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APPROXIMATION PROPERTIES OF PAIRS OF SUBSPACES

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Abstract. This study is concerned with the approximation properties of pairs. For $\lambda \geq 1$, we prove that given a Banach space X and a closed subspace Z_0, if the pair (X, Z_0) has the λ-bounded approximation property (λ-BAP), then for every ideal Z containing Z_0, the pair (Z, Z_0) has the λ-BAP; further, if Z is a closed subspace of X and the pair (X, Z) has the λ-BAP, then for every separable subspace Y_0 of X, there exists a separable closed subspace Y containing Y_0 such that the pair $(Y, Y \cap Z)$ has the λ-BAP. We also prove that if Z is a separable closed subspace of X, then the pair (X, Z) has the λ-BAP if and only if for every separable subspace Y_0 of X, there exists a separable closed subspace Y containing $Y_0 \cup Z$ such that the pair (Y, Z) has the λ-BAP.

1. Introduction

A Banach space X is said to have the *approximation property* (AP) if for every compact subset K of X and every $\varepsilon > 0$, there exists a finite rank operator S on X such that $\sup_{x \in K} \|Sx - x\| \leq \varepsilon$; briefly, $id_X \in \overline{\mathcal{F}(X)}^{\tau_c}$, where id_X is the identity operator on X, $\mathcal{F}(X)$ is the space of all finite rank operators on X, and τ_c is the topology of uniformly compact convergence on the space $\mathcal{L}(X)$ of all operators on X. For $\lambda \geq 1$, if $id_X \in \overline{\{S \in \mathcal{F}(X) : \|S\| \leq \lambda\}}^{\tau_c}$, then it can be said that X has the λ-*bounded approximation property* (λ-BAP). Lissitsin and Oja considered the AP of pairs [9]. For a closed subspace Y of a Banach space X, the pair (X, Y) is said to have the AP if for every compact subset K of X and $\varepsilon > 0$, there exists $S \in \mathcal{F}(X)$ such that $\|Sx - x\| < \varepsilon$ for all $x \in K$ and $S(Y) \subseteq Y$. Then, Figiel, Johnson and Pełczyński [2] introduced and studied the BAP of pairs. For $\lambda \geq 1$ and a closed subspace Y of a Banach space X, the pair (X, Y) is said to have the λ-BAP if for every finite dimensional subspace E of X and for every $\varepsilon > 0$, there exists $S \in \mathcal{F}(X)$ with $\|S\| \leq \lambda + \varepsilon$ such that $Sx = x$ for all $x \in E$ and $S(Y) \subseteq Y$.

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For $\mu \geq 1$, a closed subspace Z of a Banach space X is called *locally μ-complemented* in X if for every finite-dimensional subspace E of X and every $\varepsilon > 0$, there exists an operator $T : E \to Z$ with $\|T\| \leq \mu + \varepsilon$ such that $Tx = x$ for all $x \in E \cap Z$. It is well known that the AP and the BAP are inherited by locally complemented subspaces (cf. [3, Theorem 2.4]). For a closed subspace Z of a Banach space X, an operator $\Phi : Z^* \to X^*$ is called an *extension operator* if $(\Phi z^*)(z) = z^*(z)$ for every $z^* \in Z^*$ and $z \in Z$. A closed subspace Z of X is called an *ideal* if there exists an extension operator Φ from Z^* to X^* with $\|\Phi\| = 1$. The operator Φ is called a *Hahn-Banach extension operator*. It is well known that Z is an ideal in X if and only if Z is locally 1-complemented in X, and X has the λ-BAP (resp. AP) if and only if every separable ideal in X has the λ-BAP (resp. AP) (cf. [7, Proposition 4.3 and Theorem 2.2] and [3, Theorem 2.4]). In this work, we have:

Theorem 1.1. *Let Z_0 be a closed subspace of a Banach space X and $\lambda, \mu \geqslant 1$. If the pair (X, Z_0) has the λ-BAP (resp. AP) and Z is a locally μ-complemented subspace of X containing Z_0, then the pair (Z, Z_0) has the $\mu\lambda$-BAP (resp. AP). Hence, if the pair (X, Z_0) has the λ-BAP (resp. AP), then for every ideal Z containing Z_0, the pair (Z, Z_0) has the λ-BAP (resp. AP).*

Johnson proved that a Banach space X has the λ-BAP if and only if every separable subspace of X is contained in a separable subspace with the λ-BAP [4, Lemma 3]. In this work, we have:

Theorem 1.2. *Let Z be a closed subspace of a Banach space X and $\lambda \geqslant 1$. If the pair (X, Z) has the λ-BAP, then for every separable subspace Y_0 of X, there exists a separable closed subspace Y containing Y_0 such that the pair $(Y, Y \cap Z)$ has the λ-BAP.*

Theorem 1.3. *Let Z be a separable closed subspace of a Banach space X and $\lambda \geqslant 1$. The pair (X, Z) has the λ-BAP if and only if for every separable subspace Y_0 of X, there exists a separable closed subspace Y containing Y_0 and Z such that the pair (Y, Z) has the λ-BAP.*

2. Proof of Theorem 1.1

Lemma 2.1 ([5, Lemma 3.2])**.** *Let X be a Banach space and $\mu \geqslant 1$ and Z be a locally μ-complemented subspace of X. Then for every closed subspace Y of X containing Z with $\dim Y/Z < \infty$, there is a projection $P : Y \to Z$ with $\|P\| \leq 2\mu + 4$.*

Proof of Theorem 1.1. Suppose that the pair (X, Z_0) has the λ-BAP. Let F be a finite dimensional subspace of Z and $\varepsilon > 0$. By the assumption, there exists $S_0 \in \mathcal{F}(X)$ with $\|S_0\| \leq \lambda + \varepsilon$ such that $S_0 x = x$ for all $x \in F$ and $S_0(Z_0) \subseteq Z_0$. Let $E = S_0(X)$. Because E is a finite dimensional subspace of X and Z is an ideal in X, there exists an operator $T_E : E \to Z$ with $\|T_E\| \leq \mu + \varepsilon$ such that

$T_E x = x$ for all $x \in E \cap Z$. Let $S := T_E S_0$. Then we have

$$\|S\| \le \|T_E\| \|S_0\| \le (\mu + \varepsilon)(\lambda + \varepsilon).$$

Since $S_0(x) = x$ for all $x \in F$, we have $F \subseteq E \cap Z$; hence, we obtain that

$$S(x) = T_E S_0(x) = T_E(x) = x$$

for all $x \in F$. Finally, since $S_0(Z_0) \subseteq E$ and $S_0(Z_0) \subseteq Z_0 \subseteq Z$, we have $S_0(Z_0) \subseteq E \cap Z$, it follows that

$$S(Z_0) = T_E S_0(Z_0) \subseteq Z_0.$$

Suppose that the pair (X, Z_0) has the AP. Let K be a compact set of Z and $\varepsilon > 0$. Then, there exists $S_0 \in \mathcal{F}(X)$ such that

$$\sup_{z \in K} \|S_0 z - z\| \le \frac{\varepsilon}{2\mu + 4}$$

and $S_0(Z_0) \subseteq Z_0$. Let $Z_1 = Z + S_0(X)$. Clearly, we have $\dim Z_1/Z < \infty$. Since Z is locally μ-complemented in X, by Lemma 2.1, there is a projection $P_{Z_1} : Z_1 \to Z$ with $\|P_{Z_1}\| \le 2\mu + 4$. Define $S = P_{Z_1} S_0|_Z$. Clearly, we have $S \in \mathcal{F}(Z)$. Then, for every $z \in K$

$$\|Sz - z\| = \|P_{Z_1} S_0 z - P_{Z_1} z\| = \|P_{Z_1}\| \|S_0 z - z\| \le \varepsilon.$$

Clearly, we have $S(Z_0) \subseteq Z_0$. □

3. Proofs of Theorem 1.2 and Theorem 1.3

We need the following lemma to show Theorem 1.2.

Lemma 3.1 ([10, Theorem 4.1]). *Let $\lambda \ge 1$ and Z be a closed subspace of a Banach space X. Then the pair (X, Z) has the λ-BAP if and only if there exists a net (S_α) in $\mathcal{F}(X)$ such that $S_\alpha \to id_X$ strongly, $\sup_\alpha \|S_\alpha\| \le \lambda$, and $S_\alpha(Z) \subseteq Z$ for all α.*

Proof of Theorem 1.2. Let $\lambda \ge 1$ and Z be a closed subspace of X. Suppose that the pair (X, Z) has the λ-BAP. By Lemma 3.1, there exists a net (S_α) in $\mathcal{F}(X)$ such that $S_\alpha \to id_X$ strongly, $\sup_\alpha \|S_\alpha\| \le \lambda$, and $S_\alpha(Z) \subseteq Z$ for all α. Assume that $Y_0 = \overline{\{y_1, y_2, \ldots\}}$. Since we now have the extra assumption that $S_\alpha(Z) \subseteq Z$ for all α, by a verbatim proof to [11, Theorem 3.1] (we only change that the projections P's chosen by [11, Theorem 3.1] satisfy $P(Z) \subseteq Z$), there exists a sequence $(T_n) \subset \mathcal{F}(X)$ with the following properties:

(a) $T_n y_m = y_m$ for all $n \ge m$,
(b) $T_n T_m = T_m$ for all $n > m$,
(c) $\limsup_n \|T_n\| \leqslant \lambda$,
(d) $T_n(Z) \subseteq Z$.

Denote

$$Y = \{y \in X : y = \lim_{n \to \infty} T_n y\}.$$

Then, for all m, we have $y_m \in Y$, so Y is a non-empty set. Now, take any $y^{'}, y^{''} \in Y$ and $\alpha \in \mathbf{R}$. Then, we have

$$\lim_{n\to\infty} T_n(y^{'} + y^{''}) = \lim_{n\to\infty} T_n(y^{'}) + \lim_{n\to\infty} T_n(y^{''}) = y^{'} + y^{''},$$

$$\lim_{n\to\infty} T_n(\alpha y^{'}) = \alpha \lim_{n\to\infty} T_n(y^{'}) = \alpha y^{'},$$

hence, Y is a subspace of X. By (a) and (b), we have $Y_0 \subset Y$ and

$$Y = \overline{\text{span}}(\bigcup_{n=1}^{\infty} T_n(X));$$

therefore, Y is a separable subspace of X. By (d), we have

$$T_n(Z \cap Y) \subseteq T_n(Z) \subseteq Z \cap Y.$$

Additionally, by definition, we obtain $T_n|_Y \to id_Y$, strongly. By Lemma 3.1, the pair $(Y, Z \cap Y)$ has the λ-BAP. □

Proof of Theorem 1.3. Sufficiency. Let Y_0 be a separable subspace of X. By [12, Theorem], there exists a separable ideal Y of X containing Y_0 and Z_0. By Theorem 1.1, the pair (Y, Z_0) has the λ-BAP.

Necessity. Let F be a finite-dimensional subspace of X and $\varepsilon > 0$. By [10, Theorem 4.1], it is sufficient to show that there exists a finite rank operator $T \in \mathcal{F}(X)$ such that $\|T\| \leq \lambda$ and

$$\|Tx - x\| < \varepsilon\|x\|$$

for all $x \in F$ and $T(Z) \subseteq Z$. Let $(x_n)_{n=1}^{\infty}$ be a dense sequence of Z. By the argument of [12, Theorem], there exist a separable subspace Y_0 of X containing $F \cup Z$ and a directed set I consisting of finite subsets of X such that for each $G \in I$ with $F \cup \{x_1, x_2, \ldots, x_n\} \subseteq G$, there exists a finite rank operator $T_G : G \to Y_0$ with $T_G|_{F\cup\{x_1,x_2,\ldots,x_n\}} = id_{F\cup\{x_1,x_2,\ldots,x_n\}}$. Here, the net (T_G) satisfies $\limsup_G \|T_G\| \leqslant 1$.

Now, by the assumption, there exists a separable subspace Y of X with $Y_0 \subset Y$ such that (Y, Z) has the λ-BAP. Then there exists $S \in \mathcal{F}(Y)$ such that $\|S\| \leq \lambda$ and $\|Sx - x\| < \varepsilon\|x\|$ for all $x \in F$ and $S(Z) \subseteq Z$. Consider $S_G : X \to S(Y)$ defined by

$$S_G x = ST_G x \ \text{ if } x \in G, \ S_G = 0 \text{ otherwise}.$$

Consider the net $(S_G)_{G\in I}$. Then, by a compactness argument, there exists a subnet (S_H) of (S_G) such that $\widetilde{S}x := \lim_H S_H x$ exists for each $x \in X$. We can see that $\widetilde{S}$ is linear and of finite rank with $\|\widetilde{S}\| \leqslant \lambda$. Further, for every $x \in F$, we have

$$\|\widetilde{S}x - x\| = \lim_H \|S_H x - x\| = \lim_H \|ST_H x - x\| = \|Sx - x\| \leq \varepsilon\|x\|.$$

Moreover, we can easily see that $\{\widetilde{S}(x_n) : n \in \mathbf{N}\} \subseteq Z$. Indeed, take any x_n and $G \in I$ with $x_n \in G$. Then, we have

$$S_G(x_n) = ST_G(x_n) = S(x_n) \in S(Z) \subseteq Z;$$

therefore, $\widetilde{S}(x_n) = \lim_H S_H(x_n) \in Z$. Since $(x_n)_{n=1}^{\infty}$ is dense in Z and Z is closed, we have $\widetilde{S}(Z) \subseteq Z$. □

4. Open problems

Lima and Oja [8] introduced a weaker notion of the BAP. A Banach space X is said to have the *weak* BAP if there exists a $\lambda \geq 1$ such that for every Banach space Y and every $T \in \mathcal{W}(X,Y)$, $id_X \in \overline{\{S \in \mathcal{F}(X,X) : \|TS\| \leq \lambda\|T\|\}}^{\tau_c}$ where $\mathcal{W}$ is the space of all weakly compact operators between Banach spaces. Recently, Chen, Kim and Zheng [1] introduced a weaker property of the BAP of pairs. Let Y be a closed subspace of a Banach space X. Let $\lambda \geq 1$. The pair (X,Y) is said to have the weak λ-BAP if for every Banach space Z and every $R \in \mathcal{W}(X,Z)$, for every finite dimensional subspace E of X and for every $\varepsilon > 0$ there exists $S \in \mathcal{F}(X)$ with $\|RS\| \leq (\lambda+\varepsilon)\|R\|$ such that $Sx = x$ for all $x \in E$ and $S(Y) \subseteq Y$. Kim [6, Theorem 4.1] proved that X has the weak λ-BAP if and only if every ideal in X has the weak λ-BAP. One may naturally ask:

Problem 1. Let X be a Banach space and Y be a closed subspace of X, and let $\lambda \geq 1$. If the pair (X,Y) has the weak λ-BAP, then for every ideal Z in X containing Y, does the pair (Z,Y) have the weak λ-BAP?

By [12, Theorem] and the proof of [4, Lemma 3], it follows that a Banach space X has the weak λ-BAP if and only if every separable subspace of X is contained in a separable subspace with the weak λ-BAP. Considering the weak BAP version of Theorem 1.3, we ask:

Problem 2. Let X be a Banach space and Z be a separable closed subspace of X, and let $\lambda \geq 1$. If the pair (X,Z) has the weak λ-BAP, then for every separable subspace Y_0 of X, is there a separable closed subspace Y containing Y_0 and Z for which the pair (Y,Z) has the weak λ-BAP?

The converse part of Problem 2 can be easily derived by the proof of Theorem 1.3 and [1, Proposition 2.2].

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