, the Lagrangian and its variants have been a useful tool in hypergraph extremal problems, hypergraph clustering, and social networks. For example, Frankl-Füredi [5] used Lagrangian of hypergraphs in finding

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Turán densities of hypergraphs. Frankl and Rödl [6] used it in disproving Erdős long standing jumping constant conjecture. The variants of Lagrangian of hypergraphs are used to detect communities in social networks in [4, 27] and hypergraph clustering [9, 19]. In most applications, we need an upper bound for the Lagrangian of a hypergraph. The obvious generalization of Motzkin and Straus' result to hypergraphs is false. In fact, there are many examples of hypergraphs that do not achieve their Lagrangians on any proper subhypergraph. An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [22]. In [17] and [18] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques other than Lagrangian of hypergraphs.

In this paper, we attempt to explore the relationship between the Lagrangian of a hypergraph and the order of its maximum cliques when the number of edges is in a certain range though the obvious generalization of Motzkin and Straus' result to hypergraphs is false. Specifically, we obtain upper bounds for Lagrangian of a hypergraph when the number of edges is in a certain range. These results further provide substantial evidence for two conjectures in [16] and extend some known results in the literature ([16] and [23]). We also establish an upper bound for the clique number in terms of Lagrangians for hypergraphs. The presented results establish connections between a continuous optimization problem and the maximum clique problem of hypergraphs. Since practical problems such as social networks [4, 27] and clustering [9, 19] are related to the maximum clique problems, this type of results opens a door to such practical applications.

2. Definitions and main result

An r-uniform hypergraph (r-graph) consists of a set of vertices $V(G)$ and a set $E(G)$ of r-subsets of V, called edges. When $V(G)$ is not defined explicitly, it is assumed that $V(G) = [n] = \{1, 2, \ldots, n\}$. An edge $\{a_1, a_2, \ldots, a_r\}$ in G will be simply denoted by $a_1a_2 \cdots a_r$. Let $K_t^{(r)}$ denote the complete r-graph on t vertices, that is the r-graph on t vertices containing all possible edges. A complete r-graph on t vertices is also called a clique with order t. A clique is said to be maximum if it has maximum cardinality. The clique number of an r-graph G is defined as the cardinality of the maximum clique of G. Let $[t]^{(r)}$ represent the complete r-graph on the vertex set $[t]$.

Definition 1. For an r-graph $G = ([n], E(G))$ and a vector $\vec{x} = (x_1, \ldots, x_n) \in R^n$, define polynomial form $P_G(\vec{x}) : R^n \to R^1$ as

$$P_G(\vec{x}) := \sum_{i_1 i_2 \cdots i_r \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Let $S := \{\vec{x} = (x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i = 1, x_i \geq 0 \text{ for } i = 1, 2, \ldots, n\}$. The Lagrangian of G, denoted by $\lambda(G)$, is the maximum of the above homogeneous function over the standard simplex S.

$$\lambda(G) := \max\{P_G(\vec{x}) : \vec{x} \in S\}.$$

The value x_i is called the weight of the vertex i. A vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ is called a feasible weighting for G if $\vec{x} \in S$. A vector $\vec{y} \in S$ is called an optimal weighting for G if $\lambda(G, \vec{y}) = \lambda(G)$. The following fact is easily implied by the definition of the Lagrangian.

Fact 1. Let G_1, G_2 be r-uniform graphs and $G_1 \subseteq G_2$. Then $\lambda(G_1) \leq \lambda(G_2)$.

The maximum clique problem is a classical problem in combinatorial optimization which has important applications in various domains. In [10], Motzkin and Straus established a remarkable connection between the clique number and the Lagrangian of a graph.

Theorem 2.1 ([10, Theorem 1])**.** *If G is a 2-graph in which a largest clique has order t, then $\lambda(G) = \lambda\left(K_t^{(2)}\right) = \frac{1}{2}\left(1 - \frac{1}{t}\right)$.*

Motzkin-Straus theorem has been proved to be a useful tool in various domains such as maximum clique problem (see [1–3, 7]). It has been also generalized to vertex-weighted graphs [7] and edge-weighted graphs with applications to clustering and pattern recognition in image analysis (see [1–3, 7, 12–14, 20]). Lagrangians of hypergraphs have been proved to be a useful tool in hypergraph extremal problems, clustering and social networks. For example, it has been used in finding Turán densities of hypergraphs in [5, 11, 21]. However, the obvious generalization of Motzkin and Straus' result to hypergraphs is false. In fact, there are many examples of hypergraphs that do not achieve their Lagrangians on any proper subhypergraph. An attempt to generalize the Motzkin-Straus theorem to hypergraphs is due to Sós and Straus [22]. Recently, in [17] and [18] Rota Buló and Pelillo generalized the Motzkin and Straus' result to r-graphs in some way using a continuous characterization of maximal cliques other than Lagrangians of hypergraphs. We attempt to explore the connection between the Lagrangian of a hypergraph and the order of its maximum cliques for hypergraphs when the number of edges is in a certain range though the obvious generalization of Motzkin and Straus' result to hypergraphs is false. In [16], the following two conjectures are proposed.

Conjecture 1 ([16, Conjecture 1.3])**.** *Let m and t be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let G be an r-graph with m edges and contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(r)})$.*

Conjecture 2 ([16, Conjecture 1.4])**.** *Let m and t be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1}$. Let G be an r-graph with m edges and contain no clique of order $t-1$. Then $\lambda(G) < \lambda([t-1]^{(r)})$.*

In [16], Peng and Zhao proved that Conjecture 1 holds for $r = 3$.

Theorem 2.2 ([16, Theorem 1.8]). *Let m and t be positive integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2}$. Let G be a 3-graph with m edges and G contain a clique of order $t-1$. Then $\lambda(G) = \lambda([t-1]^{(3)})$.*

Lagrangians of hypergraphs and its variants have been proved to be a useful tool in various domains such as hypergraph extremal problems [5, 6, 8, 11, 21], hypergraph clustering [9, 19] and social networks [4, 27]. In most applications, an upper bound is needed. Frankl and Füredi [5] asked the following question. Given $r \ge 3$ and $m \in \mathbb{N}$ how large can the Lagrangian of an r-graph with m edges be?

For distinct $A, B \in \mathbb{N}^{(r)}$ we say that A is less than B in the *colex ordering* if $\max(A \triangle B) \in B$, where $A \triangle B = (A \setminus B) \cup (B \setminus A)$. For example, the first $\binom{t}{r}$ r-tuples in the colex ordering of $\mathbb{N}^{(r)}$ are the edges of $[t]^{(r)}$. The following conjecture of Frankl and Füredi (if it is true) proposes a solution to the question mentioned above.

Conjecture 3 ([5, Conjecture 4.1]). *The r-graph with m edges formed by taking the first m sets in the colex ordering of $\mathbb{N}^{(r)}$ has the largest Lagrangian of all r-graphs with m edges. In particular, the r-graph with $\binom{t}{r}$ edges and the largest Lagrangian is $[t]^{(r)}$.*

This conjecture is true when $r = 2$ by Theorem 2.1. Talbot [23] has proved that for positive integers m, t and r satisfying $\binom{t-1}{r} \le m \le \binom{t-1}{r} + \binom{t-2}{r-1}$, then $\lambda(C_{r,m}) = \lambda([t-1]^{(r)})$. So if Conjectures 1 and 2 are true, then Conjecture 3 is true for this range of m.

There are also some partial results for Conjecture 3 for $r = 3$. In [23], Talbot proved the following:

Theorem 2.3 ([23, Theorem 2.1]). *Let t, m and r be positive integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2} - (t-1)$. Let G be a 3-graph with m edges. Then $\lambda(G) \le \lambda([t-1]^{(3)})$.*

In [25], Tang et al. proved the following:

Theorem 2.4 ([25, Theorem 4]). *Let t, m and r be positive integers satisfying $\binom{t-1}{3} \le m \le \binom{t-1}{3} + \binom{t-2}{2} - \frac{t-2}{2}$. Let G be a 3-graph with m edges and without containing a clique of order $t-1$. Then $\lambda(G) < \lambda([t-1]^{(3)})$.*

However for general r, this conjecture is every challenging in extremal combinatorics and very few results on this conjecture are known. The following asymptotic result proved by Talbot is the evidence for Conjecture 3 for r-graphs on exactly t vertices.

Theorem 2.5 ([23, Theorem 3.1]). *For any $r \ge 4$ there exist constants γ_r and $\kappa_0(r)$ such that if m satisfies*

$$\binom{t-1}{r} \le m \le \binom{t-1}{r} + \binom{t-2}{r-1} - \gamma_r (t-1)^{r-2},$$

with $t \geq \kappa_0(r)$, let G be an r-graph on t vertices with m edges, then $\lambda(G) \leq \lambda([t-1]^{(r)})$.

Very recently, Tyomkyn obtained further asymptotic result for Conjecture 3 for r-graphs in [26]. In [15], the following result is obtained for r-graphs.

Theorem 2.6 ([15, Theorem 1.10])**.** *Let t, m and r be positive integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - (2^{r-3}-1)\left(\binom{t-2}{r-2}-1\right)$. Let G be an r-graph with t vertices and m edges and contain a clique of order $t-1$. Then $\lambda(G) = \lambda\left([t-1]^{(r)}\right)$.*

The main result in this paper is Theorem 2.7 which is an accompany result of Theorem 2.6.

Theorem 2.7. *Let m, t, and $r \geq 4$ be integers satisfying*

$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - [(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1]\left(\binom{t-2}{r-2} - 1\right).$$

Let G be an r-graph on t vertices with m edges and without containing a clique of order $t-1$. Then $\lambda(G) < \lambda\left([t-1]^{(r)}\right)$.

Theorem 2.7 supports Conjecture 2. Combing Theorems 2.6 and 2.7, we have the following result immediately.

Theorem 2.8. *Let m, t, and $r \geq 4$ be integers satisfying*

$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - [(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1]\left(\binom{t-2}{r-2} - 1\right).$$

Let G be an r-graph with t vertices and m edges. Then $\lambda(G) \leq \lambda\left([t-1]^{(r)}\right)$.

Theorem 2.8 provides further evidence for Conjecture 3. The contribution of Theorem 2.8 is that the method developed in the proof of Theorem 2.7 is simpler and different from that in Theorem 2.5 in some ways. The upper bound in Theorem 2.8 for the number of edges m is more explicit and an improvement comparing to the bound in Theorem 2.5. We remark that, in the proof of Theorem 2.5, we see that $\gamma_r = 2^{2^r}$ and $t \geq \kappa_0(r)$, where $\kappa_0(r)$ is a sufficiently large integer such that $\binom{t-2}{r-1} > \gamma_r(t-1)^{r-2} = 2^{2^r}(t-1)^{r-2}$ for $t \geq \kappa_0(r)$. In Theorem 2.8, we improve the upper bound for m from $\binom{t-1}{r} + \binom{t-2}{r-1} - \gamma_r(t-1)^{r-2}$ to

$$\binom{t-1}{r} + \binom{t-2}{r-1} - \left[(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1\right]\left(\binom{t-2}{r-2} - 1\right).$$

Note that $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \binom{t-2}{r-1} - [(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1]\left(\binom{t-2}{r-2} - 1\right)$ implies t should satisfy $\binom{t-2}{r-1} > [(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1]\left(\binom{t-2}{r-2} - 1\right)$. We also improve the condition on t from $\binom{t-2}{r-1} > 2^{2^r}(t-1)^{r-2}$ to this value.

At this moment, we cannot get rid of the restriction on the vertex number in general. As an attempt, we obtain the following weaker result without the restriction on the vertex number.

Theorem 2.9. *Let m, t, and $r \geq 4$ be integers satisfying*
$$\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \frac{(r-1)^{r-1}}{(r-1)!} \frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}.$$
Let G be an r-graph with m edges containing a clique of order $t-1$. Then $\lambda(G) = \lambda\left([t-1]^{(r)}\right)$.

The proof of Theorem 2.7 will be given in Section 4. The proofs of Theorem 2.9 will be given in Section 5. Further remarks and conclusions are given in Section 6. Next, let us give some useful results.

3. Useful results

For an r-graph $G = (V, E)$, denote the $(r-1)$-neighborhood of a vertex $i \in V$ by $E_i := \{A \in V^{(r-1)} : A \cup \{i\} \in E\}$. Similarly, denote the $(r-2)$-neighborhood of a pair of vertices $i, j \in V$ by $E_{ij} := \{B \in V^{(r-2)} : B \cup \{i,j\} \in E\}$. Denote the complement of E_i by $E_i^c = \{A \in V^{(r-1)} : A \cup \{i\} \in V^{(r)} \backslash E\}$. Also, denote the complement of E_{ij} by $E_{ij}^c := \{B \in V^{(r-2)} : B \cup \{i,j\} \in V^{(r)} \backslash E\}$ and $E_{i \backslash j} := E_i \cap E_j^c$.

We will impose one additional condition on any optimal weighting $\vec{x} = (x_1, x_2, \ldots, x_n)$ for an r-graph G:

$$|\{i : x_i > 0\}| \text{ is minimal, i.e., if } \vec{y} \text{ is a feasible weighting for } G \text{ satisfying}$$
$$|\{i : y_i > 0\}| < |\{i : x_i > 0\}|, \text{ then } \lambda(G, \vec{y}) < \lambda(G). \quad (1)$$

When the theory of Lagrange multipliers is used to find the optimum of $\lambda(G, \vec{x})$, subject to $\sum_{i=1}^{n} x_i = 1$, notice that $\lambda(E_i, \vec{x})$ corresponds to the partial derivative of $\lambda(G, \vec{x})$ with respect to x_i. The following lemma gives some necessary conditions of an optimal weighting for G.

Lemma 3.1 ([6, Theorem 2.1])**.** *Let $G = (V, E)$ be an r-graph on the vertex set $[n]$ and $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for G with k ($\leq n$) non-zero weights x_1, x_2, $\ldots$, x_k satisfying condition* (1)*. Then for every $\{i, j\} \in [k]^{(2)}$,* (a) $\lambda(E_i, \vec{x}) = \lambda(E_j, \vec{x}) = r\lambda(G)$, (b) *there is an edge in E containing both i and j.*

Definition 2. An r-graph $G = (V, E)$ on the vertex set $[n]$ is left-compressed if $j_1 j_2 \cdots j_r \in E$ implies $i_1 i_2 \cdots i_r \in E$ whenever $i_k \leq j_k, 1 \leq k \leq r$. Equivalently, an r-graph $G = (V, E)$ on the vertex set $[n]$ is left-compressed if $E_{j \backslash i} = \emptyset$ for any $1 \leq i < j \leq n$.

Remark 3.2. (a) In Lemma 3.1, part (a) implies that $x_j \lambda(E_{ij}, \vec{x}) + \lambda(E_{i \backslash j}, \vec{x}) = x_i \lambda(E_{ij}, \vec{x}) + \lambda(E_{j \backslash i}, \vec{x})$. In particular, if G is left-compressed, then
$$(x_i - x_j)\lambda(E_{ij}, \vec{x}) = \lambda(E_{i \backslash j}, \vec{x})$$

for any i, j satisfying $1 \le i < j \le k$ since $E_{j\setminus i} = \emptyset$.

(b) If G is left-compressed, then for any i, j satisfying $1 \le i < j \le k$,

$$x_i - x_j = \frac{\lambda(E_{i\setminus j}, \vec{x})}{\lambda(E_{ij}, \vec{x})} \tag{2}$$

holds. If G is left-compressed and $E_{i\setminus j} = \emptyset$ for i, j satisfying $1 \le i < j \le k$, then $x_i = x_j$.

(c) By (2), if G is left-compressed, then an optimal weighting $\vec{x} = (x_1, x_2, \dots, x_n)$ for G must satisfy $x_1 \ge x_2 \ge \cdots \ge x_n \ge 0$.

Denote $\lambda^r_{(m,t)} = \max\{\lambda(G) : G$ is an r-graph with t vertices and m edges$\}$. The following lemma is proved in [23].

Lemma 3.3 ([23, Lemma 2.3])**.** *There exists a left-compressed r-graph G with t vertices and m edges such that $\lambda(G) = \lambda^r_{(m,t)}$.*

Remark 3.4. Since the only left-compressed r-graph with t vertices and $m = \binom{t}{r}$ edges is $[t]^{(r)}$. Hence by Lemma 3.3 and Fact 1, we have $\lambda^r_{(m,t)} \le \lambda\left([t]^{(r)}\right)$.

4. Proofs of Theorem 2.7

Denote $\lambda^{r-}_{(m,t-1,t)} = \max\{\lambda(G) : G$ is an r-graph with t vertices and m edges not containing a clique of order $t-1\}$. The following lemma implies that we only need to consider left-compressed r-graphs G when we prove Theorem 2.7.

Lemma 4.1. *Let m, t and r be integers satisfying*

$$\begin{aligned}\binom{t-1}{r} &\le m \\ &\le \binom{t-1}{r} + \binom{t-2}{r-1} \\ &\quad - \left[(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1\right]\left(\binom{t-2}{r-2} - 1\right).\end{aligned}$$

There exists a left-compressed r-graph G on vertex set $[t]$ with m edges without containing $[t-1]^{(r)}$ such that $\lambda(G) = \lambda^{r-}_{(m,t-1,t)}$.

The proof of Lemma 4.1 is similar to Lemma 4.1 in [24]. However Lemma 4.1 in [24] cannot be used directly here. For completeness, we give the proof here. In the proof of Lemma 4.1, we need to define some partial order relation. An r-tuple $i_1 i_2 \cdots i_r$ is called a descendant of an r-tuple $j_1 j_2 \cdots j_r$ if $i_s \le j_s$ for each $1 \le s \le r$, and $i_1 + i_2 + \cdots + i_r < j_1 + j_2 + \cdots + j_r$. In this case, the r-tuple $j_1 j_2 \cdots j_r$ is called an ancestor of $i_1 i_2 \cdots i_r$. The r-tuple $i_1 i_2 \cdots i_r$ is called a direct descendant of $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$ and $j_1 + j_2 + \cdots + j_r = i_1 + i_2 + \cdots + i_r + 1$. We say that $i_1 i_2 \cdots i_r$ has lower hierarchy than $j_1 j_2 \cdots j_r$ if $i_1 i_2 \cdots i_r$ is a descendant of $j_1 j_2 \cdots j_r$. This is a partial order on the set of all r-tuples.

Remark 4.2. In view of Fig. 1, an r-graph G is left-compressed if and only if all descendants of an edge of G are edges of G. Equivalently, if an r-tuple is not an edge of G, then none of its ancestors will be an edge of G.

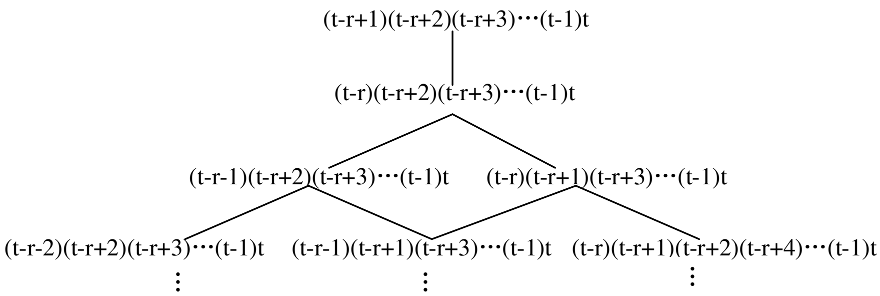

FIGURE 1. Hessian Diagram on $[t]^{(r)}$

Proof of Lemma 4.1. Let G be an r-graph with t vertices and m edges without containing a clique of order $t-1$ such that $\lambda(G) = \lambda^{r-}_{(m,t-1,t)}$. We call G an extremal r-graph for m, $t-1$ and t. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting of G. We can assume that $x_i \geq x_j$ when $i < j$ since otherwise we can just relabel the vertices of G and obtain another extremal r-graph for m, $t-1$ and t with an optimal weighting $\vec{x} = (x_1, x_2, \ldots, x_t)$ satisfying $x_i \geq x_j$ when $i < j$. Next we obtain a new r-graph H from G by performing the following:

(1) If $(t-r)\cdots(t-1) \in E(G)$, then there is at least one r-tuple in $[t-1]^{(r)} \setminus E(G)$, we replacing $(t-r)\cdots(t-1)$ by this r-tuple;
(2) If an edge in G has a descendant other than $(t-r)\cdots(t-1)$ that is not in $E(G)$, then replace this edge by a descendant other than $(t-r)\cdots(t-1)$ with the lowest hierarchy. Repeat this until there is no such an edge.

Then H satisfies the following properties:

(1) The number of edges in H is the same as the number of edges in G.
(2) $\lambda(G) = \lambda(G, \vec{x}) \leq \lambda(H, \vec{x}) \leq \lambda(H)$.
(3) $(t-r)\cdots(t-1) \notin E(H)$.
(4) For any edge in $E(H)$, all its descendants other than $(t-r)\cdots(t-1)$ will be in $E(H)$.

If H is not left-compressed, then there is an ancestor of $(t-r)\cdots(t-1)$, says e, such that $e \in E(H)$. Hence $(t-r)\cdots(t-2)t$ and all the descendants of $(t-r)\cdots(t-2)t$ other than $(t-r)\cdots(t-1)$ will be in $E(H)$. Then

$$\begin{aligned} m &\geq \binom{t-1}{r} - 1 + \binom{t-2}{r-1} \\ &> \binom{t-1}{r} + \binom{t-2}{r-1} \\ &\quad - \left[(2r-6)\times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1\right]\left(\binom{t-2}{r-2} - 1\right) \end{aligned}$$

which is a contradiction. H does not contain $[t-1]^{(r)}$ since H does not contain $(t-r)\cdots(t-1)$. Clearly H is on vertex set $[t]$. So we complete the proof of Lemma 4.1. □

In the rest of this section, we assume that $r \geq 4$ be an integer. In the following three lemmas, Lemma 4.3 implies the maximum weight of G should distribute 'uniform' on the t vertices if $\lambda(G) \geq \lambda\left([t-1]^{(r)}\right)$, and Lemma 4.5 implies G contains most of the first $\binom{t-2r+6}{r}$ edges in colex ordering of $N^{(r)}$ if $\lambda(G) \geq \lambda\left([t-1]^{(r)}\right)$, while Lemma 4.4 implies G also contains most of the first $\binom{t-2r+6}{r-1}$ edges containing $t-1$. Since G is left-compressed, G also contains most of the first $\binom{t-2r+6}{r-1}$ edges containing vertex i, where $t-2r+7 \leq i \leq t-1$. So G contains most edges of $[t-1]^{(r)}$.

Lemma 4.3. (a) *Let G be an r-graph on vertex set $[t]$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. Then $x_1 < x_{t-2r+3} + x_{t-2r+4}$ or*

$$\lambda(G) \leq \frac{1}{r!}\frac{(t-r)^{r-1}\prod_{i=t-r+2}^{t-2} i}{(t-r+1)^{r-2}(t-1)^{r-2}} < \frac{1}{r!}\frac{\prod_{i=t-r}^{t-1} i}{(t-1)^r} = \lambda\left([t-1]^{(r)}\right).$$

(b) *Let G be an r-graph on vertex set $[t]$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. Then $x_1 < 2(x_{t-2r+4} + x_{t-2r+5})$ or*

$$\lambda(G) \leq \frac{1}{r!}\frac{(t-r)^{r-1}\prod_{i=t-r+2}^{t-2} i}{(t-r+1)^{r-2}(t-1)^{r-2}} < \frac{1}{r!}\frac{\prod_{i=t-r}^{t-1} i}{(t-1)^r} = \lambda\left([t-1]^{(r)}\right).$$

Proof. (a) If $x_1 \geq x_{t-2r+3} + x_{t-2r+4}$, then $rx_1 + x_2 + \cdots + x_{t-2r+2} \geq 1$. Recalling that $x_1 \geq x_2 \geq \cdots \geq x_{t-2r+2}$, we have $x_1 \geq \frac{1}{t-r+1}$. Using Lemma 3.1, we have $\lambda(G) = \frac{1}{r}\lambda(E_1, x)$. Note that E_1 is an $(r-1)$-graph with $t-1$ vertices and total weights at most $1 - \frac{1}{t-r+1}$. Hence by Remark 3.4 (replace the total weights 1 with $1 - \frac{1}{t-r+1}$), we have

$$\lambda(G) = \frac{1}{r}\lambda(E_1, x) \leq \frac{1}{r}\binom{t-1}{r-1}\left(\frac{1 - \frac{1}{t-r+1}}{t-1}\right)^{r-1}$$

$$= \frac{1}{r!} \frac{(t-r)^{r-1} \prod_{i=t-r+2}^{t-2} i}{(t-r+1)^{r-2}(t-1)^{r-2}}.$$

Next we prove

$$\frac{1}{r!} \frac{(t-r)^{r-1} \prod_{i=t-r+2}^{t-2} i}{(t-r+1)^{r-2}(t-1)^{r-2}} < \frac{1}{r!} \frac{\prod_{i=t-r}^{t-1} i}{(t-1)^r} = \lambda\left([t-1]^{(r)}\right).$$

To show this, we only need to prove

$$(t-r)^{r-2}(t-1) < (t-r+1)^{r-1}. \tag{3}$$

If $t = r, r+1$, (3) clearly holds. Assuming $t \geq r+2$, we prove this inequality by induction. Now we suppose that (3) holds for some $r \geq 4$, we will show it also holds for $r+1$. Replacing t by $t-1$ in (3), we have

$$[t-(r+1)]^{r-2}(t-2) < (t-r)^{r-1}.$$

Multiplying $t-(r+1)$ to the above inequality, we have

$$[t-(r+1)]^{r-1}(t-2) < (t-r)^{r-1}[t-(r+1)].$$

Adding $[t-(r+1)]^{r-1}$ to the above inequality, we obtain

$$\begin{aligned}[t-(r+1)]^{r-1}(t-1) &< (t-r)^{r-1}[t-(r+1)] + [t-(r+1)]^{r-1} \\ &= (t-r)^r - (t-r)^{r-1} + [t-(r+1)]^{r-1} < (t-r)^r.\end{aligned}$$

Hence (3) also holds for $r+1$ and the induction is complete.

(b) If $x_1 \geq 2(x_{t-2r+5} + x_{t-2r+6})$, then $x_1 + x_2 + \cdots + x_{t-2r+4} + (r-2)\frac{x_1}{2} \geq x_1 + x_2 + \cdots + x_{t-2r+4} + x_{t-3} + x_{t-2r+6} + x_{t-1} + x_t = 1$. Recalling that $x_1 \geq x_2 \geq \cdots \geq x_{t-2r+4}$ and $r \geq 4$, we have $x_1 \geq \frac{1}{t-2r+4+\frac{r-2}{2}} \geq \frac{1}{t-r+1}$. The rest of the proof is identical to that in part (a), we omit the computation details here. □

Lemma 4.4. *Let G be a left-compressed r-graph on the vertex set $[t]$ without containing $[t-1]^{(r)}$. Then $|[t-2r+6]^{(r-1)} \backslash E_{t-1}| \leq 2^{r-1}|E_{(t-1)t}|$ or $\lambda(G) < \lambda([t-1]^{(r)})$.*

Proof. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G. Since G is left-compressed, by Remark 3.2(a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(G) = \lambda(G, \vec{x}) < \lambda([t-1]^{(r)})$ since G does not contain $[t-1]^{(r)}$. So we assume that $x_t > 0$.

Consider a new weighting for G, $\vec{y} = (y_1, y_2, \ldots, y_t)$ given by $y_i = x_i$ for $i \neq t-1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 3.1(a), $\lambda(E_{t-1}, \vec{x}) = \lambda(E_t, \vec{x})$, so

$$\begin{aligned}\lambda(G, \vec{y}) - \lambda(G, \vec{x}) = x_t &\left(\lambda(E_{t-1}, \vec{x}) - x_t \lambda(E_{(t-1)t}, \vec{x})\right) \\ &- x_t\left(\lambda(E_t, \vec{x}) - x_{t-1}\lambda(E_{(t-1)t}, \vec{x})\right) - x_{t-1}x_t\lambda(E_{(t-1)t}, \vec{x}))\end{aligned}$$

$$= x_t\left(\lambda(E_{t-1},\vec{x})-\lambda(E_t,\vec{x})\right)-x_t^2\lambda(E_{(t-1)t},\vec{x}\right)$$

$$= -x_t^2\lambda(E_{(t-1)t},\vec{x}). \tag{4}$$

Assume that $|[t-2r+6]^{(r-1)}\backslash E_{t-1}| > 2^{r-1}|E_{(t-1)t}|$. If $\lambda(G) < \lambda([t-1]^{(r)})$ we are done. Otherwise if $\lambda(G) \geq \lambda\left([t-1]^{(r)}\right)$ we will show that there exists a set of edges $F \subset [t-1]^{(r)} \setminus E$ satisfying

$$\lambda(F,\vec{y}) > x_t^2\lambda(E_{(t-1)t},\vec{x}). \tag{5}$$

Then using (4) and (5), the r-graph $G^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*,\vec{y}) = \lambda(G,\vec{y}) + \lambda(F,\vec{y}) > \lambda(G,\vec{x}) = \lambda(G)$. Since $\vec{y}$ has only $t-1$ positive weights, then $\lambda(G^*,\vec{y}) \leq \lambda\left([t-1]^{(r)}\right)$, and consequently, $\lambda(G) < \lambda\left([t-1]^{(r)}\right)$. This is a contradiction.

We now construct the set of edges F. Let $C = [t-2r+6]^{(r-1)} \setminus E_{t-1}$. Then by the assumption,

$$|C| > 2^{r-1}|E_{(t-1)t}| \text{ and } \lambda(C,\vec{x}) \geq 2^{r-1}|E_{(t-1)t}|x_{t-3r+8}\cdots x_{t-2r+6}.$$

Let F consist of those edges in $[t-1]^{(r)} \setminus E$ containing the vertex $t-1$. Since $\lambda(G) \geq \lambda([t-1]^{(r)})$ then $x_{t-2r+3} > \frac{x_1}{2}$ by Lemma 4.3(a) and $x_{t-2r+4} \geq x_{t-2r+5} > \frac{x_1}{4}$ by Lemma 4.3(b). Hence

$$\begin{aligned}
\lambda(F,\vec{y}) &= (x_{t-1}+x_t)\lambda(C,\vec{x}) \\
&> 2x_t \cdot 2^{r-1}|E_{(t-1)t}|x_{t-3r+8}\cdots x_{t-2r+6} \\
&\geq x_t^2|E_{(t-1)t}|(x_1)^{r-2} \\
&\geq x_t^2 \sum_{i_1\cdots i_{r-2}\in E_{(t-1)t}} x_{i_1}\cdots x_{i_{r-2}} \\
&= x_t^2\lambda(E_{(t-1)t},\vec{x}).
\end{aligned}$$

Hence F satisfies (5). This proves Lemma 4.4. □

Lemma 4.5. *Let G be a left-compressed r-graph on the vertex set $[t]$ without containing $[t-1]^{(r)}$. Then $|[t-2r+6]^{(r)}\backslash E| \leq 2^{r-1}|E_{(t-1)t}|$ or $\lambda(G) < \lambda\left([t-1]^{(r)}\right)$.*

Proof. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G. Since G is left-compressed, by Remark 3.2(a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(G) < \lambda([t-1]^{(r)})$ since G does not contain $[t-1]^{(r)}$. So we assume that $x_t > 0$.

Consider a new weighting for G, $\vec{y} = (y_1, y_2, \ldots, y_t)$ given by $y_i = x_i$ for $i \neq t-1, t$, $y_{t-1} = x_{t-1} + x_t$ and $y_t = 0$. By Lemma 3.1(a), $\lambda(E_{t-1},\vec{x}) = \lambda(E_t,\vec{x})$, similar to (4), we have

$$\lambda(G,\vec{y}) - \lambda(G,\vec{x}) = -x_t^2\lambda(E_{(t-1)t},\vec{x}). \tag{6}$$

Assume that $|[t-2r+6]^{(r)}\backslash E| > 2^{r-1}|E_{(t-1)t}|$. If $\lambda(G) < \lambda([t-1]^{(r)})$ we are done. Otherwise if $\lambda(G) \geq \lambda([t-1]^{(r)})$ we will show that there exists a set of edges $F \subset [t-2r+6]^{(4)} \setminus E$ satisfying

$$\lambda(F, \vec{y}) > x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \tag{7}$$

Then using (6) and (7), the r-graph $G^* = ([t], E^*)$, where $E^* = E \cup F$, satisfies $\lambda(G^*, \vec{y}) = \lambda(G, \vec{y}) + \lambda(F, \vec{y}) > \lambda(G, \vec{x}) = \lambda(G)$. Since $\vec{y}$ has only $t-1$ positive weights, then $\lambda(G^*, \vec{y}) \leq \lambda\left([t-1]^{(r)}\right)$, and consequently, $\lambda(G) < \lambda\left([t-1]^{(r)}\right)$. This is a contradiction.

We now construct the set of edges F. Let $C = [t-2r+6]^{(r)} \setminus E$. Then by the assumption,

$$|C| > 2^{r-1}|E_{(t-1)t}| \text{ and } \lambda(C, \vec{x}) \geq 2^{r-1}|E_{(t-1)t}| x_{t-3r+7} \cdots x_{t-2r+6}.$$

Let $F = C$. Since $\lambda(G) \geq \lambda([t-1]^{(r)})$ then $x_{t-2r+3} \geq \frac{x_1}{2}$ by Lemma 4.3(a) and $x_{t-2r+4} \geq x_{t-2r+5} > \frac{x_1}{4}$ by Lemma 4.3(b). Hence

$$\begin{aligned} \lambda(F, \vec{y}) = \lambda(C, \vec{x}) &> 2^{r-1}|E_{(t-1)t}| x_{t-3r+7} \cdots x_{t-2r+6} \geq x_t^2 |E_{(t-1)t}| (x_1)^{r-2} \\ &\geq x_t^2 \sum_{i_1 \cdots i_{r-2} \in E_{(t-1)t}} x_{i_1} \cdots x_{i_{r-2}} = x_t^2 \lambda(E_{(t-1)t}, \vec{x}). \end{aligned}$$

Hence F satisfies (7). This proves Lemma 4.5. $\square$

Now we are ready to prove Theorem 2.7.

Proof of Theorem 2.7. Let m and t be integers satisfying

$$\begin{aligned} \binom{t-1}{r} \leq m & \\ \leq & \binom{t-1}{r} + \binom{t-2}{r-1} \\ & - \left[(2r-6) \times 2^{r-1} + 2^{r-3} + (r-4)(2r-7) - 1\right] \left(\binom{t-2}{r-2} - 1\right). \end{aligned}$$

Let G be an r-graph with t vertices and m edges without containing a clique of order $t-1$ such that $\lambda(G) = \lambda^{r-}_{(m,t-1,t)}$. Then by Lemma 4.1, we can assume that G is left-compressed and does not contain $[t-1]^{(r)}$. Let $\vec{x} = (x_1, x_2, \ldots, x_t)$ be an optimal weighting for G. Since G is left-compressed, by Remark 3.2(a), $x_1 \geq x_2 \geq \cdots \geq x_t \geq 0$. If $x_t = 0$, then $\lambda(G) < \lambda([t-1]^{(r)})$ since G does not contain $[t-1]^{(r)}$. So we assume that $x_t > 0$.

If $\lambda(G) < \lambda\left([t-1]^{(r)}\right)$ we are done. Otherwise $|[t-2r+6]^{(r-1)} \setminus E_{t-1}| \leq 2^{r-1}|E_{(t-1)t}|$ by Lemma 4.4. Recalling that G is left-compressed, we have $|[t-2r+6]^{(r-1)} \setminus E_i| \leq 2^{r-1}|E_{(t-1)t}|$ for $t-2r+7 \leq i \leq t-1$. We also have $|[t-2r+6]^{(4)} \setminus E| \leq 2^{r-1}|E_{(t-1)t}|$ by Lemma 4.5. Note that $|E_{(t-1)t}| \leq \binom{t-2}{r-2} - 1$, then

$$|[t-1]^{(r)} \bigcap E| \geq |[t-2r+6]^{(r)} \bigcap E| + \sum_{i=t-2r+7}^{t-1} |[t-2r+6]^{(r-1)} \bigcap E_i|$$

$$\begin{aligned}
&\geq \binom{t-2r+6}{r} - 2^{r-1}|E_{(t-1)t}| \\
&\quad + (2r-7)\left(\binom{t-2r+6}{r-1} - (2r-7)\times 2^{r-1}|E_{(t-1)t}|\right) \\
&\geq \binom{t-2r+6}{r} + (2r-7)\binom{t-2r+6}{r-1} \\
&\quad - (2r-6)\times 2^{r-1}\left(\binom{t-2}{r-2} - 1\right).
\end{aligned}$$

Repeated using the equality $\binom{m+1}{n} = \binom{m}{n} + \binom{m}{n-1}$ to the above inequality, we have

$$|[t-1]^{(r)} \bigcap E| \geq \binom{t-1}{r} - \left[(2r-6)\times 2^{r-1} + (r-4)(2r-7)\right]\left(\binom{t-2}{r-2} - 1\right).$$

So

$$0 < |[t-1]^{(r)} \backslash E| \leq \left[(2r-6)\times 2^{r-1} + (r-4)(2r-7)\right]\left(\binom{t-2}{r-2} - 1\right).$$

Since G does not contain $[t-1]^{(r)}$. Let $E^* = E\bigcup[t-1]^{(r)}$ and $G^* = ([t], E^*)$. Denote the number of edges of G^* by m^*, then $\binom{t-1}{r} \leq m^* \leq \binom{t-1}{r} + \binom{t-2}{r-1} - 2^{r-3}(\binom{t-2}{r-2} - 1)$. So $\lambda(G^*) = \lambda\left([t-1]^{(r)}\right)$ by Theorem 2.6. Clearly, $\lambda(G^*, \vec{x}) - \lambda(G, \vec{x}) > 0$ since $x_1 \geq x_2 \geq \cdots \geq x_t > 0$ and $|[t-1]^{(r)} \backslash E| > 0$. Hence $\lambda(G) = \lambda(G, \vec{x}) < \lambda(G^*, \vec{x}) \leq \lambda(G^*) = \lambda\left([t-1]^{(r)}\right)$. This completes the proof of Theorem 2.7. □

5. Proofs of Theorem 2.9

Denote $\lambda^r_{(m,t-1)} = \max\{\lambda(G) : G$ is an r-graph with m edges containing a clique of order $t-1\}$. The following lemma implies that we only need to consider left-compressed r-graphs G when we prove Theorem 2.9.

Lemma 5.1 ([16, Lemma 3.1]). *Let m, t and r be integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t}{r} - 1$. There exists a left-compressed r-graph G with m edges containing $[t-1]^{(r)}$ such that $\lambda(G) = \lambda^r_{(m,t-1)}$.*

Proof of Theorem 2.9. Let m, t and r be integers satisfying $\binom{t-1}{r} \leq m \leq \binom{t-1}{r} + \frac{(r-1)^{r-1}}{(r-1)!}\frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}$. Let G be an r-graph with m edges containing a clique of order $t-1$ such that $\lambda(G) = \lambda^r_{(m,t-1)}$. Then by Lemma 5.1, we can assume that G is left-compressed and contains $[t-1]^{(r)}$. Clearly, $\lambda(G) \geq \lambda([t-1]^{(r)})$ since G contains $[t-1]^{(r)}$. Next we show that $\lambda(G) \leq \lambda([t-1]^{(r)})$. Let $\vec{x} = (x_1, x_2, \ldots, x_n)$ be an optimal weighting for G satisfying $x_1 \geq x_2 \geq \cdots \geq x_k > x_{k+1} = \cdots = x_n = 0$. If $k \leq t-1$, then $\lambda(G) \geq \lambda([t-1]^{(r)})$ since G has only $t-1$ positive weights. So we assume that $k \geq t$. Since $m \leq \binom{t-1}{r} + \frac{(r-1)^{r-1}}{(r-1)!}\frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}$ and G contains $[t-1]^r$, we have $|E_k| \leq \frac{(r-1)^{r-1}}{(r-1)!}\frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}$. Then by Lemma 3.1(a) and the Arithmetic Mean-Geometric Mean inequality (AM-GM inequality), we have

$$\begin{aligned}\lambda(G) &= \frac{1}{r}\lambda(E_k, x) \leq \frac{1}{r}\frac{(r-1)^{r-1}}{(r-1)!}\frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}x_1x_2\cdots x_{r-1}\\ &\leq \frac{1}{r}\frac{(r-1)^{r-1}}{(r-1)!}\frac{(t-2)\cdots(t-r)}{(t-1)^{r-1}}\left(\frac{x_1+x_2+\cdots+x_{r-1}}{r-1}\right)^{r-1}\\ &\quad \text{(By AM-GM inequality)}\\ &\leq \frac{1}{r!}\frac{(t-1)\cdots(t-r)}{(t-1)^r} = \lambda\left([t-1]^{(r)}\right).\end{aligned}$$

This completes the proof of Theorem 2.9. □

6. Remarks and conclusions

The method developed in the proof of Theorem 2.7 can also be used to deal with the case for $r = 3$ (see [25]). The upper bound for m in Theorem 2.7 and Theorem 2.8 are not the best possible. Another question in the future study is how to prove similar results as Theorem 2.7 and Theorem 2.8 without restriction of vertex number in general.

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