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# RESULTS ON UNIQUENESS OF DIFFERENTIAL MONOMIAL AND DIFFERENTIAL POLYNOMIAL

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Abstract. In this paper, we study the results concerning uniqueness of meromorphic functions sharing a small function and present one theorem which extend and improves previous results.

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## 1. Introduction

In this paper, a meromorphic function always mean a function which is meromorphic in the complex plane $\mathbb{C}$. We use the standard notations of Nevanlinna's theory such as $m(r, f)$, $N(r, f)$, $T(r, f)$ etc. (See [6], [8]). By $S(r, f)$, we mean any quantity satisfying $S(r, f) = o\{T(r, f)\}$ as $r \to \infty$ possibly outside a set of finite linear measure. A meromorphic function $a = a(z)$ is called a small function of $f$ if $T(r, a) = S(r, f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions, $a \in \mathbb{C} \cup \{\infty\}$. We say that, $f$ and $g$ share the value $a$ CM if $f - a$ and $g - a$ have the same zeros with same multiplicities. Also, $f$ and $g$ share the value $a$ IM if $f - a$ and $g - a$ have the same zeros ignoring multiplicities.

**Definition 1.1.** Let $E(a, f) = \{z \in \mathbb{C} : f(z) - a(z) = 0\}$, where a zero of $f - a$ is counted according to its multiplicity. Also, $\overline{E}(a, f)$, is the zeros of $f - a$, where a zero is counted only once. For a non-negative integer $k$, we denote by $E_l(a, f)$ the set of all zeros of $f - a$, where a zero of multiplicity $'m'$ is counted $'m'$ times if $m \leq l$ and $l + 1$ times if $m > l$. If $E_l(a, f) = E_l(a, g)$, then $f$ and $g$ share the function $a$ with weight $l$.

We write $f$ and $g$ share $(a, l)$ to mean that "$f$ and $g$ share the function $a$ with weight $l$". Since $E_l(a, f) = E_l(a, g)$ implies that $E_p(a, f) = E_p(a, g)$ for any

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integer $p$ $(0 \leq p < l)$, if $f$ and $g$ share $(a, l)$, then $f$ and $g$ share $(a, p)$, $0 \leq p < l$. We also note that $f$ and $g$ share $a$ IM(ignoring multiplicity) or CM(counting multiplicity) iff $f$ and $g$ share $(a, 0)$ or $(a, \infty)$, respectively.

We use the following notations:

Let $f$ and $g$ share 1 IM, and let $z_0$ be a zero of $f-1$ of order $p$ and a zero of $g-1$ of order $q$. We denote by $N_E^{1)}\left(r, \frac{1}{f-1}\right)$ the counting function of those 1-points of $f$ and $g$, where $p = q = 1$. By $\overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right)$ we denote the counting function of those 1-points of $f$ and $g$, where $p = q \geq 2$. Also, $\overline{N}_L\left(r, \frac{1}{f-1}\right)$ denotes the counting function of those 1-points of both $f$ and $g$, where $p > q \geq 1$; each point in these counting functions is counted only once. Similarly, we denote the terms $N_E^{1)}\left(r, \frac{1}{g-1}\right)$, $\overline{N}_E^{(2}\left(r, \frac{1}{g-1}\right)$ and $\overline{N}_L\left(r, \frac{1}{g-1}\right)$. In addition, we denote by $\overline{N}_{f>k}\left(r, \frac{1}{g-1}\right)$ the reduced counting function of those zeros of $f-1$ and $g-1$ such that $p > q = k$, and similarly the term $\overline{N}_{g>k}\left(r, \frac{1}{f-1}\right)$ is defined.

**Definition 1.2.** Let $n_0, n_1, ..., n_k$ be non-negative integers. The expression $M[f] = (f)^{n_0}\left(f^{(1)}\right)^{n_1} ... \left(f^{(k)}\right)^{n_k}$ is called a differential monomial generated by $f$ of degree $d_M = \sum_{i=0}^{k} n_i$ and weight $\Gamma_M = \sum_{i=0}^{k}(i+1)n_i$ and $k$ is the highest derivative in $M[f]$. We also denote by, $\lambda = \Gamma_M - d_M = \sum_{i=0}^{k} in_i$.

A differential polynomial $P[f]$ of a non-constant meromorphic function $f$ is defined as $P[f] = \sum_{i=1}^{m} M_i[f]$, where $M_i[f] = a_i \prod_{j=0}^{k}\left(f^{(j)}\right)^{n_{ij}}$ with $n_{i0}, n_{i1}, ..., n_{ik}$ as non-negative integers and $a_i(\not\equiv 0)$ are meromorphic functions satisfying $T(r, a_i)$ $= o(T(r, f))$ as $r \to \infty$. The numbers $\overline{d}(P) = max_{1 \leq i \leq m} \sum_{j=0}^{k} n_{ij}$ and $\underline{d}(P) = min_{1 \leq i \leq m} \sum_{j=0}^{k} n_{ij}$ are respectively called the degree and lower degree of $P[f]$.

If $\overline{d}(P) = \underline{d}(P) = d$(say), then $P[f]$ is called homogeneous differential polynomial of degree $d$. Also, we denote by $Q = max\{\Gamma_{M_j} - d(M_j) : 0 \leq j \leq k\} = max_{0 \leq j \leq k}\{n_{1j} + 2n_{2j} + ... + kn_{kj}\}$.

In 2014, Banerjee and Majumder [3] considered the weighted sharing of $f^n$ and $(f^m)^{(k)}$ and proved the following:

**Theorem 1.3.**[3] *Let $f$ be a non-constant meromorphic function , $k, n, m \in \mathbb{N}$ and $l$ be a non-negative integer. Suppose $a(\not\equiv 0, \infty)$ is a meromorphic function satisfying $T(r, a) = o(T(r, f))$ as $r \to \infty$ such that $f^n$ and $(f^m)^{(k)}$ share $(a, l)$. If $l \geq 2$ and*

$$(k+3)\Theta(\infty, f) + (k+4)\Theta(0, f) > 2k + 7 - n,$$

*or $l = 1$ and*

$$\left(k + \frac{7}{2}\right)\Theta(\infty, f) + \left(k + \frac{9}{2}\right)\Theta(0, f) > 2k + 8 - n,$$

*or* $l=0$ *and*

$$(2k+6)\Theta(\infty,f)+(2k+7)\Theta(0,f)>4k+13-n,$$

*then* $f^n=(f^m)^{(k)}$.

In 2016, the authors Kuldeep Singh Charak and Banarsi Lal [5], proved the following result on uniqueness of $p(f)$ and $P[f]$:

**Theorem 1.4.**[5] Let $f$ be a non-constant meromorphic function, $a(\not\equiv 0,\infty)$ be a meromorphic function satisfying $T(r,a)=o(T(r,f))$ as $r\to\infty$, and $p(z)$ be a polynomial of degree $n\geq 1$ with $p(0)=0$. Let $P[f]$ be a non-constant differential polynomial of $f$. Suppose $p(f)$ and $P[f]$ share $(a,l)$ with one of the following conditions:

(i) $l\geq 2$ and

$$(Q+3)\Theta(\infty,f)+2n\Theta(0,p(f))+\overline{d}(P)\delta(0,f)>Q+3+2\overline{d}(P)-\underline{d}(P)+n,$$

(ii) $l=1$ and

$$\left(Q+\frac{7}{2}\right)\Theta(\infty,f)+\frac{5n}{2}\Theta(0,p(f))+\overline{d}(P)\delta(0,f)>Q+\frac{7}{2}+2\overline{d}(P)-\underline{d}(P)+\frac{3n}{2},$$

(iii) $l=0$ and

$$(2Q+6)\Theta(\infty,f)+4n\Theta(0,p(f))+2\overline{d}(P)\delta(0,f)>2Q+6+4\overline{d}(P)-2\underline{d}(P)+3n.$$

Then, $p(f)\equiv P[f]$.

In 2018, Harina P. Waghamore and Naveenkumar S.H. [7], proved the uniqueness of meromorphic functions of the form $\mathcal{P}(f)=f_1^pP(f_1)-a$ and $H[f]-a$ and obtained the following result:

**Theorem 1.5.**[7] Let $k(\geq 1),n(\geq 1),p(\geq 1)$ and $m(\geq 0)$ be integers and $f$ and $f_1=f-w_p$ be two non-constant meromorphic functions and $H[f]$ be a non-constant differential polynomial generated by $f$. Let $\mathcal{P}(z)=a_{m+n}z^{m+n}+...+a_nz^n+...+a_0$, $a_{m+n}\neq 0$ be a polynomial in $z$ of degree $m+n$ such that $\mathcal{P}(f)=f_1^pP(f_1)$. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. Suppose $\mathcal{P}(f)-a$ and $H[f]-a$ share $(0,l)$. If $l\geq 2$ and

$$(Q+3)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}(0,f)>Q+3+\mu_2+\overline{d}(H)-p,$$

or $l=1$ and

$$\left(Q+\frac{7}{2}\right)\Theta(\infty,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}\left(r,\frac{1}{f}\right)+\frac{1}{2}\Theta(w_p,f)$$
$$>Q+4+\mu_2+\overline{d}(H)+\frac{m+n-3p}{2},$$

or $l=0$ and

$$(2Q+6)\Theta(\infty,f)+2\Theta(w_p,f)+\mu_2\delta_{\mu_2^*}(w_p,f)+\overline{d}(H)\delta_{k+2}(0,f)$$
$$+\overline{d}(H)\delta_{k+1}(0,f)>2Q+8+\mu_2+2\overline{d}(H)+2(m+n)-3p,$$

then $\mathcal{P}(f) \equiv H[f]$.

## 2. Main results

In this paper, our interest is to investigate the uniqueness of a monomial and differential polynomial. In general this is not true, but under essential conditions, we prove the result, which extend and improves the previous results.

The following theorem is our main result:

**Theorem 2.1.** *Let $f$ be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree $d_M$ and weight $\Gamma_M$ and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant differential polynomial of $f$. Also, let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to $f$. Suppose $M[f]-a$ and $P[f]-a$ share $(0, l)$. If*
*$l \geq 2$ and*

$$\begin{aligned}(Q+3\lambda+3)\Theta(\infty, f)+\lambda\Theta(0, f)+2d_M\delta_{1+k}(0, f)+\overline{d}(P)\delta(0, f)\\ > Q+4\lambda+3+d_M+2\overline{d}(P)-\underline{d}(P),\end{aligned} \tag{1}$$

*or $l = 1$ and*

$$\begin{aligned}\left(\frac{2Q+7\lambda+7}{2}\right)\Theta(\infty, f)+\lambda\Theta(0, f)+\frac{5d_M}{2}\delta_{1+k}(0, f)+\overline{d}(P)\delta(0, f)\\ > \frac{2Q+9\lambda+7}{2}+\frac{3d_M}{2}+2\overline{d}(P)-\underline{d}(P),\end{aligned} \tag{2}$$

*or $l = 0$ and*

$$\begin{aligned}(2Q+5\lambda+6)\Theta(\infty, f)+\lambda\Theta(0, f)+4d_M\delta_{1+k}(0, f)+2\overline{d}(P)\delta(0, f)\\ > 2Q+6\lambda+6+3d_M+4\overline{d}(P)-2\underline{d}(P).\end{aligned} \tag{3}$$

*Then, $M[f] \equiv P[f]$.*

*Proof.* Let $F = \frac{M[f]}{a}$ and $G = \frac{P[f]}{a}$. Then,

$$F-1 = \frac{M[f]-a}{a}, \quad G-1 = \frac{P[f]-a}{a}.$$

Since $M[f]$ and $P[f]$ share $(a, l)$, it follows that $F$ and $G$ share $(1, l)$ except the zeros and poles of $a$. Also note that,

$$\overline{N}(r, F) = \overline{N}(r, f)+S(r, f) \quad \textit{and} \quad \overline{N}(r, G) = \overline{N}(r, f)+S(r, f).$$

Define

$$\psi = \left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right). \tag{4}$$

**Claim.** $\psi \equiv 0$.
Suppose on contrary, we have $\psi \not\equiv 0$. Then from (4), we have $m(r,\psi) = S(r,f)$. By the second fundamental theorem of Nevanlinna, we have

$$T(r,F) + T(r,G) \le 2\overline{N}(r,f) + \overline{N}\left(r,\frac{1}{F}\right) + \overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) - N_0\left(r,\frac{1}{F'}\right) - N_0\left(r,\frac{1}{G'}\right) + S(r,f), \quad (5)$$

where $N_0\left(r,\frac{1}{F'}\right)$ is the counting function of the zeros of $F^{'}$, which are not the zeros of $F(F-1)$ and $N_0\left(r,\frac{1}{G'}\right)$ denotes the counting function of the zeros of $G^{'}$, which are not the zeros of $G(G-1)$.

**Case 1.** When $l \ge 1$.
Then, from (4), we have

$$N_E^{1)}\left(r,\frac{1}{F-1}\right) \le N\left(r,\frac{1}{\psi}\right) + S(r,f) \le T(r,\psi) + S(r,f) \le N(r,\psi) + S(r,f),$$

where

$$N(r,\psi) \le \overline{N}(r,F) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + N_0\left(r,\frac{1}{F'}\right) + N_0\left(r,\frac{1}{G'}\right) + S(r,f)$$

and so,

$$\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) = N_E^{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) + S(r,f).$$

Since

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right) &= N_E^{1)}\left(r,\frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{F-1}\right) + \overline{N}_L\left(r,\frac{1}{G-1}\right) \\ &= \overline{N}\left(r,\frac{1}{G-1}\right).\end{aligned}$$

$$\overline{N}\left(r,\frac{1}{F-1}\right) + \overline{N}\left(r,\frac{1}{G-1}\right) \le \overline{N}(r,f) + \overline{N}_{(2}\left(r,\frac{1}{F}\right) + \overline{N}_{(2}\left(r,\frac{1}{G}\right) + 2\overline{N}_L\left(r,\frac{1}{F-1}\right) + 2\overline{N}_L\left(r,\frac{1}{G-1}\right) + \overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)$$

$$+\overline{N}\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f). \tag{6}$$

**Subcase 1.1.** When $l=1$.
In this case, we have

$$\overline{N}_L\left(r,\frac{1}{F-1}\right)\leq\frac{1}{2}N\left(r,\frac{1}{F'}|F\neq 0\right)\leq\frac{1}{2}\overline{N}(r,F)+\frac{1}{2}\overline{N}\left(r,\frac{1}{F}\right), \tag{7}$$

where $N\left(r,\frac{1}{F'}|F\neq 0\right)$ denotes the zeros of $F^{'}$, which are not the zeros of $F$. By using Lemma 2.7, $F$ and $G$ share $(1,1)$, then

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)-\overline{N}_{F>2}\left(r,\frac{1}{G-1}\right)\\ \leq N\left(r,\frac{1}{G-1}\right)-\overline{N}\left(r,\frac{1}{G-1}\right).\end{aligned} \tag{8}$$

By (7) and (8), we have

$$\begin{aligned}2\overline{N}_L\left(r,\frac{1}{F-1}\right)&+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)+\overline{N}_{F>2}\left(r,\frac{1}{G-1}\right)\\ &\leq N\left(r,\frac{1}{G-1}\right)+\overline{N}_L\left(r,\frac{1}{F-1}\right)+S(r,f)\\ &\leq N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}N_1\left(r,\frac{1}{M[f]}\right)+S(r,f).\end{aligned} \tag{9}$$

Then, from (6) and (9), we have

$$\begin{aligned}\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\ &+N\left(r,\frac{1}{G-1}\right)+\frac{1}{2}\overline{N}(r,f)+\frac{1}{2}N_1\left(r,\frac{1}{M[f]}\right)\\ &+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\\ &\leq\frac{3}{2}\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+T(r,G)\\ &+\frac{1}{2}N_1\left(r,\frac{1}{M[f]}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\end{aligned} \tag{10}$$

From (4) and (10), we obtain

$$T(r,F)\leq\frac{7}{2}\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}\left(r,\frac{1}{G}\right)$$

$$+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+\frac{1}{2}N_1\left(r,\frac{1}{M[f]}\right)+S(r,f)$$

$$T(r,M)\leq\frac{7}{2}\overline{N}(r,f)+\frac{5}{2}N_1\left(r,\frac{1}{M[f]}\right)+N\left(r,\frac{1}{P[f]}\right)+S(r,f).$$

By using Lemmas 2.5, 2.6 and 2.8, we get

$$\begin{aligned}
&d_MT(r,f)-\lambda\overline{N}(r,f)-\lambda\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)\\
&\leq\left(Q+\frac{7}{2}\right)\overline{N}(r,f)+\frac{5d_M}{2}N_{1+k}\left(r,\frac{1}{f}\right)+\frac{5\lambda}{2}\overline{N}(r,f)\\
&+(\overline{d}(P)-\underline{d}(P))T(r,f)+\overline{d}(P)N\left(r,\frac{1}{f}\right)+S(r,f)
\end{aligned}$$

$$\begin{aligned}
\left(\frac{2Q+7\lambda+7}{2}\right)\Theta(\infty,f)+\lambda\Theta(0,f)+\frac{5d_M}{2}\delta_{1+k}(0,f)+\overline{d}(P)\delta(0,f)&\\
\leq\frac{2Q+9\lambda+7}{2}+\frac{3d_M}{2}+2\overline{d}(P)-\underline{d}(P),&
\end{aligned}$$

which is a contradiction to (2).

**Subcase 1.2.** When $l\geq 2$.
In this case, we have

$$\begin{aligned}
2\overline{N}_L\left(r,\frac{1}{F-1}\right)+2\overline{N}_L\left(r,\frac{1}{G-1}\right)+\overline{N}_E^{(2}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\\
\leq N\left(r,\frac{1}{G-1}\right)+S(r,f).&
\end{aligned}$$

Thus, from (6), we have

$$\begin{aligned}
\overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{G-1}\right)&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)\\
&+N\left(r,\frac{1}{G-1}\right)+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f)\\
&\leq\overline{N}(r,f)+\overline{N}_{(2}\left(r,\frac{1}{F}\right)+\overline{N}_{(2}\left(r,\frac{1}{G}\right)+T(r,G)\\
&+N_0\left(r,\frac{1}{F'}\right)+N_0\left(r,\frac{1}{G'}\right)+S(r,f) \qquad (11)
\end{aligned}$$

From (5) and (11), we have

$$T(r,F)\leq 3\overline{N}(r,f)+2\overline{N}\left(r,\frac{1}{F}\right)+N\left(r,\frac{1}{G}\right)+S(r,f).$$

By using Lemmas 2.5, 2.6 and 2.8, we get

$$d_MT(r,f)-\lambda\overline{N}(r,f)-\lambda\overline{N}\left(r,\frac{1}{f}\right)+S(r,f)$$

$$\le 3\overline{N}(r,f) + 2N_1\left(r, \frac{1}{M[f]}\right) + N\left(r, \frac{1}{P[f]}\right) + S(r,f).$$
$$\le (Q+3\lambda+3)\overline{N}(r,f) + 2d_M N_{1+k}\left(r, \frac{1}{f}\right) + \overline{d}(P) N\left(r, \frac{1}{f}\right)$$
$$+ S(r,f)$$

$$(Q+3\lambda+3)\Theta(\infty,f) + \lambda\Theta(0,f) + 2d_M\delta_{1+k}(0,f) + \overline{d}(P)\delta(0,f)$$
$$\le Q + 4\lambda + 3 + d_M + 2\overline{d}(P) - \underline{\mathrm{d}}(P),$$

which is a contradiction to (1).

**Case 2.** When $l = 0$.
Then, we have

$$N_E^{1)}\left(r, \frac{1}{F-1}\right) = N_E^{1)}\left(r, \frac{1}{G-1}\right) + S(r,f),$$
$$\overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) = \overline{N}_E^{(2}\left(r, \frac{1}{G-1}\right) + S(r,f).$$

Also, from (4), we have

$$\overline{N}\left(r, \frac{1}{F-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right)$$
$$\le N_E^{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right)$$
$$+ \overline{N}_L\left(r, \frac{1}{G-1}\right) + \overline{N}\left(r, \frac{1}{G-1}\right) + S(r,f)$$
$$\le N_E^{1)}\left(r, \frac{1}{F-1}\right) + \overline{N}_L\left(r, \frac{1}{F-1}\right) + N\left(r, \frac{1}{G-1}\right) + S(r,f)$$
$$\le \overline{N}(r,f) + \overline{N}_{(2}\left(r, \frac{1}{F}\right) + \overline{N}_{(2}\left(r, \frac{1}{G}\right) + 2\overline{N}_L\left(r, \frac{1}{F-1}\right)$$
$$+ \overline{N}_L\left(r, \frac{1}{G-1}\right) + N\left(r, \frac{1}{G-1}\right) + N_0\left(r, \frac{1}{F'}\right) + N_0\left(r, \frac{1}{G'}\right) + S(r,f). \tag{12}$$

From Lemma 2.7 and (5), (12), we obtain

$$T(r,F) \le 6\overline{N}(r,f) + 4N_1\left(r, \frac{1}{M[f]}\right) + 2N\left(r, \frac{1}{P[f]}\right) + S(r,f).$$

By using Lemmas 2.5, 2.6 and 2.8, we get

$$d_M T(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r, \frac{1}{f}\right) + S(r,f)$$
$$\le 6\overline{N}(r,f) + 4d_M N_{1+k}\left(r, \frac{1}{f}\right) + 4\lambda\overline{N}(r,f) + 2Q\overline{N}(r,f)$$

$$+ 2(\overline{d}(P) - \underline{d}(P))T(r,f) + 2\overline{d}(P)N\left(r,\frac{1}{f}\right) + S(r,f).$$

$$(2Q+5\lambda+6)\Theta(\infty,f) + \lambda\Theta(0,f) + 4d_M\delta_{1+k}(0,f) + 2\overline{d}(P)\delta(0,f)$$
$$\leq 2Q + 6\lambda + 6 + 3d_M + 4\overline{d}(P) - 2\underline{d}(P),$$

which is a contradiction to (3).

Thus, when $\psi \not\equiv 0$, we get contradiction.

Hence, $\psi \equiv 0$.

By (4), we have

$$\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F-1} = \frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G-1},$$

and by integrating twice, we get

$$\frac{1}{F-1} = \frac{C}{G-1} + D, \tag{13}$$

where $C \neq 0$ and $D$ are constants.

Then, we have the following three cases :

**Case(i).** When $D \neq 0, -1$.
Rewriting (13) as

$$\frac{G-1}{C} = \frac{F-1}{D+1-DF}.$$

Then, we have

$$\overline{N}(r,G) = \overline{N}\left(r, \frac{1}{F - \frac{D+1}{D}}\right).$$

In this case, by using second fundamental theorem and lemmas 2.5, 2.8, we get

$$d_MT(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r,\frac{1}{f}\right) + S(r,f)$$
$$\leq \overline{N}(r,f) + \overline{N}(r,\frac{1}{F}) + \overline{N}\left(r,\frac{1}{F-\frac{D+1}{D}}\right) + S(r,f)$$
$$\leq \overline{N}(r,f) + N_1(r,\frac{1}{F}) + \overline{N}(r,G) + S(r,f)$$

$$d_MT(r,f) \leq (2\lambda+2)\overline{N}(r,f) + d_MN_{1+k}\left(r,\frac{1}{f}\right) + \lambda\overline{N}\left(r,\frac{1}{f}\right) + S(r,f)$$

$$(2\lambda+2)\Theta(\infty,f) + \lambda\Theta(0,f) + d_M\delta_{1+k}(0,f) \leq 3\lambda + 2,$$

which contradicts (1), (2) and (3).

**Case (ii).** When $D = 0$.
Then from (13), we have

$$G = CF - (C-1) \tag{14}$$

$$Therefore,\ \ if\ C \neq 1,\ \ then\ \ \overline{N}(r, \frac{1}{G}) = \overline{N}\left(r, \frac{1}{F - \frac{(C-1)}{C}}\right).$$

Now, by using second fundamental theorem and lemmas 2.5, 2.8, we get

$$\begin{aligned} d_M T(r,f) - \lambda \overline{N}(r,f) - \lambda \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C-1}{C}}\right) + S(r,f). \\ d_M T(r,f) &\leq (Q + 2\lambda + 1)\overline{N}(r,f) + \lambda \overline{N}\left(r, \frac{1}{f}\right) + d_M N_{1+k}\left(r, \frac{1}{f}\right) \\ &+ (\overline{d}(P) - \underline{d}(P))T(r,f) + \overline{d}(P) N\left(r, \frac{1}{f}\right) + S(r,f). \end{aligned}$$

$$\begin{aligned} (Q + 2\lambda + 1)\Theta(\infty, f) + \lambda\Theta(0,f) + d_M \delta_{1+k}(0,f) + \overline{d}(P)\delta(0,f) \\ \leq Q + 3\lambda + 1 + 2\overline{d}(P) - \underline{d}(P), \end{aligned}$$

which contradicts (1), (2) and (3).
Thus, $C = 1$ and so in this case from (14), we have $F \equiv G$.
Therefore, $M[f] \equiv P[f]$.

**Case(iii).** When $D = -1$.
Then, from (13), we have

$$\frac{1}{F-1} = \frac{C}{G-1} - 1 \tag{15}$$

Therefore if $C \neq -1$, then

$$\overline{N}(r, \frac{1}{G}) = \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right).$$

Now, by using second fundamental theorem and lemmas 2.5, 2.8, we get

$$\begin{aligned} d_M T(r,f) - \lambda \overline{N}(r,f) - \lambda \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \\ \leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{F}) + \overline{N}\left(r, \frac{1}{F - \frac{C}{C+1}}\right) + S(r,f). \end{aligned}$$

i.e.,

$$\begin{aligned} d_M T(r,f) - \lambda \overline{N}(r,f) - \lambda \overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \\ \leq \overline{N}(r,f) + N_1(r, \frac{1}{M[f]}) + \overline{N}\left(r, \frac{1}{P[f]}\right) + S(r,f), \end{aligned}$$

which is same as case (ii), which gives

$$(Q+2\lambda+1)\Theta(\infty,f)+\lambda\Theta(0,f)+d_M\delta_{1+k}(0,f)+\overline{d}(P)\delta(0,f)$$
$$\leq Q+3\lambda+1+2\overline{d}(P)-\underline{d}(P),$$

which contradicts (1), (2) and (3).

Therefore, $FG=1$.

By Lemma 2.9, we get a contradiction to $FG\not\equiv 1$.

Therefore, $F\equiv G$

i.e., $M[f]\equiv P[f]$.

Hence the proof of Theorem 2.1. □

**Example 2.1.** Let $f(z)=cos\alpha z+1-\frac{1}{\alpha^4}$, where $\alpha\neq 0,\pm 1,\pm i$. Let $M[f]=f^{(1)}$ and $P[f]=2f^{(1)}$. Then, we have $d_M=1,\Gamma_M=2$ and $\lambda=1$, also $\overline{d}(P)=1$, $\underline{d}(P)=1$, $\Gamma_{M_j}=2$, $d(M_j)=1$ and $Q=1$. Also, since $\overline{N}(r,f)=S(r,f)$, $\Theta(\infty,f)=1$ and $\overline{N}\left(r,\frac{1}{f}\right)=\overline{N}\left(r,\frac{1}{cos\alpha z+1-\frac{1}{\alpha^4}}\right)\sim T(r,f)$, $\Theta(0,f)=0$. Therefore $\delta(0,f)=\delta_{1+k}(0,f)=0$. Also, we know that $M[f]$ and $P[f]$ share $(a,l)$, $l\geq 0$, but none of the inequalities (1), (2) and (3) are satisfied and $M[f]\not\equiv P[f]$.

Hence, For $M[f]\equiv P[f]$, the conditions of Theorem 2.1 are essential.

**Remark 2.1.** As a particular case, we obtain the uniqueness of monomial $M[f]$ and homogeneous differential polynomial $P[f]$ (i.e., $\overline{d}(P)=\underline{d}(P)=d$) as follows:

**Theorem 2.2.** *Let $f$ be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree $d_M$ and weight $\Gamma_M$ and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant homogeneous differential polynomial of $f$. Also, let $a(z)(\not\equiv 0,\infty)$ be a small function with respect to $f$. Suppose $M[f]-a$ and $P[f]-a$ share $(0,l)$. If*
*$l\geq 2$ and*

$$(Q+3\lambda+3)\Theta(\infty,f)+\lambda\Theta(0,f)+2d_M\delta_{1+k}(0,f)+d\delta(0,f)>Q+4\lambda+3+d_M+d, \tag{16}$$

*or $l=1$ and*

$$\left(\frac{2Q+7\lambda+7}{2}\right)\Theta(\infty,f)+\lambda\Theta(0,f)+\frac{5d_M}{2}\delta_{1+k}(0,f)+d\delta(0,f)>\frac{2Q+9\lambda+7}{2}+\frac{3d_M}{2}+d, \tag{17}$$

*or $l=0$ and*

$$(2Q+5\lambda+6)\Theta(\infty,f)+\lambda\Theta(0,f)+4d_M\delta_{1+k}(0,f)+2d\delta(0,f)>2Q+6\lambda+6+3d_M+2d. \tag{18}$$

*Then, $M[f]\equiv P[f]$.*

*Proof.* By taking $\overline{d}(P) = \underline{d}(P) = d$ and proceeding as in the lines of proof of theorem 2.1, we get the proof of theorem 2.2. □

**Theorem 2.3.** *Let $f$ be a non-constant entire function and $M[f]$ be a differential monomial of degree $d_M$ and weight $\Gamma_M$ and $k(\geq 1)$ is the highest derivative in $M[f]$. Let $P[f]$ be a non-constant differential polynomial of $f$. Also, let $a(z)(\not\equiv 0, \infty)$ be a small function with respect to $f$. Suppose $M[f] - a$ and $P[f] - a$ share $(0, l)$. If*
*$l \geq 2$ and*

$$\delta_{1+k}(0, f) > \frac{d_M + 2\overline{d}(P) - \underline{d}(P) + \lambda}{2d_M + \overline{d}(P) + \lambda},$$

*or $l = 1$ and*

$$\delta_{1+k}(0, f) > \frac{\frac{3d_M}{2} + 2\overline{d}(P) - \underline{d}(P) + \lambda}{\frac{5d_M}{2} + \overline{d}(P) + \lambda},$$

*or $l = 0$ and*

$$\delta_{1+k}(0, f) > \frac{3d_M + 4\overline{d}(P) - 2\underline{d}(P) + \lambda}{4d_M + 2\overline{d}(P) + \lambda}.$$

*Then, $M[f] \equiv P[f]$.*

*Proof.* By taking $\overline{N}(r, f) = S(r, f)$ and proceeding as in the lines of proof of theorem 2.1, we get the proof of theorem 2.3. □

We use the following lemmas in our result:

**Lemma 2.4.** [2] *Let $f$ be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree $d_M$ and weight $\Gamma_M$. Then,*

$$T(r, M) \leq d_M T(r, f) + \lambda\overline{N}(r, \infty; f) + S(r, f).$$

**Lemma 2.5.** [2] *For the differential monomial $M[f]$,*

$$N_p(r, 0, M[f]) \leq d_M N_{p+k}(r, 0; f) + \lambda\overline{N}(r, \infty; f) + S(r, f).$$

**Lemma 2.6.** [4] *Let $f$ be a non-constant meromorphic function and $P[f]$ be a differential polynomial of $f$. Then,*

$$m\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + S(r, f),$$

$$N\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) \leq (\overline{d}(P) - \underline{d}(P))N\left(r, \frac{1}{f}\right) + Q\left[\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right)\right] + S(r, f),$$

$$N\left(r, \frac{1}{P[f]}\right) \leq Q\overline{N}(r, f) + (\overline{d}(P) - \underline{d}(P))m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{\overline{d}(P)}}\right) + S(r, f),$$

*where $Q = max_{1\leq i\leq m}\{n_{i0} + n_{i1} + ... + kn_{ik}\}$.*

**Lemma 2.7.** [1] *Let $f$ and $g$ be two non-constant meromorphic functions.*

*(i) If $f$ and $g$ share $(1,0)$, then*

$$\overline{N}_L\left(r, \frac{1}{f-1}\right) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + S(r),$$

*where $S(r) = o(T(r))$ as $r \to \infty$ with $T(r) = max\,\{T(r,f); T(r,g)\}$.*

*(ii) If $f$ and $g$ share $(1,1)$, then*

$$2\overline{N}_L\left(r, \frac{1}{f-1}\right) + 2\overline{N}_L\left(r, \frac{1}{g-1}\right) + \overline{N}_E^{(2}\left(r, \frac{1}{f-1}\right) - \overline{N}_{f>2}\left(r, \frac{1}{g-1}\right)$$
$$\leq N\left(r, \frac{1}{g-1}\right) - \overline{N}\left(r, \frac{1}{g-1}\right).$$

**Lemma 2.8.** *Let $f$ be a non-constant meromorphic function and $M[f]$ be a differential monomial of degree $d_M$ and weight $\Gamma_M$. Then,*

$$T(r,M) \geq d_M T(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

*Proof.* By using the first fundamental theorem and lemma 2.6, we have

$$d_M T(r,f) = T(r, f^{d_M}) \leq T\left(r, \frac{M}{f^{d_M}}\right) + T(r,M) + S(r,f)$$
$$\leq N\left(r, \frac{M}{f^{d_M}}\right) + T(r,M) + S(r,f)$$

$$d_M T(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \leq T(r,M)$$

$$i.e.,\; T(r,M) \geq d_M T(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r, \frac{1}{f}\right) + S(r,f).$$

Hence the proof of lemma. □

**Remark 2.2.** In view of the Lemma 2.4 and Lemma 2.8, we get the following inequality

$$d_M T(r,f) - \lambda\overline{N}(r,f) - \lambda\overline{N}\left(r, \frac{1}{f}\right) + S(r,f) \leq T(r,M)$$
$$\leq d_M T(r,f) + \lambda\overline{N}(r,f) + S(r,f).$$

**Lemma 2.9.** *Let $f$ be a non-constant meromorphic function and $a(z)$ be a small function of $f$. Let us define $F = \frac{M[f]}{a}$ and $G = \frac{P[f]}{a}$. Then, $FG \not\equiv 1$.*

*Proof.* On contrary, let $FG = 1$. i.e., $M[f]P[f] = a^2$.
Here, $f$ can't have any zero and poles.
Therefore, $\overline{N}(r,0;f) = S(r,f) = \overline{N}(r,\infty;f)$.

By using the first fundamental theorem and lemma 2.6, we give the proof.
Let us consider,

$$\begin{aligned}(d_M + \overline{d}(P))T(r, f) = T\left(r, f^{d_M+\overline{d}(P)}\right) &\leq T\left(r, \frac{f^{d_M} f^{\overline{d}(P)}}{M[f]P[f]}\right) + S(r, f)\\ &\leq T\left(r, \frac{M[f]}{f^{d_M}}\right) + T\left(r, \frac{P[f]}{f^{\overline{d}(P)}}\right) + S(r, f)\\ &\leq (\overline{d}(P) - \underline{d}(P))T(r, f) + S(r, f)\end{aligned}$$

$i.e.,\ (d_M + \underline{d}(P))T(r, f) \leq S(r, f),$

which is not possible.
Hence, $FG \not\equiv 1$.
Hence the proof of lemma. □

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