J. Appl. Math. & Informatics Vol. **37**(2019), No. 3 - 4, pp. 295 - 305
https://doi.org/10.14317/jami.2019.295

ON HIGHER ORDER (p, q)-FROBENIUS-GENOCCHI NUMBERS AND POLYNOMIALS†

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Abstract. In the present paper, we introduce (p, q)-Frobenius-Genocchi numbers and polynomials and investigate some basic identities and properties for these polynomials and numbers including addition theorems, difference equations, derivative properties, recurrence relations and so on. Then, we provide integral representations, implicit and explicit formulas and relations for these polynomials and numbers. We consider some relationships for (p, q)-Frobenius-Genocchi polynomials of order α associated with (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials and (p, q)-Genocchi polynomials.

AMS Mathematics Subject Classification : Primary 05A30, Secondary 11B68, 11B73.
Key words and phrases : (p, q)-Bernoulli polynomials, (p, q)-Euler polynomials, (p, q)-Genocchi polynomials, (p, q)-Frobenius-Genocchi polynomials.

1. Introduction

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1, -2, \cdots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The (p, q)-numbers are defined as

$$[n]_{p,q} = p^{n-1} + p^{n-2}q + p^{n-3}q^2 + \cdots + pq^{n-2} + q^{n-1} = \frac{p^n - q^n}{p - q}.$$

Received December 9, 2018. Revised April 18, 2019. Accepted April 20, 2019. *Corresponding author.
†This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Science, ICT and Future Planning(No. 2017R1E1A1A03070483).

We can write easily that $[n]_{p,q} = p^{n-1}[n]_{q/p}$, where $[n]_{q/p}$ is the q-number in q-calculus given by $[n]_{q/p} = \frac{(q/p)^n-1}{(q/p)-1}$. Thereby this implies that (p,q)-numbers and q-numbers are different, that is, we cannot obtain (p,q)-numbers just by substituting q by q/p in the definition of q-numbers. In the case of $p = 1$, (p,q)-numbers reduce to q-numbers,(see [6, 7]).

The (p,q)-derivative of a function f with respect to x is defined by

$$D_{p,q}f(x) = D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, (x \neq 0) \tag{1.1}$$

and $(D_{p,q}f(0)) = f^{'}(0)$, provided that f is differentiable at 0. The number (p,q)-derivative operator holds the following properties

$$D_{p,q}(f(x)g(x)) = g(p(x))D_{p,q}f(x) + f(qx)D_{p,q}g(x), \tag{1.2}$$

and

$$D_{p,q}\left(\frac{f(x)}{g(x)}\right) = \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)}. \tag{1.3}$$

The (p,q)-analogue of $(x+a)^n$ is given by

$$(x+a)^n_{p,q} = (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}), n \geq 1$$

$$= \sum_{k=0}^{n} \binom{n}{k}_{p,q} p^{\binom{n}{2}} q^{\binom{n-k}{2}} x^k a^{n-k},$$

where the (p,q)-Gauss Binomial coefficients$\binom{n}{k}_{p,q}$ and (p,q)-factorial $[n]_{p,q}!$ are defined by

$\binom{n}{k}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}(n \geq k)$ and $[n]_{p,q}! = [n]_{p,q}\cdots[2]_{p,q}[1]_{p,q}, (n \in \mathbb{N})$.

The (p,q)-exponential function are defined by

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{\binom{n}{2}}x^n}{[n]_{p,q}!} \; and \; E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}x^n}{[n]_{p,q}!},$$

holds the identities

$$e_{p,q}(x)E_{p,q}(-x) = 1 \; and \; e_{p^-q^-}(x) = E_{p,q}(x), \tag{1.4}$$

and have the (p,q)-derivatives

$$D_{p,q}e_{p,q}(x) = e_{p,q}(px) \; and \; D_{p,q}E_{p,q}(x) = E_{p,q}(qx). \tag{1.5}$$

The definition (p,q)-integral is defined by

$$\int_0^a f(x)d_{p,q}x = (p-q)a\sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(a\frac{p^k}{q^{k+1}}\right),$$

in conjunction with

$$\int_a^b f(x)d_{p,q}x = \int_0^b f(x)d_{p,q}x - \int_0^a f(x)d_{p,q}x, \quad \text{(see [19])}. \tag{1.6}$$

The generalized (p, q)-Bernoulli polynomials, the generalized (p, q)-Euler polynomials and the generalized (p, q)-Genocchi polynomials are defined by means of the following generating function as follows (see[1-20]):

$$\left(\frac{t}{e_{p,q}(t) - 1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x : p, q)\frac{t^n}{[n]_{p,q}!} \ , \mid t \mid < 2\pi, \tag{1.7}$$

$$\left(\frac{2}{e_{p,q}(t) + 1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x : p, q)\frac{t^n}{[n]_{p,q}!} \ , \mid t \mid < \pi, \tag{1.8}$$

and

$$\left(\frac{2t}{e_{p,q}(t) + 1}\right)^{\alpha} e_{p,q}(xt) = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x : p, q)\frac{t^n}{[n]_{p,q}!} \ , \mid t \mid < \pi. \tag{1.9}$$

It is clear that

$$B_n^{(\alpha)}(0 : p, q) = B_n^{(\alpha)}(p, q) \ , \ E_n^{(\alpha)}(0 : p, q) = E_n^{(\alpha)}(p, q),$$

and

$$G_n^{(\alpha)}(0 : p, q) = G_n^{(\alpha)}(p, q) \ (n \in \mathbb{N}).$$

Very recently, Yaşar and Özarslan [20] introduced Frobenius-Genocchi polynomials are defined by means of the following generating relation:

$$\frac{(1-\lambda)t}{e^t - \lambda} e^{xt} = \sum_{n=0}^{\infty} G_n^F(x;\lambda)\frac{t^n}{n!}. \tag{1.10}$$

Taking $\lambda = -1$ in (1.10), we get the Genocchi polynomials

$$\frac{2t}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \mid t \mid < \pi. \tag{1.11}$$

The following section provides some identities and properties of (p, q)-Frobenius-Genocchi numbers of order α involving addition property, difference equations, derivative properties, recurrence relationships. We also provide integral representations, implicit and explicit formulas and relations for mentioned polynomials and numbers. By using generating function of the polynomial stated in Definition (2.1), we derive some relationship for (p, q)-Frobenius Genocchi polynomials of order α related to (p, q)-Bernoulli polynomials, the (p, q)-Euler polynomials and the (p, q)-Genocchi polynomials.

2. Definition and properties of the (p,q)-Frobenius-Genocchi polynomials of order α, $g_n^{(\alpha)}(x;u:p,q)$

In this section, we introduce and investigate (p,q)-Frobenius-Genocchi polynomials of order α and its properties.

Definition 2.1. The (p,q)-Frobenius-Genocchi polynomials $g_n^{(\alpha)}(x;u:p,q)$ of order α are defined by means of the following generating function:

$$\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} e_{p,q}(xt)=\sum_{n=0}^{\infty} g_n^{(\alpha)}(x;u:p,q)\frac{t^n}{[n]_{p,q}!}, \tag{2.1}$$

where α is suitable (real or complex) parameter, $p,q\in\mathbb{C}$ with $0<\mid q\mid<\mid p\mid\leq 1$ and $u\in\mathbb{C}/\{1\}$.

Remark 2.1. For $x=0$ and $\alpha=1$ in (2.1), the result reduces to

$$\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)=\sum_{n=0}^{\infty} g_n(u:p,q)\frac{t^n}{[n]_{p,q}!}, \tag{2.2}$$

where $g_n(u:p,q)$ denotes the (p,q)-Frobenius-Genocchi number.

Remark 2.2. On setting $u=-1$, equation (2.1) reduces to

$$\left(\frac{2t}{e_{p,q}(t)+1}\right)^{\alpha} e_{p,q}(xt)=\sum_{n=0}^{\infty} G_n^{(\alpha)}(x:p,q)\frac{t^n}{[n]_{p,q}!}, \tag{2.3}$$

where $G_n^{(\alpha)}(x:p,q)$ denotes the (p,q)-Genocchi polynomials of order α, (see [6]).

From (2.1), we have

$$\begin{aligned}
&g_n^{(1)}(x;u:p,q):=g_n(x;u:p,q),\\
&g_n^{(\alpha)}(x;u:p,q)|_{p=1}:=g_{n,q}^{(\alpha)}(x;u),\ \text{(see [17])},\\
&\lim_{\substack{p=1\\ q\to 1^-}} g_n^{(\alpha)}(x;u:p,q)=G_n^{(\alpha)}(x;u),\ \text{(see [13, 14])}.
\end{aligned}$$

From Definition (2.1), we give the following theorems:

Theorem 2.2. *The following relationship holds true:*

$$g_{p,q}^{(\alpha)}(D_{p,q})x^n=\sum_{k=0}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)_{p,q} g_k^{(\alpha)}(u:p,q)x^{n-k}. \tag{2.4}$$

Proof. By means of the (p,q)-derivative operator $D_{p,q}$, we have

$$g_{p,q}^{(\alpha)}(D_{p,q})x^n = g_{p,q}^{(\alpha)}\left(\frac{\partial}{\partial_{p,q}x}\right)x^n = \sum_{k=0}^{\infty}\frac{g_n^{(\alpha)}(u:p,q)}{[k]_{p,q!}}\left(\frac{\partial}{\partial_{p,q}x}\right)^k x^n$$
$$= \sum_{k=0}^{n} g_k^{(\alpha)}(u:p,q)\frac{[n]_{p,q}!}{[k]_{p,q!}[n-k]_{p,q}!}x^{n-k}$$
$$= \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_{p,q} g_k^{(\alpha)}(u:p,q)x^{n-k}.$$

Therefore, we complete proof ot Theorem 2.2. $\square$

Here, we state a relationship of (p,q)-Frobenius-Genocchi polynomials of order α and (p,q)-Frobenius-Genocchi numbers of order α.

Theorem 2.3. *The following relationship holds true:*

$$g_n^{(\alpha)}(x;u:p,q) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_{p,q} p^{\binom{n-k}{2}} g_k^{(\alpha)}(u:p,q)x^{n-k}. \tag{2.5}$$

Proof. By using (2.1), we have

$$\sum_{n=0}^{\infty} g_n^{(\alpha)}(x;u:p,q)\frac{t^n}{[n]_{p,q}!} = \left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} e_{p,q}(xt)$$
$$= \sum_{k=0}^{n} g_k^{(\alpha)}(u:p,q)\frac{t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty} p^{\binom{n}{2}} x^n \frac{t^n}{[n]_{p,q}!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_{p,q} p^{\binom{n-k}{2}} g_k^{(\alpha)}(u:p,q)x^{n-k}\right)\frac{t^n}{[n]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$ of both sides, we arrive at the desired result (2.5).

$\square$

Corollary 2.4. *In the case $x=1$ in Theorem 2.3, we have*

$$g_n^{(\alpha)}(1;u:p,q) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_{p,q} p^{\binom{n-k}{2}} g_k^{(\alpha)}(u:p,q). \tag{2.6}$$

Remark 2.3. The (p,q)-generalization of the following formula:

$$g_n^{(\alpha)}(1;u) = \sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right) g_k^{(\alpha)}(u). \tag{2.7}$$

Note that

$$g_n^{(0)}(x;u:p,q) = p^{\binom{n}{2}} x^n.$$

Theorem 2.5. *The following formula holds true:*

$$g_n^{(\alpha)}((x+y)_{p,q};u:p,q)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}y^{n-k}p^{\binom{n-k}{2}}g_k^{(\alpha)}(x;u:p,q). \tag{2.8}$$

$$g_n^{(\alpha+\beta)}(x;u:p,q)=\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}g_k^{(\alpha)}(x;u:p,q)g_{n-k}^{(\beta)}(u:p,q). \tag{2.9}$$

$$\frac{\partial}{\partial_{p,q}x}g_n^{(\alpha)}(x;u:p,q)=[n]_{p,q}g_{n-1}^{(\alpha)}(px;u:p,q). \tag{2.10}$$

Proof. Using definition 2.1 and differentiating generating function (2.1) with respect to x with the help of equation (1.2) and then simplifying with the help of the Cauchy product, formulas (2.8)-(2.10) are obtained. □

Theorem 2.6. *(Difference equation) For $n \geq 1$, we have*

$$(1-u)g_{n-1}^{(\alpha-1)}(u:p,q)=g_n^{(\alpha)}(1;u:p,q)-ug_n^{(\alpha)}(u:p,q). \tag{2.11}$$

Proof. We can easily derive by using the equation (2.1). We omit the proof. □

Theorem 2.7. *(Recurrence relationship) $g_n^{(\alpha)}(x;u:p,q)$ fulfills the following equality:*

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}p^{\binom{n-k}{2}}g_n^{(\alpha)}(x;u:p,q)-ug_n^{(\alpha)}(x;u:p,q)=(1-u)g_n^{(\alpha-1)}(x;u:p,q). \tag{2.12}$$

Proof. Using generating function (2.1), we get (2.12).

□

Corollary 2.8. *For $\alpha = 1$ in Theorem 2.7, we have*

$$\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)_{p,q}p^{\binom{n-k}{2}}g_n(x;u:p,q)-ug_n(x;u:p,q)=(1-u)x^np^{\binom{n}{2}}. \tag{2.13}$$

3. Main results

In this section, we derive implicit and explicit formulas, integral representations, some identities for $g_n^{(\alpha)}(x;u:p,q)$. Also, we present new theorems and some (p,q)-extensions of known results in Carlitz [1], Kurt [9], Simsek [16, 17] and so on. We start with the following explicit formula for (p,q)-Frobenius-Genocchi polynomials of order α by the following theorem.

Theorem 3.1. *The following implicit summation formula holds true:*

$$\begin{aligned} &g_{k+l}^{(\alpha)}(z;u:p,q)\\ &=\sum_{n,m=0}^{k,l}\begin{pmatrix} l\\ m\end{pmatrix}_{p,q}\begin{pmatrix} k\\ n\end{pmatrix}_{p,q}p^{\binom{n+m}{2}}(z-x)^{m+n}g_{k-n,l-m}^{(\alpha)}(x;u:p,q). \end{aligned}\tag{3.1}$$

Proof. Replacing t with $(t+w)$ in (2.1) and using result [18, p.52, Eq.2], we get

$$\left(\frac{(1-u)(t+w)}{e_{p,q}(t+w)-u}\right)^{\alpha}=e_{p,q}(-x(t+w))\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(x;u:p,q)\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}.\tag{3.2}$$

Replacing x by z, and equating the obtained equation with the above equation, we arrive at

$$\begin{aligned} &e_{p,q}((z-x)(t+w))\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(x;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}\\ &=\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(z;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}. \end{aligned}\tag{3.3}$$

Expanding the exponent part in the above equation, we have

$$\begin{aligned} &\sum_{N=0}^{\infty}\frac{[(z-x)(t+w)]^N p^{\binom{N}{2}}}{[N]_{p,q}!}\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(x;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}\\ &=\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(z;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}. \end{aligned}\tag{3.4}$$

From equation (3.4) we can derive the following equation.

$$\begin{aligned} &\sum_{n,m=0}^{\infty}\frac{(z-x)^{(n+m)}p^{\binom{n+m}{2}}t^n w^m}{[n]_{p,q}![m]_{p,q}!}\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(x;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}\\ &=\sum_{k,l=0}^{\infty}g_{k+l}^{(\alpha)}(z;u:p,q))\frac{t^k}{[k]_{p,q}!}\frac{w^l}{[l]_{p,q}!}. \end{aligned}\tag{3.5}$$

Using use of Lemma [18, p.100, Eq.2] and then on comparing the coefficients of t^k and w^l, we get the required result. □

Corollary 3.2. *For $l=0$ in Theorem 3.1, we get*

$$g_{k}^{(\alpha)}(z;u:p,q)=\sum_{n=0}^{k}\begin{pmatrix} k\\ n\end{pmatrix}_{p,q}p^{\binom{n}{2}}(z-x)^{n}g_{k-n}^{(\alpha)}(x;u:p,q).\tag{3.6}$$

Theorem 3.3. *The following (p,q)-integral is valid*

$$\int_a^b g_n^{(\alpha)}(x;u:p,q)d_{p,q}x=\frac{g_{n+1}^{(\alpha)}(\frac{b}{p};u:p,q)-g_{n+1}^{(\alpha)}(\frac{a}{p};u:p,q)}{[n+1]_{p,q}}.\tag{3.7}$$

Proof. Since

$$\int_a^b \frac{\partial}{\partial_{p,q}x} g_n^{(\alpha)}(x;u:p,q)d_{p,q}x = f(b)-f(a), \text{ (see [19])}$$

in terms of equation (2.10) and equations (1.5) and (1.6), we arrive at the asserted result

$$\int_a^b \frac{\partial}{\partial_{p,q}x} g_n^{(\alpha)}(x;u:p,q)d_{p,q}x = \frac{1}{[n+1]_{p,q}}\int_a^b g_n^{(\alpha)}(\frac{x}{p};u:p,q)d_{p,q}x$$
$$= \frac{g_{n+1}^{(\alpha)}(\frac{b}{p};u:p,q) - g_{n+1}^{(\alpha)}(\frac{a}{p};u:p,q)}{[n+1]_{p,q}}.$$

This completes the proof of this theorem. □

Theorem 3.4. *The following result holds true:*

$$(2u-1)\sum_{k=0}^{n}\binom{n}{k}_{p,q} g_k(u:p,q)g_{n-k}(x;1-u:p,q)$$
$$= ug_n(x;u:p,q) - (1-u)g_n(x;1-u:p,q). \tag{3.8}$$

Proof. By utilizing the same method of Duran et al. [5] and Kurt [9], we first consider the identity

$$\frac{2u-1}{(e_{p,q}(t)-u)(e_{p,q}(t)-(1-u))} = \frac{1}{e_{p,q}(t)-u} - \frac{1}{e_{p,q}(t)-(1-u)},$$

then we have

$$(2u-1)\frac{(1-u)e_{p,q}(xt)(1-(1-u)t)}{(e_{p,q}(t)-u)(e_{p,q}(t)-(1-u))}$$
$$= u\frac{(1-u)te_{p,q}(xt)}{e_{p,q}(t)-u} - \frac{(1-u)te_{p,q}(xt)(1-(1-u)t)}{e_{p,q}(t)-(1-u)}.$$

Using generating function, we can represent

$$(2u-1)\sum_{k=0}^{\infty} g_k(u:p,q)\frac{t^k}{[k]_{p,q}!}\sum_{n=0}^{\infty} g_n(x;1-u:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$= u\sum_{n=0}^{\infty} g_n(x;u:p,q)\frac{t^n}{[n]_{p,q}!} - (1-u)\sum_{n=0}^{\infty} g_n(x;1-u:p,q)\frac{t^n}{[n]_{p,q}!}.$$

On comparing the coefficient of t^n, we arrive at the required result (3.8). □

Theorem 3.5. *Each of the following relationships holds true:*

$$g_n^{(\alpha)}(x;u:p,q)$$
$$= \sum_{s=0}^{n+1}\binom{n+1}{s}_{p,q}\left[\sum_{k=0}^{s}\binom{s}{k}_{p,q} B_{s-k}(x;p,q)p^{\binom{k}{2}} - B_s(x;p,q)\right]\frac{g_{n+1-s}^{(\alpha)}(u:p,q)}{[n+1]_{p,q}}, \tag{3.9}$$

where $B_n(x;p,q)$ is (p,q)-Bernoulli polynomials.

Proof. By using definition (2.1), we have

$$\begin{aligned}
&\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} e_{p,q}(xt)\\
&=\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} \frac{t}{e_{p,q}(t)-u}\frac{e_{p,q}(t)-u}{t}e_{p,q}(xt)\\
&=\frac{1}{t}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)_{p,q} B_{n-k}(x;p,q)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} g_n^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}\\
&-\frac{1}{t}\sum_{n=0}^{\infty}B_n(x;p,q)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty} g_n^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}\\
&=\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n\\ s\end{array}\right)_{p,q}\sum_{k=0}^{s}\left(\begin{array}{c} s\\ k\end{array}\right)_{p,q} B_{s-k}(x;p,q)p^{\binom{k}{2}}\right] g_{n-s}^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}\\
&-\frac{1}{t}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n\\ s\end{array}\right)_{p,q} B_s(x;p,q)\right] g_{n-s}^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

By using Cauchy product and comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the required result (3.9). □

Theorem 3.6. *Each of the following relationships holds true:*

$$\begin{aligned}
&g_n^{(\alpha)}(x;u:p,q)\\
&=\sum_{s=0}^{n}\left(\begin{array}{c} n\\ s\end{array}\right)_{p,q}\left[\sum_{k=0}^{s}\left(\begin{array}{c} s\\ k\end{array}\right)_{p,q} E_{s-k}(x;p,q)p^{\binom{k}{2}}+E_s(x;p,q)\right]\frac{g_{n-s}^{(\alpha)}(u:p,q)}{[2]_{p,q}},
\end{aligned} \tag{3.10}$$

where $E_n(x;p,q)$ is (p,q)-Euler polynomials.

Proof. By using definition (2.1), we have

$$\begin{aligned}
&\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} e_{p,q}(xt)=\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha}\frac{[2]_{p,q}}{e_{p,q}(t)+1}\frac{e_{p,q}(t)+1}{[2]_{p,q}}e_{p,q}(xt)\\
&=\frac{1}{[2]_{p,q}}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n\\ k\end{array}\right)_{p,q} E_{n-k}(x;p,q)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}E_n(x;p,q)\frac{t^n}{[n]_{p,q}!}\right]\\
&\times\sum_{n=0}^{\infty} g_n^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}\\
&=\frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n\\ s\end{array}\right)_{p,q}\sum_{k=0}^{s}\left(\begin{array}{c} s\\ k\end{array}\right)_{p,q} E_{s-k}(x;p,q)p^{\binom{k}{2}}+\sum_{s=0}^{n}\left(\begin{array}{c} n\\ s\end{array}\right)_{p,q} E_s(x;p,q)\right]\\
&\times g_{n-s}^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, we arrive at the desired result (3.10). $\square$

Theorem 3.7. *Each of the following relationships holds true:*

$$g_n^{(\alpha)}(x;u:p,q)$$
$$=\sum_{s=0}^{n}\left(\begin{array}{c} n+1 \\ s \end{array}\right)_{p,q}\left[\sum_{k=0}^{s}\left(\begin{array}{c} s \\ k \end{array}\right)_{p,q} G_{s-k}(x;p,q)p^{\binom{k}{2}}+G_s(x;p,q)\right]\frac{g_{n-s}^{(\alpha)}(u:p,q)}{[2]_{p,q}[n+1]_{p,q}}, \tag{3.11}$$

where $G_n(x;p,q)$ is (p,q)-Genocchi polynomials.

Proof. By using definition (2.1), we have

$$\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha} e_{p,q}(xt)=\left(\frac{(1-u)t}{e_{p,q}(t)-u}\right)^{\alpha}\frac{[2]_{p,q}t}{e_{p,q}(t)+1}\frac{e_{p,q}(t)+1}{[2]_{p,q}t}e_{p,q}(xt)$$
$$=\frac{1}{[2]_{p,q}t}\left[\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)_{p,q} G_{n-k}(x;p,q)p^{\binom{k}{2}}\right)\frac{t^n}{[n]_{p,q}!}+\sum_{n=0}^{\infty}G_n(x;p,q)\frac{t^n}{[n]_{p,q}!}\right]$$
$$\times\sum_{n=0}^{\infty}g_n^{(\alpha)}(u:p,q)\frac{t^n}{[n]_{p,q}!}$$
$$=\frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left[\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)_{p,q}\sum_{k=0}^{s}\left(\begin{array}{c} s \\ k \end{array}\right)_{p,q} G_{s-k}(x;p,q)p^{\binom{n}{2}}+\sum_{s=0}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)_{p,q} G_s(x;p,q)\right]$$
$$\times\, g_{n+1-s}^{(\alpha)}(u:p,q)\frac{t^n}{[n+1]_{p,q}!}.$$

Comparing the coefficients of $\frac{t^n}{[n]_{p,q}!}$, then we have the asserted result (3.11). $\square$

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