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# SYMMETRIC IDENTITIES FOR DEGENERATE CARLITZ-TYPE $q$-EULER NUMBERS AND POLYNOMIALS†

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Abstract. In this paper we define the degenerate Carlitz-type $q$-Euler polynomials by generalizing the degenerate Euler numbers and polynomials, degenerate Carlitz-type Euler numbers and polynomials. We also give some interesting properties, explicit formulas, a connection with degenerate Carlitz-type $q$-Euler numbers and polynomials.

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## 1. Introduction

Many mathematicians have studied in the area of the degenerate Bernoulli numbers and polynomials, degenerate Euler numbers and polynomials, degenerate Genocchi numbers and polynomials, degenerate tangent numbers and polynomials(see [1-16]). In this paper, we define the degenerate Carlitz-type $q$-Euler numbers and polynomials and study some properties of the degenerate Carlitz-type $q$-Euler numbers and polynomials.

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ denotes the set of nonnegative integers, $\mathbb{Z}_0^- = \{0, -1, -2, -3, \ldots\}$ denotes the set of nonpositive integers, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{C}$ denotes the set of complex numbers. We remember that the classical degenerate Euler numbers $\mathcal{E}_n(\lambda)$ and Euler polynomials $\mathcal{E}_n(x, \lambda)$ are defined by the following generating

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functions(see [2, 16])

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}=\sum_{n=0}^{\infty}\mathcal{E}_n(\lambda)\frac{t^n}{n!}, \tag{1}$$

and

$$\frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}\mathcal{E}_n(x,\lambda)\frac{t^n}{n!}, \tag{2}$$

respectively.

Some interesting properties of the classical degenerate Euler numbers and polynomials were first investigated by Carlitz[2]. We recall that the classical Stirling numbers of the first kind $S_1(n,k)$ and $S_2(n,k)$ are defined by the relations(see [16])

$$(x)_n=\sum_{k=0}^{n}S_1(n,k)x^k \text{ and } x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k,$$

respectively. Here $(x)_n = x(x-1)\cdots(x-n+1)$ denotes the falling factorial polynomial of order $n$. The numbers $S_2(n,m)$ also admit a representation in terms of a generating function

$$\sum_{n=m}^{\infty}S_2(n,m)\frac{t^n}{n!}=\frac{(e^t-1)^m}{m!}.$$

We also have

$$\sum_{n=m}^{\infty}S_1(n,m)\frac{t^n}{n!}=\frac{(\log(1+t))^m}{m!}.$$

The generalized falling factorial $(x|\lambda)_n$ with increment $\lambda$ is defined by

$$(x|\lambda)_n=\prod_{k=0}^{n-1}(x-\lambda k)$$

for positive integer $n$, with the convention $(x|\lambda)_0=1$; we may also write

$$(x|\lambda)_n=\sum_{k=0}^{n}S_1(n,k)\lambda^{n-k}x^k.$$

Note that $(x|\lambda)$ is a homogeneous polynomials in $\lambda$ and $x$ of degree $n$, so if $\lambda\neq 0$ then $(x|\lambda)_n=\lambda^n(\lambda^{-1}x|1)_n$. Clearly $(x|0)_n=x^n$. We also need the binomial theorem: for a variable $x$,

$$(1+\lambda t)^{x/\lambda}=\sum_{n=0}^{\infty}(x|\lambda)_n\frac{t^n}{n!}.$$

The $q$-number is defined as

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+\cdots+q^{n-3}+q^{n-2}+q^{n-1}.$$

By using $q$-number, we define the degenerate Carlitz-type $q$-Euler numbers and polynomials, which generalized the previously known numbers and polynomials, including the degenerate Euler numbers and polynomials. We begin by recalling here the Carlitz-type $q$-Euler numbers and polynomials.

**Definition 1.1.** The Carlitz-type $q$-Euler numbers $E_{n,q}$ and $q$-Euler polynomials $E_{n,q}(x)$ are defined by means of the generating functions

$$\begin{aligned}\sum_{n=0}^{\infty} E_{n,q}\frac{t^n}{n!} &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m e^{[m]_q t},\\ \sum_{n=0}^{\infty} E_{n,q}(x)\frac{t^n}{n!} &= [2]_q \sum_{m=0}^{\infty}(-1)^m q^m e^{[m+x]_q t},\end{aligned} \tag{3}$$

respectively.

Many kinds of generalizations of these polynomials and numbers have been presented in the literature(see [1-16]). Based on this idea, we construct the degenerate Carlitz-type $q$-Euler number $\mathcal{E}_{n,q}(\lambda)$ and $q$-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$. In the following section, we introduce the Carlitz-type $q$-Euler polynomials and numbers. After that we will investigate some their properties.

## 2. Degenerate Carlitz-type $q$-Euler numbers and polynomials

In this section, we define the degenerate Carlitz-type $q$-Euler numbers and polynomials and provide some of their relevant properties.

**Definition 2.1.** For $|q| < 1$, the degenerate Carlitz-type $q$-Euler numbers $\mathcal{E}_{n,q}(\lambda)$ and polynomials $\mathcal{E}_{n,q}(x,\lambda)$ are defined by means of the generating functions

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(\lambda)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m (1+\lambda t)^{\frac{[m]_q}{\lambda}}, \tag{4}$$

and

$$\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} = [2]_q \sum_{m=0}^{\infty}(-1)^m q^m (1+\lambda t)^{\frac{[m+x]_q}{\lambda}}, \tag{5}$$

respectively.

Obviously, if $q \to 1$, then we have

$$\mathcal{E}_{n,q}(x,\lambda) = \mathcal{E}_n(x,\lambda), \quad \mathcal{E}_{n,q}(\lambda) = \mathcal{E}_n(\lambda).$$

On the other hand, we observe that

$$
\begin{aligned}
(1+\lambda t)^{\frac{[x+y]_q}{\lambda}} &= e^{\frac{[x+y]_q}{\lambda}\log(1+\lambda t)} \\
&= \sum_{n=0}^{\infty}\left(\frac{[x+y]_q}{\lambda}\right)^n \frac{(\log(1+\lambda t))^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} S_1(n,m)\lambda^{n-m}[x+y]_q^m\right)\frac{t^n}{n!}.
\end{aligned}
\tag{6}
$$

By (5), we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x,\lambda)\frac{t^n}{n!} \\
&= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m (1+\lambda t)^{\frac{[m+x]_q}{\lambda}} \\
&= [2]_q\sum_{m=0}^{\infty}(-1)^m q^m \\
&\quad \times \sum_{n=0}^{\infty}\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}\frac{\sum_{j=0}^{l}\binom{l}{j}(-1)^j q^{(x+m)j}}{(1-q)^l}\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left([2]_q\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{S_1(n,l)\lambda^{n-l}\binom{l}{j}(-1)^j q^{xj}}{(1-q)^l}\frac{1}{1+q^{j+1}}\right)\frac{t^n}{n!}.
\end{aligned}
\tag{7}
$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

**Theorem 2.2.** *For $n \in \mathbb{Z}_+$, we have*

$$
\begin{aligned}
\mathcal{E}_{n,q}(x,\lambda) &= [2]_q\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{\binom{l}{j}(-1)^j S_1(n,l)\lambda^{n-l}q^{xj}}{(1-q)^l}\frac{1}{1+q^{j+1}} \\
&= [2]_q\sum_{m=0}^{\infty}\sum_{l=0}^{n}(-1)^m S_1(n,l)\lambda^{n-l}q^m[x+m]_q^l, \\
\mathcal{E}_{n,q}(\lambda) &= [2]_q\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{\binom{l}{j}(-1)^j S_1(n,l)\lambda^{n-l}}{(1-q)^l}\frac{1}{1+q^{j+1}} \\
&= [2]_q\sum_{m=0}^{\infty}\sum_{l=0}^{n}(-1)^m S_1(n,l)\lambda^{n-l}q^m[m]_q^l.
\end{aligned}
$$

The degenerate Carlitz-type $q$-Euler number $\mathcal{E}_{n,q}(\lambda)$ can be determined explicitly. A few of them are

$$\begin{aligned}
\mathcal{E}_{0,q}(\lambda) &= 1,\\
\mathcal{E}_{1,q}(\lambda) &= \frac{[2]_q}{(1-q)(1+q)} - \frac{[2]_q}{(1-q)(1+q^2)},\\
\mathcal{E}_{2,q}(\lambda) &= \frac{[2]_q}{(1-q)^2(1+q)} - \frac{[2]_q\lambda}{(1-q)(1+q)} + \frac{[2]_q\lambda}{(1-q)(1+q^2)}\\
&\quad - \frac{2[2]_q}{(1-q)^2(1+q^2)} + \frac{[2]_q}{(1-q)^2(1+q^3)},\\
\mathcal{E}_{3,q}(\lambda) &= \frac{[2]_q}{(1-q)^3(1+q)} + \frac{2[2]_q\lambda^2}{(1-q)(1+q)} - \frac{3[2]_q\lambda}{(1-q)^2(1+q)}\\
&\quad - \frac{2[2]_q\lambda^2}{(1-q)(1+q^2)} + \frac{6[2]_q\lambda}{(1-q)^2(1+q^2)} - \frac{3[2]_q}{(1-q)^3(1+q^2)}\\
&\quad - \frac{3[2]_q\lambda}{(1-q)^2(1+q^3)} + \frac{3[2]_q}{(1-q)^3(1+q^3)} - \frac{[2]_q}{(1-q)^3(1+q^4)}.
\end{aligned}$$

By replacing $t$ by $\dfrac{e^{\lambda t}-1}{\lambda}$ in (5), we have

$$\begin{aligned}
\sum_{m=0}^{\infty} E_{m,q}(x)\frac{t^m}{m!} &= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\left(\frac{e^{\lambda t}-1}{\lambda}\right)^n \frac{1}{n!}\\
&= \sum_{n=0}^{\infty} \mathcal{E}_{n,q}(x,\lambda)\lambda^{-n} \sum_{m=n}^{\infty} S_2(m,n)\lambda^m \frac{t^m}{m!} \qquad (8)\\
&= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \mathcal{E}_{n,q}(x,\lambda)\lambda^{m-n} S_2(m,n)\right)\frac{t^m}{m!}.
\end{aligned}$$

Thus, we have the following theorem.

**Theorem 2.3.** *For $m \in \mathbb{Z}_+$, we have*

$$E_{m,q}(x) = \sum_{n=0}^{m} \mathcal{E}_{n,q}(x,\lambda)\lambda^{m-n} S_2(m,n).$$

By replacing $t$ by $\log(1+\lambda t)^{1/\lambda}$ in (3), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^n \frac{1}{n!} &= [2]_q \sum_{m=0}^{\infty} (-1)^m q^m (1+\lambda t)^{\frac{[m+x]_q}{\lambda}}\\
&= \sum_{m=0}^{\infty} \mathcal{E}_{m,q}(x,\lambda)\frac{t^m}{m!}, \qquad (9)
\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty} E_{n,q}(x)\left(\log(1+\lambda t)^{1/\lambda}\right)^{n}\frac{1}{n!}\\&=\sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} E_{n,q}(x)\lambda^{m-n}S_1(m,n)\right)\frac{t^m}{m!}.\end{aligned}\tag{10}$$

Thus, by (9) and (10), we have the following theorem.

**Theorem 2.4.** *For $m \in \mathbb{Z}_+$, we have*

$$\mathcal{E}_{m,q}(x,\lambda)=\sum_{n=0}^{m} E_{n,q}(x)\lambda^{m-n}S_1(m,n)$$

The degenerate Carlitz-type $q$-Euler polynomials $\mathcal{E}_{n,q}(x,\lambda)$ can be determined explicitly. A few of them are

$$\begin{aligned}
\mathcal{E}_{0,q}(x,\lambda)&=1,\\
\mathcal{E}_{1,q}(x,\lambda)&=\frac{[2]_q}{(1-q)(1+q)}-\frac{[2]_q q^x}{(1-q)(1+q^2)},\\
\mathcal{E}_{2,q}(x,\lambda)&=-\frac{[2]_q\lambda}{(1-q)(1+q)}+\frac{[2]_q}{(1-q)^2(1+q)}+\frac{[2]_q\lambda q^x}{(1-q)(1+q^2)}\\
&\quad-\frac{2[2]_q q^x}{(1-q)^2(1+q^2)}+\frac{[2]_q q^{2x}}{(1-q)^2(1+q^3)},\\
\mathcal{E}_{3,q}(x,\lambda)&=\frac{2[2]_q\lambda^2}{(1-q)(1+q)}-\frac{3[2]_q\lambda}{(1-q)^2(1+q)}+\frac{[2]_q}{(1-q)^3(1+q)}\\
&\quad-\frac{2[2]_q\lambda^2 q^x}{(1-q)(1+q^2)}+\frac{6[2]_q\lambda q^x}{(1-q)^2(1+q^2)}-\frac{3[2]_q q^x}{(1-q)^3(1+q^2)}\\
&\quad-\frac{3[2]_q\lambda}{(1-q)^2(1+q^3)}+\frac{3[2]_q q^{2x}}{(1-q)^3(1+q^3)}-\frac{[2]_q q^{3x}}{(1-q)^3(1+q^4)}.
\end{aligned}$$

We introduce a $q$-analogue of the generalized falling factorial $(x|\lambda)_n$ with increment $\lambda$. The generalized $q$-falling factorial $([x]_q|\lambda)_n$ with increment $\lambda$ is defined by

$$([x]_q|\lambda)_n=\prod_{k=0}^{n-1}([x]_q-\lambda k)$$

for positive integer $n$, with the convention $([x]_q|\lambda)_0=1$.

By (4) and (5), we get

$$\begin{aligned}&-[2]_q(-1)^n q^n\sum_{l=0}^{\infty}(-1)^l q^l(1+\lambda t)^{\frac{[l+n]_q}{\lambda}}+[2]_q\sum_{l=0}^{\infty}(-1)^l q^l(1+\lambda t)^{\frac{[l+n]_q}{\lambda}}\\&=[2]_q\sum_{l=0}^{n-1}(-1)^l q^l(1+\lambda t)^{\frac{[l]_q}{\lambda}}.\end{aligned}$$

Hence we have

$$\begin{aligned}
&(-1)^{n+1}q^n\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(n,\lambda)\frac{t^m}{m!}+\sum_{m=0}^{\infty}\mathcal{E}_{m,q}(\lambda)\frac{t^m}{m!}\\
&=\sum_{m=0}^{\infty}\left([2]_q\sum_{l=0}^{n-1}(-1)^l q^l([l]_q|\lambda)_m\right)\frac{t^m}{m!}.
\end{aligned}\tag{11}$$

By comparing the coefficients $\frac{t^m}{m!}$ on both sides of (11), we have the following theorem.

**Theorem 2.5.** *For $n \in \mathbb{Z}_+$, we have*

$$\sum_{l=0}^{n-1}(-1)^l q^l([l]_q|\lambda)_m=\frac{(-1)^{n+1}q^n\mathcal{E}_{m,q}(n,\lambda)+\mathcal{E}_{m,q}(\lambda)}{[2]_q}.$$

We observe that

$$\begin{aligned}
(1+\lambda t)^{\frac{[x+y]_q}{\lambda}}&=(1+\lambda t)^{\frac{[x]_q}{\lambda}}(1+\lambda t)^{\frac{q^x[y]_q}{\lambda}}\\
&=\sum_{m=0}^{\infty}([x]_q|\lambda)_m\frac{t^m}{m!}e^{\log(1+\lambda t)\frac{q^x[y]_q}{\lambda}}\\
&=\sum_{m=0}^{\infty}([x]_q|\lambda)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^x[y]_q}{\lambda}\right)^l\frac{\log(1+\lambda t)^l}{l!}\\
&=\sum_{m=0}^{\infty}([x]_q|\lambda)_m\frac{t^m}{m!}\sum_{l=0}^{\infty}\left(\frac{q^x[y]_q}{\lambda}\right)^l\sum_{k=l}^{\infty}S_1(k,l)\lambda^k\frac{t^k}{k!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([x]_q|\lambda)_{n-k}\lambda^{k-l}q^{xl}[y]_q^l S_1(k,l)\right)\frac{t^n}{n!}.
\end{aligned}\tag{12}$$

From (5) and (12), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\mathcal{E}_{n,q\zeta}(x,\lambda)\frac{t^n}{n!}=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m(1+\lambda t)^{\frac{[m+x]_q}{\lambda}}\\
&=[2]_q\sum_{m=0}^{\infty}(-1)^m q^m\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([x]_q|\lambda)_{n-k}\lambda^{k-l}q^{xl}[m]_q^l S_1(k,l)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}[2]_q\sum_{m=0}^{\infty}(-1)^m q^m[m]_q^l([x]_q|\lambda)_{n-k}\lambda^{k-l}q^{xl}S_1(k,l)\right)\frac{t^n}{n!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}E_{l,q}([x]_q|\lambda)_{n-k}\lambda^{k-l}q^{xl}S_1(k,l)\right)\frac{t^n}{n!}
\end{aligned}$$

By comparing the coefficients $\frac{t^n}{n!}$ in the above equation, we have the following theorem.

**Theorem 2.6.** *For $n \in \mathbb{Z}_+$, we have*

$$\mathcal{E}_{n,q}(x,\lambda) = \sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}([x]_q|\lambda)_{n-k}\lambda^{k-l}q^{xl}S_1(k,l)E_{l,q}.$$

Taking $x=0$ in Theorem 2.3, Theorem 2.4, and Theorem 2.6, we have the following corollary.

**Corollary 2.7.** *For $m \in \mathbb{Z}_+$, we have*

$$\mathcal{E}_{m,q}(\lambda) = \sum_{n=0}^{m} E_{n,q}\lambda^{m-n}S_1(m,n), \quad E_{m,q} = \sum_{n=0}^{m}\mathcal{E}_{n,q}(\lambda)\lambda^{m-n}S_2(m,n).$$

## 3. Symmetric properties about degenerate Carlitz-type $q$-Euler numbers and polynomials

In this section, we are going to obtain the main results of degenerate Carlitz-type $q$-Euler numbers and polynomials. We also establish some interesting symmetric identities for degenerate Carlitz-type $q$-Euler numbers and polynomials. Let $w_1$ and $w_2$ be odd positive integers. Observe that $[xy]_q = [x]_{q^y}[y]_q$ for any $x, y \in \mathbb{C}$.
By substitute $w_1x + \frac{w_1 i}{w_2}$ for $x$ in Definition 2.1, replace $q$ by $q^{w_2}$ and replace $\lambda$ by $\dfrac{\lambda}{[w_2]_q}$, respectively, we derive

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_1}}[w_2]_q^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_q}\right)\right)\frac{t^n}{n!}\\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{n=0}^{\infty}\mathcal{E}_{n,q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_q}\right)\frac{([w_2]_q t)^n}{n!}\\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}}\sum_{n=0}^{\infty}(-1)^n q^{w_2 n}\\
&\qquad \times\left(1+\frac{\lambda}{[w_2]_q}[w_2]_q t\right)^{\frac{[w_1x+\frac{w_1 i}{w_2}+n]_{q^{w_2}}}{\frac{\lambda}{[w_2]_q}}}\\
&= [2]_{q^{w_1}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}[2]_{q^{w_2}}\sum_{n=0}^{\infty}(-1)^n q^{w_2 n}\left(1+\lambda t\right)^{\frac{[w_1w_2x+w_1i+nw_2]_q}{\lambda}}.
\end{aligned}$$

Since for any non-negative integer $n$ and odd positive integer $w_1$, there exist unique non-negative integer $r$ such that $n = w_1 r + j$ with $0 \le j \le w_1 - 1$.

Hence, this can be written as

$$
\begin{aligned}
&[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{n=0}^{\infty}(-1)^n q^{w_2 n} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_1i+nw_2]_q}{\lambda}}. \\
&=[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{\substack{w_1r+j=0\\0\le j\le w_1-1}}^{\infty}(-1)^{w_1r+j}q^{w_2(w_1r+j)} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_1i+(w_1r+j)w_2]_q}{\lambda}}. \\
&=[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^{w_1r}(-1)^j q^{w_2w_1r}q^{w_2j} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_1i+w_1w_2r+w_2j]_q}{\lambda}} \\
&=[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_1i}q^{w_2w_1r}q^{w_2j} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_1i+w_1w_2r+w_2j]_q}{\lambda}}.
\end{aligned}
$$

It follows from the above equation that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_2}}[w_2]_q^n\sum_{i=0}^{w_2-1}(-1)^i q^{w_1i}\mathcal{E}_{n,q^{w_2}}\left(w_1x+\frac{w_1i}{w_2},\frac{\lambda}{[w_2]_q}\right)\right)\frac{t^n}{n!} \\
&=[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_1-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_1i}q^{w_2w_1r}q^{w_2j} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_1i+w_1w_2r+w_2j]_q}{\lambda}}.
\end{aligned}
\tag{13}
$$

From the similar method, we can have that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\left([2]_{q^{w_2}}[w_1]_q^n\sum_{i=0}^{w_1-1}(-1)^i q^{w_2i}\mathcal{E}_{n,q^{w_1}}\left(w_2x+\frac{w_2i}{w_1},\frac{\lambda}{[w_1]_q}\right)\right)\frac{t^n}{n!} \\
&=[2]_{q^{w_1}}[2]_{q^{w_2}}\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}\sum_{r=0}^{\infty}(-1)^i(-1)^r(-1)^j q^{w_2i}q^{w_1w_1r}q^{w_1j} \\
&\qquad\times(1+\lambda t)^{\frac{[w_1w_2x+w_2i+w_1w_2r+w_1j]_q}{\lambda}}.
\end{aligned}
\tag{14}
$$

Thus, we have the following theorem from (13) and (14).

**Theorem 3.1.** *Let $w_1$ and $w_2$ be odd positive integers. Then one has*

$$[2]_{q^{w_1}}[w_2]_q^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,q^{w_2}}\left(w_1 x + \frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_q}\right)$$
$$= [2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,q^{w_1}}\left(w_2 x + \frac{w_2 j}{w_1}, \frac{\lambda}{[w_1]_q}\right).$$

It follows that we show some special cases of Theorem 3.1. Setting $w_2 = 1$ in Theorem 3.1, we obtain the multiplication theorem for the degenerate Carlitz-type $q$-Euler polynomials.

**Corollary 3.2.** *Let $w_1$ be odd positive integer. Then one has*

$$\mathcal{E}_{n,q}(x,\lambda) = \frac{[2]_q[w_1]_q^n}{[2]_{q^{w_1}}} \sum_{j=0}^{w_1-1}(-1)^j q^j \mathcal{E}_{n,q^{w_1}}\left(\frac{x+j}{w_1}, \frac{\lambda}{[w_1]_q}\right). \tag{15}$$

Letting $q \to 1$ in (15) leads to the familiar multiplication theorem for the degenerate Euler polynomials

$$\mathcal{E}_n(x,\lambda) = w_1^n \sum_{j=0}^{w_1-1}(-1)^j \mathcal{E}_n\left(\frac{x+i}{w_1}, \frac{\lambda}{w_1}\right). \tag{16}$$

Letting $\lambda \to 0$ in (16) leads to the familiar multiplication theorem for the Euler polynomials

$$E_n(x) = w_1^n \sum_{j=0}^{w_1-1}(-1)^j E_n\left(\frac{x+i}{w_1}\right).$$

Setting $x = 0$ in Theorem 3.1, we have the following corollary.

**Corollary 3.3.** *Let $w_1$ and $w_2$ be odd positive integers. Then one has*

$$[2]_{q^{w_1}}[w_2]_q^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,q^{w_2}}\left(\frac{w_1 i}{w_2}, \frac{\lambda}{[w_2]_q}\right)$$
$$= [2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,q^{w_1}}\left(\frac{w_2 j}{w_1}, \frac{\lambda}{[w_1]_q}\right).$$

By Theorem 2.4 and Corollary 3.3, we have the following theorem.

**Theorem 3.4.** *Let $w_1$ and $w_2$ be odd positive integers. Then one has*

$$\sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}[w_2]_q^l[2]_{q^{w_1}} \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i} E_{l,q^{w_2}}\left(\frac{w_1}{w_2} i\right)$$
$$= \sum_{l=0}^{n} S_1(n,l)\lambda^{n-l}[w_1]_q^l[2]_{q^{w_2}} \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j} E_{l,q^{w_1}}\left(\frac{w_2}{w_1} j\right).$$

We obtain another result by applying the addition theorem for the Carlitz-type $q$-Euler polynomials $E_{n,q}(x)$. From (3), Theorem 2.4, and Theorem 3.1, we have

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_q^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_q}\right)\\
&=[2]_{q^{w_1}}[w_2]_q^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{l=0}^{n}E_{l,q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2}\right)\left(\frac{\lambda}{[w_2]_q}\right)^{n-l}S_1(n,l)\\
&=[2]_{q^{w_1}}\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}[w_2]_q^l\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\sum_{k=0}^{l}q^{w_1(l-k)i}\\
&\qquad\qquad\times E_{l-k,q^{w_2}}(w_1x)\left(\frac{[w_1]_q}{[w_2]_q}\right)^k[i]_{q^{w_1}}^k\\
&=[2]_{q^{w_1}}\sum_{l=0}^{n}S_1(n,l)\lambda^{n-l}\sum_{k=0}^{l}\binom{l}{k}[w_1]_q^k[w_2]_q^{l-k}E_{l-k,q^{w_2}}(w_1x)\\
&\qquad\qquad\times\sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}q^{(l-k)w_1 i}[i]_{q^{w_1}}^k.
\end{aligned}$$

For all different integer $n \geq 0$, let $\mathcal{S}_{l,k,q}(w_1)=\sum_{i=0}^{w_1-1}(-1)^i q^{(l-k+1)i}[i]_q^k$. This sum $\mathcal{S}_{l,k,q}(w_1)=\sum_{i=0}^{w_1-1}(-1)^i q^{(l-k+1)i}[i]_q^k$ is called the $q$-powers sums. Therefore, we obtain that

$$\begin{aligned}
&[2]_{q^{w_1}}[w_2]_q^n \sum_{i=0}^{w_2-1}(-1)^i q^{w_1 i}\mathcal{E}_{n,q^{w_2}}\left(w_1x+\frac{w_1 i}{w_2},\frac{\lambda}{[w_2]_q}\right)\\
&=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}[2]_{q^{w_1}}[w_1]_q^k[w_2]_q^{l-k}E_{l-k,q^{w_2}}(w_1x)\mathcal{S}_{l,k,q^{w_1}}(w_2),
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&[2]_{q^{w_2}}[w_1]_q^n \sum_{j=0}^{w_1-1}(-1)^j q^{w_2 j}\mathcal{E}_{n,q^{w_1}}\left(w_2x+\frac{w_2 j}{w_1},\frac{\lambda}{[w_1]_q}\right)\\
&=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}[2]_{q^{w_2}}[w_2]_q^k[w_1]_q^{l-k}E_{l-k,q^{w_1}}(w_2x)\mathcal{S}_{l,k,q^{w_2}}(w_1).
\end{aligned}\tag{18}$$

By (17) and (18), we obtain the following symmetric identity.

**Theorem 3.5.** *Let $w_1$ and $w_2$ be odd positive integers. Then one has*

$$\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}[2]_{q^{w_1}}[w_1]_q^k[w_2]_q^{l-k}E_{l-k,q^{w_2}}(w_1x)S_{l,k,q^{w_1}}(w_2)$$
$$=\sum_{l=0}^{n}\sum_{k=0}^{l}\binom{l}{k}S_1(n,l)\lambda^{n-l}[2]_{q^{w_2}}[w_2]_q^k[w_1]_q^{l-k}E_{l-k,q^{w_1}}(w_2x)S_{l,k,q^{w_2}}(w_1).$$

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