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ON CHARACTERIZATIONS OF THE CONTINUOUS DISTRIBUTIONS BY INDEPENDENCE PROPERTY OF THE QUOTIENT-TYPE UPPER RECORD VALUES

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Abstract. In this paper we obtain characterizations of a family of continuous probability distribution by independence property of upper record values. Also, we introduce some examples of the characterizations of distributions from these general classes of continuous distributions.

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1. Introduction

Record values are important in many real-life situations involving data relating to weather, sports, economics, and life-tests.

Many researchers have studied the characterizations by the independence of the upper record values. For example Ahsanullah [2] studied the characterizations of Pareto distribution by upper record values. Also, Juhás and Skřivánková [5] showed characterization of general classes of distributions with the independent property that the random variables $g(L_n)$ and $g(L_{n+1})-g(L_n)$. For further various characterizations by the independence of the record values the interested readers are referred to Jin and Lee [4], Lee and Lim [6, 7] among others.

In this paper, we investigate new characterizations of a family of continuous probability distribution by independence property of the quotient-type upper record values.

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2. Main results

In this section, we give our main results. For proving our main theorems we need the following Lemma 2.1. and 2.2..

Lemma 2.1. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 1$ as $x \to \alpha^+$ and $g(x) \to \infty$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Suppose that $\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))} = e^{-q(u,v)}$ and $h(u,v) = \big(q(u,v)\big)^r e^{-q(u,v)} \cdot \big(\frac{\partial}{\partial v} q(u,v)\big)$, for $r \geq 0$, where $h(u,v) \neq 0$ and $\frac{\partial}{\partial v} q(u,v) \neq 0$ for any positive u and v. If $h(u,v)$ is independent of u, then $q(u,v)$ is a function of v only.*

Proof. Let

$$\begin{aligned} h(u,v) &= \big(q(u,v)\big)^r e^{-q(u,v)} \big(\frac{\partial}{\partial v} q(u,v)\big) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \big(q(u,v)\big)^{k+r} \big(\frac{\partial}{\partial v} q(u,v)\big). \end{aligned} \tag{1}$$

Since $h(u,v)$ is independent of u, we obtain

$$h(u,v) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \big(q(u,v)\big)^{k+r} \big(\frac{\partial}{\partial v} q(u,v)\big) = l(v). \tag{2}$$

Integrating (2) with respect to v, we get

$$\begin{aligned} &\int h(u,v) dv = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(k+r+1)} \big(q(u,v)\big)^{k+r+1} \\ &= \int l(v) dv + c = L(v). \end{aligned} \tag{3}$$

Here L is a function of v only and c is independent of v but may depend on u.

Now taking $v \to 1$, $q(u,v) \to 0$, we have c independently of u from (3). Differentiating (3) with respect to u, we get

$$\begin{aligned} \frac{\partial}{\partial u} L(v) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \big(q(u,v)\big)^{k+r} \big(\frac{\partial}{\partial u} q(u,v)\big) \\ &= l(v) \big(\frac{\partial}{\partial v} q(u,v)\big)^{-1} \big(\frac{\partial}{\partial u} q(u,v)\big) = 0. \end{aligned} \tag{4}$$

Now we know $l(v) \neq 0$ and $\frac{\partial}{\partial v} q(u,v) \neq 0$, so we must have

$$\frac{\partial}{\partial u} q(u,v) = 0. \tag{5}$$

Hence $q(u,v)$ is a function of v only.

This completes the proof.

□

Lemma 2.2. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 0$ as $x \to \alpha^+$ and $g(x) \to \infty$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Suppose that $\frac{R(g^{-1}(uv))}{R(g^{-1}(v))} = q(u,v)$ and $h(u,v) = \big(1 - q(u,v)\big)^{n-m-1} \times \ (q(u,v))^{m-1}\big(\frac{\partial}{\partial u} q(u,v)\big)$ for $1 \leq m < n$, where $R(x) = -ln\bar{F}(x)$, $h(u,v) \neq 0$ and $\frac{\partial}{\partial u} q(u,v) \neq 0$ for any positive u and v. If $h(u,v)$ is independent of v, then $q(u,v)$ is a function of u only.*

Proof. The proof is similar to the proof of Lemma 2.1.. □

Theorem 2.3. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 1$ as $x \to \alpha^+$ and $g(x) \to \infty$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Then $F(x) = 1 - (g(x))^{-\lambda}$ for all $g(x) \geq 1, \lambda > 0$, if and only if $\frac{g(X_{U(n)})}{g(X_{U(m)})}$ and $g(X_{U(m)})$ are independent for $1 \leq m < n$.*

Proof. If $F(x) = 1 - (g(x))^{-\lambda}$, then it is easy to see that $\frac{g(X_{U(n)})}{g(X_{U(m)})}$ and $g(X_{U(m)})$ are independent for $1 \leq m < n$.

Let us use the transformation $U = g(X_{U(m)})$ and $V = \frac{g(X_{U(n)})}{g(X_{U(m)})}$. The Jacobian of the transformation is $J = \frac{\partial}{\partial u}(g^{-1}(u)) \times \frac{\partial}{\partial v}(g^{-1}(uv))$. Since $g(x)$ is an increasing and differentiable function, both $\frac{\partial}{\partial u}(g^{-1}(u))$ and $\frac{\partial}{\partial v}(g^{-1}(uv))$ are positive. Thus we can write the joint pdf $f_{U,V}(u,v)$ of U and V as

$$\begin{aligned} f_{U,V}(u,v) =& \frac{1}{\Gamma(m)\Gamma(n-m)} (R(g^{-1}(u)))^{m-1} r(g^{-1}(u)) f(g^{-1}(uv)) \\ & \times \big(R(g^{-1}(uv)) - R(g^{-1}(u))\big)^{n-m-1} \frac{\partial}{\partial u}(g^{-1}(u)) \frac{\partial}{\partial v}(g^{-1}(uv)) \end{aligned} \tag{6}$$

for all $u > 1$ and $v > 1$, where $R(x) = -ln\bar{F}(x)$ and $r(x) = \frac{d}{dx}R(x)$.

The pdf $f_U(u)$ of U is given by

$$f_U(u) = \frac{R(g^{-1}(u)))^{m-1}}{\Gamma(m)} f(g^{-1}(u)) \frac{\partial}{\partial u}(g^{-1}(u)) \tag{7}$$

for all $u > 1$ and $m \geq 1$.

From (6) and (7), we can get the conditional pdf of $f_V(v|u)$ as

$$f_V(v|u) = \frac{\big(R(g^{-1}(uv)) - R(g^{-1}(u))\big)^{n-m-1}}{\Gamma(n-m)\bar{F}(g^{-1}(u))} f(g^{-1}(uv))\frac{\partial}{\partial v}(g^{-1}(uv))$$
$$= \frac{1}{\Gamma(n-m)}\bigg(-ln\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))}\bigg)^{n-m-1}\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))}\bigg(-\frac{\partial}{\partial v}\big(-ln\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))}\big)\bigg), \tag{8}$$

for all $u > 1$ and $v > 1$.

Since U and V are independent and using Lemma 2.1., $q(u, v) = -ln\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))}$ is a function of v only. Thus

$$\frac{\bar{F}(g^{-1}(uv))}{\bar{F}(g^{-1}(u))} = L(v) \tag{9}$$

where $L(v)$ is a function of v only. Letting $u \to 1^+$, so $g^{-1}(u) \to \alpha$ and $\bar{F}(g^{-1}(u)) \to 1$, we get $L(v) = \bar{F}(g^{-1}(v))$ from $\bar{F}(\alpha) = 1$. Then, we get

$$\bar{F}(g^{-1}(uv)) = \bar{F}(g^{-1}(u))\bar{F}(g^{-1}(v)) \tag{10}$$

for all $u > 1$ and $v > 1$.

By the theory of functional equation see Aczel(1966), the only continuous solution of (10) with boundary conditions $\bar{F}(\alpha) = 1$ and $\bar{F}(\beta) = 0$ is

$$\bar{F}(x) = (g(x))^{-\lambda} \tag{11}$$

for all $g(x) > 1$ and $\lambda > 0$.

This completes the proof.

□

Remark 2.1. If we set $g(x) = x$, $x > 1$, then $F(x) = 1 - x^{-\lambda}$ for all $x > 1$, $\lambda > 0$, if and only if $\frac{X_{U(n)}}{X_{U(m)}}$ and $X_{U(m)}$ are independently distributed in [6].

Remark 2.2. If we set $g(x) = e^x$, $x > 0$, then $F(x) = 1 - e^{-\lambda x}$ for all $x > 0$, $\lambda > 0$, if and only if $X_{U(n)} - X_{U(m)}$ and $X_{U(m)}$ are independently distributed in [3].

Remark 2.3. The case when $m = k$ and $n = k + 1$, which is the special case of Theorem 2.3, have been treated in [4].

Theorem 2.4. *Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables each distributed identically with an absolutely continuous cdf $F(x)$ and pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Let $g(x)$ is an increasing and differentiable function with $g(x) \to 0$ as $x \to \alpha^+$ and $g(x) \to \infty$ as $x \to \beta^-$ for all $x \in (\alpha, \beta)$. Then $F(x) = 1 - e^{-g(x)^{\lambda}}$*

for all $g(x) > 0, \lambda > 0$, *if and only if* $\frac{g(X_{U(m)})}{g(X_{U(n)})}$ *and* $g(X_{U(n)})$ *are independent for* $1 \leq m < n$.

Proof. The proof is similar to Theorem 2.3., and is omitted. □

Remark 2.4. If we set $g(x) = x$, $x > 0$, then $F(x) = 1 - e^{-x^{\lambda}}$ for all $x > 0$, $\lambda > 0$, if and only if $\frac{X_{U(m)}}{X_{U(n)}}$ and $X_{U(n)}$ are independently distributed in [5].

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