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LARGE TIME BEHAVIOR FOR THE DEGENERATE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH DAMPING

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Abstract. In this paper we consider degenerate compressible Navier-Stokes equations giving rise to a variety of mathematical problems in many areas. We study the long time behavior for the degenerate compressible Navier-Stokes equations with damping.

1. Introduction

The models of compressible Navier-Stokes equations arise in science and a variety of engineering in many practical applications such as geophysics, astrophysics, and some engineering problems appearing in plasma confinement, liquid-metal cooling of nuclear reactors, and electromagnetic casting. Dissipative quantum models describe a quantum system together with a reservoir, which absorbs the energy lost by the system, for example, active regions and contacts in semiconductor devices, spin chains with thermal noise, or open Bose-Einstein condensates. These models are typically based on the master equation in Lindblad form, the Schrödinger-Langevin equation, or collisional Wigner equations. Due to their high numerical complexity, simpler macroscopic models have been derived in recent years. As a physical model of quantum fluids, we here consider the degenerate quantum compressible Navier-Stokes equations with damping:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \varrho^\gamma = \operatorname{div}(\varrho \mathbb{D}\mathbf{u}) - \varrho|\mathbf{u}|^2\mathbf{u} + \varrho \nabla \mathbf{G}, \tag{1.2}$$

where $\mathbf{u}$ is the vector field, $\gamma > 1$, ϱ is the density, and $\mathbb{D}\mathbf{u}$ is defined as follows:

$$\mathbb{D}\mathbf{u} = \frac{1}{2}[\nabla \mathbf{u} + \nabla^T \mathbf{u}].$$

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For this model, we will consider the case of periodic domain, $\Omega = \mathbb{T}^3$ and take $G = -x_3$ for the force potential.

The existence of global weak solution for compressible Navier-Stokes equations with density dependent viscosity was also one challenge issues in mathematics community thanks to a variety of applications to ocean physics and the shallow water equations. Recently, Vasseur and Yu [10, 11] have proved the global-in-time existence solutions for system (1.1)-(1.2) for 3D degenerate compressible Navier-Stokes equations based on Bresh and Desjardins' entropy inequality in [1, 2] and the Mellet-Vasseur type inequality given in [9].

In this paper, we study the long time behavior of degenerate compressible Navier-Stokes equations with damping based on [4] with finite energy. In general, we can derive specific compactness of velocity and density, but we need more compactness to deal with density if there is vacuum on density, we can not have any information on velocity. To overcome this problems, Vasseur and Yu [10, 11] have derived rigourous Bresh and Desjardin [1, 2] entropy inequality. This is different approach from the previous results and this paper proved Lemma 3.1.

There are recent progressive results of long time behaviour by [3, 5–8, 12] based on [4], but in this paper we use the similar regularity in [9] and we need the damping term to get Lemma 3.1. The outline of this article is as follows: In Section 2, we introduce the notion of weak solutions and the main result for degenerate compressible Navier Stokes equation with damping. In Section 3, we give the proof of the large time behavior of the solutions.

2. Main results

In this section we introduce the main result of long time behaviour for degenerate compressible Navier-Stokes equations with damping. Before this, we mention the global weak solutions of degenerate compressible Navier-Stokes equations.

Theorem 2.1 (See [10]). *Suppose that the initial data* $(\varrho_0, \mathbf{m}_0)$ *satisfy*:

$$\begin{aligned} &\varrho_0 \in L^\gamma(\Omega),\ \varrho_0 \geq 0,\ \nabla\sqrt{\varrho_0} \in L^2(\Omega), \\ &\mathbf{m}_0 \in L^1(\Omega),\ \mathbf{m}_0 := \varrho_0 \mathbf{u}_0 = 0 \text{ if } \varrho_0 = 0,\ \frac{|\mathbf{m}_0|^2}{\varrho_0} \in L^1(\Omega). \end{aligned} \tag{2.1}$$

Then, for any $\gamma > 1$ *and any* $T > 0$, *there exists a weak solution of* (1.1)-(1.2) *on* $(0, T)$ *in the sense of distribution verifying the energy inequality and the Bresch-Desjardins* (*B-D*) *entropy*:

$$E(t) + \int_0^T \int_\Omega \Big(\mu\varrho|\mathbb{D}\mathbf{u}|^2 + \varrho|\mathbf{u}|^4\Big) \mathrm{d}x\mathrm{d}t \leq E_0, \tag{2.2}$$

where

$$E(t) := \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{\gamma - 1}\varrho^\gamma + \varrho x_3\Big)\mathrm{d}x,$$

$$E_0 := \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma-1}\varrho_0^\gamma + \varrho_0 x_3\Big)\mathrm{d}x$$

and

$$\begin{aligned}
&\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u} + \nabla \ln \varrho|^2 + \frac{1}{\gamma-1}\varrho^\gamma + \varrho x_3\Big)\mathrm{d}x + \int_0^T\int_\Omega |\nabla \varrho^{\frac{\gamma}{2}}|^2 \mathrm{d}x\mathrm{d}t \\
&+ \int_0^T\int_\Omega \varrho|\nabla\mathbf{u} - \nabla^T\mathbf{u}|^2\mathrm{d}x\mathrm{d}t \le \int_\Omega \Big(\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\gamma-1}\varrho_0^\gamma + \varrho_0 x_3\Big)\mathrm{d}x.
\end{aligned} \tag{2.3}$$

Remark 2.1. In the computation of the B-D entropy inequality on the torus, we have used the following fact:

$$\int_0^T\int_\Omega \nabla\varrho\cdot\nabla G \; \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega \partial_{x_3}\varrho \; \mathrm{d}x\mathrm{d}t = 0$$

for any $T > 0$.

Before we mention the main result, let us observe the solution of the state problem which is the target solution:

$$\nabla\overline{\varrho}^\gamma = \overline{\varrho}\nabla G \text{ in } \Omega \tag{2.4}$$

in the sense of distribution, which implies that

$$\nabla\overline{\varrho}^\gamma = \frac{\gamma}{\gamma-1}\overline{\varrho}\nabla\overline{\varrho}^{\gamma-1} = \overline{\varrho}\nabla G. \tag{2.5}$$

Thus, we have

$$\overline{\varrho} = \Big(\frac{\gamma-1}{\gamma}\Big)^{\frac{1}{\gamma-1}}(\theta - x_3)_+^{\frac{1}{\gamma-1}} \tag{2.6}$$

for a certain constant $\theta > 0$ where $(g)_+$ is defined as follows:

$$(g)_+ = g \text{ if } g > 0 \text{ and } (f)_+ = 0 \text{ if } g \le 0.$$

Notice that $\overline{\varrho}$ does only depend on x_3.

We now present the result of the long time behavior of global weak solutions to the degenerate compressible Navier-Stokes equations.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^3$ be a periodic domain $\mathbb{T}^3$ and $(\varrho, \mathbf{u})$ be a weak solution to* (1.1)-(1.2) *verifying the initial data* (2.1)*. Then we have*

$$\begin{cases} \varrho(t) \to \overline{\varrho} \text{ strongly in } L^\gamma(\Omega), \\ \sqrt{\varrho(t)}\mathbf{u}(t) \to 0 \text{ strongly in } L^2(\Omega;\mathbb{R}^3), \end{cases} \tag{2.7}$$

as $t \to \infty$ where $\overline{\varrho}$ in given in (2.6)*.*

3. Proof of Theorem 2.2

In this section we prove the long time behavior of solutions of equations (1.1)-(1.2).

To investigate the behaviour of large time of weak solutions of (1.1)-(1.2), let us now define $\mathbf{u}_n, \varrho_n$ by

$$\begin{cases} \mathbf{u}_n(t,x) = \mathbf{u}(t+n,x), \\ \varrho_n(t,x) = \varrho(t+n,x) \end{cases} \tag{3.1}$$

for all integer $n \geq 1$, $t \in [0,1]$. In virtue of (2.2), we obtain that

$$\begin{aligned} &\operatorname*{ess\ sup}_{t\in[0,1]} \int_\Omega \Big(\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{1}{\gamma-1}\varrho_n^\gamma + \varrho_n x_3\Big)\mathrm{d}x \\ &\qquad + \int_0^1\int_\Omega \Big(\varrho_n|\mathbb{D}\mathbf{u}_n|^2 + \varrho_n|\mathbf{u}_n|^4\Big)\mathrm{d}x\mathrm{d}t \leq C. \end{aligned} \tag{3.2}$$

Moreover, the BD entropy inequality provides

$$\begin{aligned} &\operatorname*{ess\ sup}_{t\in[0,1]} \int_\Omega \Big(\frac{1}{2}\varrho_n|\mathbf{u}_n + \nabla \ln \varrho_n|^2 + \frac{1}{\gamma-1}\varrho_n^\gamma + \varrho_n x_3\Big)\mathrm{d}x \\ &+ \int_0^1\int_\Omega |\nabla \varrho_n^{\frac{\gamma}{2}}|^2\mathrm{d}x\mathrm{d}t + \int_0^1\int_\Omega \varrho_n|\nabla\mathbf{u}_n - \nabla^T\mathbf{u}_n|^2\mathrm{d}x\mathrm{d}t \leq C. \end{aligned} \tag{3.3}$$

It follows that

$$\begin{cases} \{\varrho_n\}_{n=1}^\infty & \text{bounded in } L^\infty(0,1;L^\gamma(\Omega)), \\ \{\sqrt{\varrho_n}\mathbf{u}_n\}_{n=1}^\infty & \text{bounded in } L^\infty(0,1;L^2(\Omega;\mathbb{R}^3)), \\ \{\varrho_n^{\frac{1}{4}}\mathbf{u}_n\}_{n=1}^\infty & \text{bounded in } L^4(0,1;L^4(\Omega;\mathbb{R}^3)). \\ \{\sqrt{\varrho_n}\nabla\mathbf{u}_n\}_{n=1}^\infty & \text{bounded in } L^2(0,1;L^2(\Omega;\mathbb{R}^{3\times 3})). \end{cases} \tag{3.4}$$

Following the B-D entropy inequality, we see that

$$\nabla\sqrt{\varrho_n} = \frac{1}{2}\frac{\nabla\varrho_n}{\sqrt{\varrho_n}} \text{ bounded in } L^\infty(0,1;L^2(\Omega)) \tag{3.5}$$

and thus the Poincare's inequality implies that

$$\varrho_n \in L^\infty(0,1;L^3(\Omega)). \tag{3.6}$$

Thus using Höllder's inequality and (3.4) imply that

$$\|\varrho_n\mathbf{u}_n\|_{L^2([0,1]\times\Omega)} \leq \|\varrho_n^{3/4}\|_{L^\infty(0,1;L^4(\Omega))}\|\varrho_n^{1/4}\mathbf{u}_n\|_{L^4([0,1]\times\Omega)} \leq C. \tag{3.7}$$

Consequently, the third estimate of (3.4), together with (3.6) and (3.7), prove the following Lemma:

Lemma 3.1. *Let* $(\varrho_n, \mathbf{u}_n)$ *be given in the* (3.1). *Then we get*

$$\begin{aligned} \lim_{n\to\infty}\int_0^1 &\Big(\|\sqrt{\varrho_n}\nabla\mathbf{u}_n\|^2_{L^2(\Omega;\mathbb{R}^{3\times3})} + \|\varrho_n\mathbf{u}_n\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\sqrt{\varrho}_n\mathbf{u}_n\|^2_{L^2(\Omega;\mathbb{R}^3)} \\ &+ \|\varrho_n|\mathbf{u}_n|^2\mathbf{u}_n\|_{L^1(\Omega)}\Big)\mathrm{d}t = 0. \end{aligned} \tag{3.8}$$

Indeed, we have used the following inequalities for the proof:

$$\|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^2([0,1]\times\Omega)} \le C\|\varrho_n\|_{L^\infty(0,1;L^3(\Omega))}\|\varrho_n^{\frac14}\mathbf{u}_n\|_{L^4([0,1]\times\Omega)} \le C,$$

and

$$\|\varrho_n|\mathbf{u}_n|^2\mathbf{u}_n\|_{L^1([0,1]\times\Omega)} \le \|\varrho_n^{\frac14}\mathbf{u}_n\|^2_{L^4([0,1]\times\Omega)}\|\varrho_n^{\frac12}\mathbf{u}_n\|_{L^2([0,1]\times\Omega)} \le C.$$

3.1. Estimates of density and pressure

To show the strong convergence of ϱ_n, we first notice that the estimate (3.6), together with (3.4), provides

$$\varrho_n\mathbf{u}_n = \sqrt{\varrho_n}\sqrt{\varrho_n}\mathbf{u}_n \text{ bounded in } L^\infty(0,1;L^{\frac32}(\Omega)). \tag{3.9}$$

Moreover, using the continuity equation with (3.9) gives us

$$\partial_t\varrho_n \text{ bounded in } L^\infty(0,1;W^{-1,\frac32}(\Omega)), \tag{3.10}$$

and the energy inequalities (3.2) and (3.3) provides that

$$\nabla\varrho_n = 2\sqrt{\varrho_n}\nabla\sqrt{\varrho_n} \text{ bounded in } L^\infty(0,1;L^{\frac32}(\Omega)). \tag{3.11}$$

Therefore, by Aubin-Lions lemma, we have

$$\varrho_n \ \in C([0,1];L^{\frac32}(\Omega)) \text{ for any } n\ge 1. \tag{3.12}$$

Consequently, we get the following Lemma:

$$\varrho_n \to \overline{\varrho} \text{ strongly in } C([0,1];L^{3/2}(\Omega)). \tag{3.13}$$

We now control the pressure. From the B-D entropy and Poincare's inequality, it is easy seen to deduce that

$$\varrho_n^\gamma \in L^1(0,1;L^3(\Omega)). \tag{3.14}$$

The energy inequality (3.2) also gives us that

$$\varrho_n^\gamma \text{ bounded in } \in L^\infty(0,1;L^1(\Omega)). \tag{3.15}$$

Thus, Hölder inequality and (3.14) with the regularity of density in (3.15) shows that

$$\varrho_n^\gamma \text{ bounded in } L^{\frac53}([0,1]\times\Omega). \tag{3.16}$$

Indeed,

$$\|\varrho_n^\gamma\|_{L^{5/3}([0,1]\times\Omega)} \le \|\varrho_n^\gamma\|^{2/5}_{L^\infty(0,1;L^1(\Omega))}\|\varrho_n^\gamma\|^{3/5}_{L^1(0,1;L^3(\Omega))} \le C.$$

Moreover, we already know that ϱ_n^γ converges strongly to $\overline{\varrho}^\gamma$ almost everywhere and so the Lebegue dominated convergence Theorem shows that

$$\varrho_n^\gamma \longrightarrow \overline{\varrho}^\gamma \text{ in } L^1([0,1]\times\Omega). \tag{3.17}$$

Consequently, passing to the limit in (1.2), together with using (3.8), (3.13), (3.17), we are able to deduce the state equation (2.4).

3.2. Convergence of the result (2.7)

Since the energy has finite value by (2.1) as t tends to ∞,

$$E_\infty := \operatorname{ess}\lim_{t\to\infty} E(t) < \infty,$$

where $\operatorname{ess}\lim_{t\to\infty}$ means that we take the limit except the measure zero for time variable. Following (3.8), we get

$$\lim_{n\to\infty}\int_n^{n+1}\int_\Omega \varrho|\mathbf{u}|^2 \mathrm{d}x\mathrm{d}t = 0,$$

which implies that

$$E_\infty = \lim_{n\to\infty}\int_n^{n+1}\int_\Omega \frac{\varrho^\gamma}{\gamma-1} + \varrho x_3 \mathrm{d}x\mathrm{d}t = \int_\Omega \frac{\overline{\varrho}^\gamma}{\gamma-1} + \overline{\varrho} x_3 \mathrm{d}x,$$

where we have also used Lemma 3.1. Consequently, we deduce that

$$\begin{aligned} E_\infty &\le \liminf_{t\to\infty}\int_\Omega \frac{\varrho^\gamma(t)}{\gamma-1} + \varrho(t)x_3\mathrm{d}x \le \limsup_{t\to\infty}\int_\Omega \frac{\varrho^\gamma(t)}{\gamma-1} + \varrho(t)x_3\mathrm{d}x \\ &\le \limsup_{t\to\infty}\int_\Omega \frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \frac{\varrho^\gamma(t)}{\gamma-1} + \varrho(t)x_3\mathrm{d}x \\ &\le \limsup_{t\to\infty} E(t) = E_\infty, \end{aligned}$$

which implies that

$$\lim_{t\to\infty}\int_\Omega \varrho(t)|\mathbf{u}(t)|^2\mathrm{d}x = 0, \text{ and } \lim_{t\to\infty}\int_\Omega \varrho^\gamma(t)\mathrm{d}x = \int_\Omega \overline{\varrho}^\gamma(t)\mathrm{d}x$$

and thus we prove the result (2.7) where we have used the convexity of L^γ.

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