J. Appl. Math. & Informatics Vol. **37***(2019), No. 3 - 4, pp. 317 - 328*
https://doi.org/10.14317/jami.2019.317

SYMMETRIC PROPERTIES OF CARLITZ'S TYPE (p, q)-GENOCCHI POLYNOMIALS

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ABSTRACT. This paper defines Carlitz's type (p, q)-Genocchi polynomials and Carlitz's type (h, p, q)-Genocchi polynomials, and explains fourteen properties which can be complemented by Carlitz's type (p, q)-Genocchi polynomials and Carlitz's type (h, p, q)-Genocchi polynomials, including distribution relation, symmetric property, and property of complement. Also, it explores alternating powers sums by proving symmetric property related to Carlitz's type (p, q)-Genocchi polynomials.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Hurwitz Euler Zeta function, Genocchi polynomials and numbers, (p, q)-Genocchi polynomials.

1. Introduction

Many researchers have investigated Genocchi numbers G_n and Genocchi polynomials $G_n(x)$. Until now, Genocchi numbers and Genocchi polynomials have been studied in many field, such as analytic number theory, theory of modular forms and quantum physics(see [1-7]).

This paper uses the following notations: $\mathbb{N} = \{1, 2, 3, \cdots\}$ denotes the natural numbers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \cdots\}$ denotes the set of non-positive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers.

q-number and (p, q)-number are defined by

$$[n]_q = \frac{1 - q^n}{1 - q} \quad \text{and} \quad [n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

It is obvious that (p, q)-number includes symmetric property, which is q-number as $p = 1$. Especially, we have $\lim_{q \to 1} [n]_{p,q} = n$ when $p = 1$.

Received June 28, 2018. Revised February 22, 2019. Accepted March 25, 2019.

We know that the classical Genocchi numbers G_n and polynomials $G_n(x)$ are defined by the following generating functions:

$$\frac{2t}{e^t+1}=\sum_{n=0}^{\infty}G_n\frac{t^n}{n!}\quad(|t|<\pi)$$

and

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}\quad(|t|<\pi).$$

Many mathematicians have studied q-Genocchi numbers and polynomials, an extension of classical Genocchi numbers and polynomials. From now on, this paper will introduce q-Genocchi, which has been studied by several researchers.

First, Kim defined q-extension of Genocchi numbers $G_{n,q}$ and q-analogue of Genocchi polynomials $G_{n,q}(x)$ in 2007, as follows(see [4]):

$$G_q(t)=\sum_{n=0}^{\infty}G_{n,q}\frac{t^n}{n!}=[2]_q t\sum_{n=0}^{\infty}(-1)^n q^n e^{[n]_q t}$$

and

$$G_q(x,t)=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}=[2]_q t\sum_{n=0}^{\infty}(-1)^n q^{n+x} e^{[n+x]_q t},$$

where $q\in\mathbb{C}$ with $|q|<1$.

Also, he examined the properties of q-analogue of Genocchi polynomials such as distribution relation. He gave some connection between q-extension of Euler polynomials and q-extension of Genocchi numbers.

Seocond, Ryoo and Kang defined q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α in 2012, as follows(see [5]):

$$F_q^{(\alpha)}(t)=\sum_{n=0}^{\infty}G_{n,q}^{(\alpha)}\frac{t^n}{n!}=[2]_{q^\alpha} t\sum_{n=0}^{\infty}(-1)^n q^{\alpha n} e^{[n]_q t}$$

and

$$F_q^{(\alpha)}(x,t)\sum_{n=0}^{\infty}G_{n,q}^{(\alpha)}(x)\frac{t^n}{n!}=[2]_{q^\alpha} t\sum_{n=0}^{\infty}(-1)^n q^{\alpha n} e^{[n+x]_q t},$$

where $\alpha\in\mathbb{C}$.

They investigeted the properties of q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α such as property of complement. Furthermore, they compared between q-Genocchi polynomials and q-Genocchi Zeta function.

Finally, Kang and Ryoo defined the twisted q-Genocchi polynomials $G_{n,q,w}(x)$ in 2014, as follows(see [2]):

$$\sum_{n=0}^{\infty}G_{n,q,w}(x)\frac{t^n}{n!}=[2]_q t\sum_{n=0}^{\infty}(-1)^n q^n w^n e^{[n+x]_q t},$$

where r is a positive integer and w is r-th root of 1.

They derived some properties of the twisted q-Genocchi polynomials such as symmetric property.

Next, it will introduce definition of Hurwitz (p,q)-Euler Zeta function, which is an extension of Hurwitz Euler Zeta function. For $s \in \mathbb{C}$ with $Re(s) > 0$ and $x \notin \mathbb{Z}_0^-$, Hurwitz (p,q)-Euler Zeta function $\zeta_{p,q}(s,x)$ defined by the following generation function(see [6, 7]):

$$\zeta_{p,q}(s,x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{[n+x]_{p,q}^s}. \tag{1.1}$$

This paper modifies q-Genocchi numbers and polynomials, which is mentioned above, and studies (p,q)-Genocchi numbers and polynomials that is extended by q-Genocchi numbers and polynomials. To be specific, it defines Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$. Also, it gives some general properties of Carlitz's type (p,q)-Genocchi polynomials. At the end, it examines symmetric properties of Carlitz's type (p,q)-Genocchi polynomials.

2. Some properties of Carlitz's type (p,q)-Genocchi numbers and polynomials

In this section, we define Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$ and Carlitz's type (h,p,q)-Genocchi polynomials $G_{n,p,q}^{(h)}(x)$. It derives some several properties.

Definition 2.1. For $0 < q < p \leq 1$, Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$ are defined as the following generating functions

$$F_{p,q}(t) = \sum_{n=0}^{\infty} G_{n,p,q} \frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m q^m e^{[m]_{p,q} t} \tag{2.1}$$

and

$$F_{p,q}(x,t) = \sum_{n=0}^{\infty} G_{n,p,q}(x) \frac{t^n}{n!} = [2]_q t \sum_{m=0}^{\infty} (-1)^m q^m e^{[m+x]_{p,q} t}, \tag{2.2}$$

respectively.

If we put $p = 1$ and let $q \to 1$ in equations (2.1) and (2.2), then Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ reduce to the classical Genocchi numbers G_n and Carlitz's type (p,q)-Genocchi polynomials $G_{n,p,q}(x)$ reduce to the classical Genocchi polynomials $G_n(x)$. In other words, they are:

$$\lim_{q\to 1} G_{n,p,q} = G_n \quad \text{and} \quad \lim_{q\to 1} G_{n,p,q}(x) = G_n(x).$$

Definition 2.2. For $0 < q < p \leq 1$, Carlitz's type (h,p,q)-Genocchi polynomials $G_{n,p,q}^{(h)}(x)$ is defined as the following generating function

$$\sum_{n=0}^{\infty} G_{n,p,q}^{(h)}(x) \frac{t^n}{n!} = [2]_q t \sum_{k=0}^{\infty} (-1)^k q^k p^{kh} e^{[x+k]_{p,q} t}.$$

In particular, Carlitz's type (h,p,q)-Genocchi polynomials are Carlitz's type (h,p,q)-Genocchi numbers when $x=0$, that is, $G_{n,p,q}^{(h)} = G_{n,p,q}^{(h)}(0)$ denotes Carlitz's type (h,p,q)-Genocchi numbers.

By using Definition 2.2, we get

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,p,q}^{(h)}(x)\frac{t^n}{n!} &= [2]_q t\sum_{m=0}^{\infty}(-1)^m q^m p^{mh} e^{[x+m]_{p,q}t}\\ &= [2]_q t\sum_{m=0}^{\infty}(-1)^m q^m p^{mh}\sum_{n=0}^{\infty}[x+m]_{p,q}^n\frac{t^n}{n!}\\ &= [2]_q\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m q^m p^{mh} n\,[x+m]_{p,q}^{n-1}\frac{t^n}{n!}.\end{aligned}\tag{2.3}$$

Hence, we get the following theorem by comparing the coefficients of $\frac{t^n}{n!}$ and then putting $n=n+1$ on both sides of the above equation (2.3).

Theorem 2.3. For non-negative integer n, we obtain

$$\frac{G_{n+1,p,q}^{(h)}(x)}{n+1} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^m p^{hm}[m+x]_{p,q}^n.$$

We find by utilizing Definition 2.2 that

$$\begin{aligned}\sum_{n=0}^{\infty} G_{n,p,q}^{(h)}(x)\frac{t^n}{n!} &= [2]_q t\sum_{k=0}^{\infty}(-1)^k q^k p^{kh} e^{[x+k]_{p,q}t}\\ &= [2]_q t\sum_{n=0}^{\infty}\left(\frac{1}{p-q}\right)^n\sum_{k=0}^{\infty}(-1)^k q^k p^{kh}\left(p^{x+k}-q^{x+k}\right)^n\frac{t^n}{n!}\\ &= [2]_q t\sum_{n=0}^{\infty}\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl} p^{x(n-l)}\\ &\quad\times\sum_{k=0}^{\infty}(-1)^k q^{k(1+l)} p^{k(n-l+h)}\frac{t^n}{n!}\\ &= [2]_q\sum_{n=0}^{\infty} n\left(\frac{1}{p-q}\right)^{n-1}\\ &\quad\times\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l q^{xl} p^{x(n-1-l)}\frac{1}{1+q^{l+1}p^{n-1-l+h}}\frac{t^n}{n!}.\end{aligned}\tag{2.4}$$

Therefore, we get the following theorem by comparing the coefficients of $\frac{t^n}{n!}$ and then putting $n=n+1$ on both sides of the equation (2.4).

Theorem 2.4. For non-negative integer n, we have

$$\frac{G_{n+1,p,q}^{(h)}(x)}{n+1} = [2]_q\left(\frac{1}{p-q}\right)^n\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{xl} p^{x(n-l)}\frac{1}{1+q^{l+1}p^{n-l+h}}.$$

Next, we get the following theorem by using equation (2.2) in the same way as in equation (2.4).

Theorem 2.5. For non-negative integer n, we obtain

$$\frac{G_{n+1,p,q}(x)}{n+1} = [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{xl} p^{x(n-l)} \frac{1}{1+q^{l+1}p^{n-l}}.$$

If we compare the above Theorem 2.4 and Theorem 2.5, we can see that Theorem 2.4 and Theorem 2.5 are the same results when $h = 0$.

By using Theorem 2.5, we obtain

$$\begin{aligned} \frac{G_{n+1,p,q}(x)}{n+1} &= [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} p^{x(n-l)} \frac{1}{1+q^{l+1}p^{n-l}} \\ &= [2]_q \left(\frac{1}{p-q}\right)^n \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} p^{x(n-l)} \frac{1}{1+q^{m(l+1)}p^{m(n-l)}} \\ &\quad \times \sum_{a=0}^{m-1} (-1)^a q^{a(l+1)} p^{a(n-l)} \\ &= \frac{[2]_q}{[2]_{q^m}} [m]_{p,q}^n \sum_{a=0}^{m-1} (-1)^a q^a \frac{G_{n+1,p^m,q^m}\left(\frac{a+x}{m}\right)}{n+1}. \end{aligned} \tag{2.5}$$

Consequently, we get the following theorem from the above equation (2.5). This theorem is called distribution relation.

Theorem 2.6. (Distribution relation) For non-negative integer n and any positive odd integer m, we have

$$G_{n+1,p,q}(x) = \frac{[2]_q}{[2]_{q^m}} [m]_{p,q}^n \sum_{a=0}^{m-1} (-1)^a q^a G_{n+1,p^m,q^m}\left(\frac{a+x}{m}\right). \tag{2.6}$$

Let us put $h = 0$ in Theorem 2.3, we have

$$\begin{aligned} \frac{G_{n+1,p,q}(x)}{n+1} &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m [m+x]_{p,q}^n \\ &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m \sum_{l=0}^{n} \binom{n}{l} q^{xl} [m]_{p,q}^l [x]_{p,q}^{n-l} p^{m(n-l)} \\ &= \sum_{l=0}^{n} \binom{n}{l} [x]_{p,q}^{n-l} q^{xl} \frac{G_{l+1,p,q}^{(n-l)}}{l+1}. \end{aligned} \tag{2.7}$$

Therefore, we get the following theorem from the above equation (2.7). It is the connection between Carlitz's type (p, q)-Genocchi polynomials $G_{n,p,q}(x)$ and Carlitz's type (h, p, q)-Genocchi polynomials $G_{n,p,q}^{(h)}(x)$.

Theorem 2.7. For non-negative integer n, we have

$$\frac{G_{n+1,p,q}(x)}{n+1}=\sum_{l=0}^{n}\binom{n}{l}[x]_{p,q}^{n-l}q^{xl}\frac{G_{l+1,p,q}^{(n-l)}}{l+1}.$$

By using Theorem 2.5, we obtain

$$\begin{aligned}
&\frac{G_{n+1,p^{-1},q^{-1}}(1-x)}{n+1}\\
&=[2]_{q^{-1}}\left(\frac{1}{p^{-1}-q^{-1}}\right)^{n}\\
&\times\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{-(1-x)l}p^{-(1-x)(n-l)}\frac{1}{1+q^{-(l+1)}p^{-(n-l)}}\\
&=\left(\frac{1}{p-q}\right)^{n}p^{n}q^{n}(-1)^{n}[2]_{q}\sum_{l=0}^{n}\binom{n}{l}(-1)^{l}q^{xl}p^{x(n-l)}\frac{1}{1+q^{l+1}p^{n-l}}\\
&=p^{n}q^{n}(-1)^{n}\frac{G_{n+1,p,q}(x)}{n+1}.
\end{aligned}\tag{2.8}$$

Hence, we get the following theorem from the above equation (2.8). This theorem is called property of complement.

Theorem 2.8. (Property of complement) For non-negative integer n, we have

$$G_{n+1,p^{-1},q^{-1}}(1-x)=p^{n}q^{n}(-1)^{n}G_{n+1,p,q}(x).$$

We can easily observe the following equation.

$$\begin{aligned}
&-[2]_{q}t\sum_{k=0}^{\infty}(-1)^{n+k}q^{n+k}e^{[n+k]_{p,q}t}+[2]_{q}t\sum_{k=0}^{\infty}(-1)^{k}q^{k}e^{[k]_{p,q}t}\\
&=[2]_{q}t\sum_{k=0}^{n-1}(-1)^{k}q^{k}e^{[k]_{p,q}t}.
\end{aligned}\tag{2.9}$$

If we express the left-hand side of the above equation (2.9) as Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$, then we have

$$\begin{aligned}
&(-1)^{n+1}q^{n}\sum_{m=0}^{\infty}G_{m,p,q}(n)\frac{t^{m}}{m!}+\sum_{m=0}^{\infty}G_{m,p,q}\frac{t^{m}}{m!}\\
&=[2]_{q}t\sum_{k=0}^{n-1}(-1)^{k}q^{k}e^{[k]_{p,q}t}\\
&=\sum_{m=0}^{\infty}\left([2]_{q}m\sum_{k=0}^{n-1}(-1)^{k}q^{k}[k]_{p,q}^{m-1}\right)\frac{t^{m}}{m!}.
\end{aligned}\tag{2.10}$$

Therefore, we get the following theorem by comparing the coefficients of $\frac{t^m}{m!}$ on both sides of the above equation (2.10).

Theorem 2.9. For $m, n \in \mathbb{N}$, we get

$$\frac{(-1)^{n+1}q^n G_{m,p,q}(n) + G_{m,p,q}}{[2]_q m} = \sum_{k=0}^{n-1}(-1)^k q^k [k]_{p,q}^{m-1}.$$

The above theorem is explained as an alternating sums using Carlitz's type (p,q)-Genocchi numbers $G_{n,p,q}$ and polynomials $G_{n,p,q}(x)$.

3. Symmtric properties of the Carlitz's type (p,q)-Genocchi polynomials

In section 3, this paper investigates the relation between Carlitz's type (p,q)-Genocchi polynomials $G_{n,p,q}(x)$ and Hurwitz (p,q)-Euler Zeta function $\zeta_{p,q}(s,x)$. After that it derives symmetric property of Carlitz's type (p,q)-Genocchi polynomials $G_{n,p,q}(x)$.

By using equation (2.2), we get

$$\begin{aligned}\left.\frac{d^k}{dt^k}F_{p,q}(x,t)\right|_{t=0} &= \left.\frac{d^k}{dt^k}[2]_q\sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_{p,q}t}\right|_{t=0}\\ &= [2]_q k\sum_{m=0}^{\infty}(-1)^m q^m [m+x]_{p,q}^{k-1}\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\left(\frac{d}{dt}\right)^k\left.\left(\sum_{n=0}^{\infty}G_{n,p,q}(x)\frac{t^n}{n!}\right)\right|_{t=0} &= \left.\sum_{n=k}^{\infty}G_{n,p,q}(x)\frac{t^{n-k}}{(n-k)!}\right|_{t=0}\\ &= G_{k,p,q}(x).\end{aligned} \tag{3.2}$$

Thus, we get the following theorem by comparing the equations (3.1) and (3.2), and then putting $n = n+1$ on both sides.

Theorem 3.1. For non-negative integer k, we have

$$\frac{G_{k+1,p,q}(x)}{k+1} = [2]_q\sum_{m=0}^{\infty}(-1)^m q^m [m+x]_{p,q}^k.$$

By using equation (1.1) and Theroem 3.1, we get

$$\begin{aligned}\zeta_{p,q}(-k,x) &= [2]_q\sum_{m=0}^{\infty}\frac{(-1)^m q^m}{[m+x]_{p,q}^{-k}}\\ &= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m [m+x]_{p,q}^k\\ &= \frac{G_{k+1,p,q}(x)}{k+1}.\end{aligned} \tag{3.3}$$

Therefore, we get the following theorem from the above equation (3.3). This theorem is connection between Carlitz's type (p,q)-Genocchi polynomials $G_{n,p,q}(x)$ and Hurwitz (p,q)-Euler Zeta function $\zeta_{p,q}(s,x)$.

Theorem 3.2. For non-negative integer k, we have

$$\zeta_{p,q}(-k,x)=\frac{G_{k+1,p,q}(x)}{k+1}.$$

By substituting $w_1x+\frac{w_1 i}{w_2}$ for x and replacing p with p^{w_2} and q with q^{w_2} in Theorem 3.1, respectively, we derive

$$\begin{aligned}
&\frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)}{n+1}\\
&=[2]_{q^{w_2}}\sum_{k=0}^{\infty}(-1)^k q^{w_2 k}\left[k+w_1x+\frac{w_1 i}{w_2}\right]^n_{p^{w_2},q^{w_2}} \qquad (3.4)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]^n_{p,q}}\sum_{k=0}^{\infty}(-1)^k q^{w_2 k}\left[w_2k+w_1w_2x+w_1 i\right]^n_{p,q}.
\end{aligned}$$

For any non-negative integer k and positive odd integer w_1, there exist unique non-negative integer r such that $k=w_1r+j$ with $0\leq j\leq w_1-1$. Hence, this can be written as

$$\begin{aligned}
&\frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)}{n+1}\\
&=\frac{[2]_{q^{w_2}}}{[w_2]^n_{p,q}}\sum_{w_1r+j=0}^{\infty}(-1)^{w_1r+j}q^{w_2(w_1r+j)}\left[w_2(w_1r+j)+w_1w_2x+w_1 i\right]^n_{p,q} \qquad (3.5)\\
&=\frac{[2]_{q^{w_2}}}{[w_2]^n_{p,q}}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^{w_1r+j}q^{w_2(w_1r+j)}\left[w_2(w_1r+j)+w_1w_2x+w_1 i\right]^n_{p,q}.
\end{aligned}$$

Let us put $\frac{[2]_{q^{w_1}}}{[w_1]^n_{p,q}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}$ on both sides of the above equation (3.5), we obtain

$$\begin{aligned}
&\frac{[2]_{q^{w_1}}}{[w_1]^n_{p,q}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)}{n+1}\\
&=\frac{[2]_{q^{w_1}}}{[w_1]^n_{p,q}}\frac{[2]_{q^{w_2}}}{[w_2]^n_{p,q}}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1} \qquad (3.6)\\
&\times\sum_{r=0}^{\infty}(-1)^{r+i+j}q^{w_1w_2r+w_1 i+w_2 j}\left[w_1w_2(x+r)+w_1 i+w_2 j\right]^n_{p,q}.
\end{aligned}$$

From similar method of the equation (3.4), we get

$$\begin{aligned}&\frac{G_{n+1,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2j}{w_1}\right)}{n+1}\\&=\frac{[2]_{q^{w_1}}}{[w_1]_{p,q}^n}\sum_{k=0}^{\infty}(-1)^k q^{w_1k}\left[w_1k+w_1w_2x+w_2j\right]_{p,q}^n.\end{aligned}\tag{3.7}$$

After some calculations in the above equation (3.7) and then applies $\frac{[2]_{q^{w_2}}}{[w_2]_{p,q}^n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}$, we have

$$\begin{aligned}&\frac{[2]_{q^{w_2}}}{[w_2]_{p,q}^n}\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}\frac{G_{n+1,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2j}{w_1}\right)}{n+1}\\&=\frac{[2]_{q^{w_1}}}{[w_1]_{p,q}^n}\frac{[2]_{q^{w_2}}}{[w_2]_{p,q}^n}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\\&\times\sum_{r=0}^{\infty}(-1)^{r+i+j}q^{w_1w_2r+w_1i+w_2j}\left[w_1w_2(r+x)+w_1i+w_2j\right]_{p,q}^n.\end{aligned}\tag{3.8}$$

Consequently, we get the following theorem by comparing the results of the equations (3.6) and (3.8).

Theorem 3.3. (Symmetric property) For non-negative integer n and any odd positive intger w_1 and w_2, we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_2]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_1]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}G_{n+1,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2j}{w_1}\right).\end{aligned}$$

By using Theroem 3.2 and Theroem 3.3, we get the following corollary.

Corollary 3.4. For any positive odd intger w_1 and w_2, we have

$$\begin{aligned}&[2]_{q^{w_1}}[w_1]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}\zeta_{p^{w_2},q^{w_2}}\left(n,w_1x+\frac{w_1i}{w_2}\right)\\&=[2]_{q^{w_2}}[w_2]_{p,q}^n\sum_{j=0}^{w_1-1}(-1)^j q^{w_2j}\zeta_{p^{w_1},q^{w_1}}\left(n,w_2x+\frac{w_2j}{w_1}\right).\end{aligned}$$

By taking $w_1=1$ and replacing x with $\frac{x}{w_2}$ in Theorem 3.3, we obtain

$$\begin{aligned}&[2]_q[w_2]_{p,q}^n\sum_{i=0}^{w_2-1}(-1)^i q^i G_{n+1,p^{w_2},q^{w_2}}\left(\frac{x}{w_2}+\frac{i}{w_2}\right)\\&=[2]_{q^{w_2}}G_{n+1,p,q}(x).\end{aligned}\tag{3.9}$$

Hence, we get the following corollary by calculating the above equation (3.9).

Corollary 3.5. For non-negative integer n and any positive odd integer w_2, we get

$$G_{n+1,p,q}(x) = \frac{[2]_q[w_2]_{p,q}^n}{[2]_{q^{w_2}}} \sum_{i=0}^{w_2-1} (-1)^i q^i G_{n+1,p^{w_2},q^{w_2}} \left(\frac{x+i}{w_2}\right). \tag{3.10}$$

Proofs of Theorem 2.6 and Corollary 3.5 are different but the results are the same. Thus, Corollary 3.5 can be called distribution relation.

If we put $w_1 = 1$ and $w_2 = 3$ in Theorem 3.3, then we get the following corllary.

Corollary 3.6. For non-negative integer n, we obtain

$$\begin{aligned}
&G_{n+1,p^3,q^3}\left(\frac{x}{3}\right) - qG_{n+1,p^3,q^3}\left(\frac{x+1}{3}\right) + q^2 G_{n+1,p^3,q^3}\left(\frac{x+2}{3}\right) \\
&= \frac{[2]_{q^3}}{[2]_q[3]_{p,q}^n} G_{n+1,p,q}(x).
\end{aligned}$$

Next, by applying toalternating series, we can get different result of symmetric property. By using Theorem 3.3, we have

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} \frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1 x + \frac{w_1 i}{w_2}\right)}{n+1} \\
&= [2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}} \sum_{m=0}^{\infty}(-1)^m q^{w_2 m}\left[m + w_1 x + \frac{w_1 i}{w_2}\right]_{p^{w_2},q^{w_2}}^n \\
&= [2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}} \\
&\times \sum_{m=0}^{\infty}(-1)^m q^{w_2 m}\left(q^{w_1 i}[m+w_1 x]_{p^{w_2},q^{w_2}} + p^{w_2(m+w_1 x)}[i]_{p^{w_1},q^{w_1}}\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^n \\
&= [2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}} \sum_{m=0}^{\infty}(-1)^m q^{w_2 m} \\
&\times \sum_{l=0}^{n}\binom{n}{l} q^{w_1 i(n-l)}[m+w_1 x]_{p^{w_2},q^{w_2}}^{n-l} p^{w_2 m l} p^{w_1 w_2 x l}[i]_{p^{w_1},q^{w_1}}^l \left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l.
\end{aligned}$$

From Theorem 2.3, we get

$$
\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)}{n+1}\\
&=[2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\\
&\times\sum_{l=0}^{n}\binom{n}{l}q^{w_1 i(n-l)}p^{w_1w_2xl}\frac{G^{(l)}_{n-l+1,p^{w_2},q^{w_2}}(w_1x)}{n-l+1}[i]^l_{p^{w_1},q^{w_1}}\left(\frac{[w_1]_{p,q}}{[w_2]_{p,q}}\right)^l \quad (3.11)\\
&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}\frac{G^{(l)}_{n-l+1,p^{w_2},q^{w_2}}(w_1x)}{n-l+1}\\
&\times\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i(n-l+1)}[i]^l_{p^{w_1},q^{w_1}}.
\end{aligned}
$$

As a result of the above the equation (3.11), we obtain

$$
\begin{aligned}
&[2]_{q^{w_1}}[w_2]_{p,q}^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\frac{G_{n+1,p^{w_2},q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)}{n+1}\\
&=[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}\frac{G_{n-l+1,p^{w_2},q^{w_2}}(w_1x)}{n-l+1}\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2)
\end{aligned}
\quad (3.12)
$$

and

$$
\begin{aligned}
&[2]_{q^{w_2}}[w_1]_{p,q}^n \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\frac{G_{n+1,p^{w_1},q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1}\right)}{n+1}\\
&=[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_{p,q}[w_1]^{n-l}_{p,q}p^{w_1w_2xl}\frac{G_{n-l+1,p^{w_1},q^{w_1}}(w_2x)}{n-l+1}\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1).
\end{aligned}
\quad (3.13)
$$

Consequently, we get the following theorem from the equations (3.12) and (3.13).

Theorem 3.7. For any positive odd integer w_1 and w_2, we have

$$
\begin{aligned}
&[2]_{q^{w_1}}\sum_{l=0}^{n}\binom{n}{l}[w_1]^l_{p,q}[w_2]^{n-l}_{p,q}p^{w_1w_2xl}\frac{G_{n-l+1,p^{w_2},q^{w_2}}(w_1x)}{n-l+1}\mathcal{E}_{n,l,p^{w_1},q^{w_1}}(w_2)\\
&=[2]_{q^{w_2}}\sum_{l=0}^{n}\binom{n}{l}[w_2]^l_{p,q}[w_1]^{n-l}_{p,q}p^{w_1w_2xl}\frac{G_{n-l+1,p^{w_1},q^{w_1}}(w_2x)}{n-l+1}\mathcal{E}_{n,l,p^{w_2},q^{w_2}}(w_1),
\end{aligned}
$$

where $\mathcal{E}_{n,l,p,q}(k)=\sum_{i=0}^{k-1}(-1)^i q^{(1+n-l)i}[i]^l_{p,q}$ is called as alternating series.

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