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$(L,*)$-FILTERS AND $(L,*,\odot)$-LIMIT SPACES†

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ABSTRACT. In this paper, we introduce the notion of the $(L,*,\odot)$-limit spaces and investigate the relations $(L,*,\odot)$-limit spaces and $(L,*)$-filters on ecl-premonoid. We give their examples.

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1. Introduction

For the case that the lattice is a frame, L-filters were introduced in [2,3]. Höhle and Sostak [4] introduced the concept of L-filters for a complete quasi-monoidal lattice L. For the case that the lattice is a stsc quantale, L-filters were introduced in [9]. Lattice-valued convergence spaces were introduced for the case that the lattice is a frame [5-8] or for the case of complete residuated lattice [11] or for the case of ecl-premonoid [10]. Jäger [5-6] developed stratified L-convergence structures based on the concepts of L-filters where L is a complete Heyting algebra. Yao [11] extended stratified L-convergence structures to complete residuated lattices and investigated between stratified L-convergence structures and L-fuzzy topological spaces. As an extension of Yao [11], Fang [7,8] introduced L-ordered convergence structures on L-ordered filters and investigated between L-ordered convergence structures and strong L-topological spaces.

In this paper, we define the $(L,*,\odot)$-limit spaces as an extension of L-convergence space on ecl-premonoid in Orpen's sense [10]. From $(L,*)$-filters, we can obtain various $(L,*,\odot)$-limit structures and give their examples.

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2. Preliminaries

Definition 2.1. [10] A complete lattice $(L, \leq, \bot, \top)$ with bottom element $\bot$ and top element $\top$ is called a GL-monoid $(L, \leq, *, \bot, \top)$ with a binary operation $* : L \times L \to L$ satisfying the following conditions:

(G1) $a * \top = a$, for all $a \in L$,
(G2) $a * b = b * a$, for all $a, b \in L$,
(G3) $a * (b * c) = (a * b) * c$, for all $a, b \in L$,
(G4) if $a \leq b$, there exists $c \in L$ such that $b * c = a$,
(G5) $a * \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a * b_i)$.

We can define an implication operator:

$$a \Rightarrow b = \bigvee\{c \mid a * c \leq b\}.$$

Example 2.2. (1) A continuous t-norm $([0, 1], \leq, *)$ is a GL-monoid.
(2) A frame $(L, \leq, \wedge)$ is a GL-monoid.

Definition 2.3. [10] A complete lattice $(L, \leq, \bot, \top)$ is called a cl-premonoid $(L, \leq, \odot)$ with a binary operation $\odot : L \times L \to L$ satisfying the following conditions:

(CL1) $a \leq a \odot \top$ and $a \leq \top \odot a$, for all $a \in L$,
(CL2) if $a \leq b$ and $c \leq d$, then $a \odot c \leq b \odot d$,
(CL3) $a \odot \bigvee_{i\in\Gamma} b_i = \bigvee_{i\in\Gamma}(a \odot b_i)$ and $\bigvee_{j\in\Gamma} a_j \odot b = \bigvee_{j\in\Gamma}(a_j \odot b)$.

We can define an implication operator:

$$a \to b = \bigvee\{c \mid a \odot c \leq b\}.$$

Example 2.4. (1) Every GL-monoid $(L, \leq, *)$ is a cl-premonoid.
(2) Define maps $\odot_i : [0, 1] \times [0, 1] \to [0, 1]$ as follows:

$$x \odot_1 y = x^{\frac{1}{p}} \cdot y^{\frac{1}{p}} (p \geq 1), x \odot_2 y = (x^p + y^p) \wedge 1 (p \geq 1).$$

Then $(L, \leq, \odot_i)$ is a cl-premonoid for $i = 1, 2$.

Definition 2.5. [10] A complete lattice $(L, \leq, \bot, \top)$ is called an ecl-premonoid $(L, \leq, \odot, *)$ with a GL-monoid $(L, \leq, *)$ and a cl-premonoid $(L, \leq, \odot)$ which satisfy the following condition:

(D) $(a \odot b) * (c \odot d) \leq (a * c) \odot (b * d)$, for all $a, b, c, d \in L$.

An ecl-premonoid $(L, \leq, \odot, *)$ is called an M-ecl-premonoid if it satisfies the following condition:

(M) $a \leq a \odot a$ for all $a \in L$.

Example 2.6. (1) Let $(L, \leq, *)$ be a GL-monoid and $(L, \leq, \wedge)$ is a cl-premonoid. Then $(L, \leq, \wedge, *)$ is an M-ecl-premonoid.

(2) Let $(L, \leq, *)$ be a GL-monoid. Then $(L, \leq, *, *)$ is an ecl-premonoid. If $* = \cdot$, $0.5 \not\leq 0.5 \cdot 0.5 = 0.25$. $(L, \leq, \cdot, \cdot)$ is not an M-ecl-premonoid.

(3) Let $(L, \leq, \cdot)$ be a GL-monoid. Define a map $\odot : [0, 1] \times [0, 1] \to [0, 1]$ as $x \odot y = (x + y) \wedge 1$. Then $(L, \leq, \odot, \cdot)$ is not an M-cl-premonoid because

$$0.7 = (0.3 \odot 0.4) \cdot (0.5 \odot 0.7) \not\leq (0.3 \cdot 0.5) \odot (0.4 \cdot 0.7) = 0.15 + 0.28 = 0.43$$

(4) Let $(L,\leq,\cdot)$ be a GL-monoid. Define a map $\odot:[0,1]\times[0,1]\to[0,1]$ as $x\odot y=x^{\frac{1}{3}}\cdot y^{\frac{1}{3}}$. Then $(L,\leq,\odot,\cdot)$ is an M-cl-premonoid.

Lemma 2.7. *Let $(L,\leq,\odot,*)$ be an ecl-premonoid. For each $a,b,c,d,a_i,b_i\in L$ and for $\uparrow\in\{\to,\Rightarrow\}$, we have the following properties.*

(1) *If $b\leq c$, then $a\odot b\leq a\odot c$ and $a*b\leq a*c$.*
(2) *$a\odot b\leq c$ iff $a\leq b\to c$. Moreover, $a*b\leq c$ iff $a\leq b\Rightarrow c$.*
(3) *If $b\leq c$, then $a\uparrow b\leq a\uparrow c$ and $c\uparrow a\leq b\uparrow a$.*
(4) *$a\leq b$ iff $a\Rightarrow b=\top$.*
(5) *$a*b\leq a\odot b$, $a\to b\leq a\Rightarrow b$ and $a*(b\odot c)\leq(a*b)\odot c$.*
(6) *$(a\uparrow b)\odot(c\uparrow d)\leq(a\odot c)\uparrow(b\odot d)$.*
(7) *$(b\uparrow c)\leq(a\odot b)\uparrow(a\odot c)$.*
(8) *$(b\uparrow c)\leq(a\uparrow b)\uparrow(a\uparrow c)$ and $(b\uparrow a)\leq(a\uparrow c)\uparrow(b\uparrow c)$.*
(9) *$(b\to c)\leq(a\uparrow b)\to(a\uparrow c)$ and $(b\uparrow a)\leq(a\to c)\to(b\uparrow c)$*
(10) *$a_i\uparrow b_i\leq(\bigwedge_{i\in\Gamma}a_i)\uparrow(\bigwedge_{i\in\Gamma}b_i)$.*
(11) *$a_i\uparrow b_i\leq(\bigvee_{i\in\Gamma}a_i)\uparrow(\bigvee_{i\in\Gamma}b_i)$.*
(12) *$(c\uparrow a)*(b\to d)\leq(a\to b)\to(c\uparrow d)$.*

Proof. (1) Let $b\leq c$. Then $b\vee c=c$. By (L4), $(a\odot b)\vee(a\odot c)=a\odot(b\vee c)=a\odot c$. Thus $(a\odot b)\leq(a\odot c)$. Similarly, $a*b\leq a*c$.

(2) and (3) follow from the definitions $\to$ and $\Rightarrow$.

(4) Let $a\leq b$. Since $a*\top=a$, then $\top\leq a\Rightarrow a\leq a\Rightarrow b$.

Let $a\Rightarrow b=\top$. Then $a=a*\top\leq b$.

(5) For $(a*c)\odot(d*b)\geq(a\odot d)*(c\odot b)$, put $c=d=\top$, then $a\odot b\geq a*b$. Thus, $a\to b\leq a\Rightarrow b$. Moreover, we have

$$a*(b\odot c)\leq(a\odot\top)*(b\odot c)\leq(a*b)\odot(\top*c)=(a*b)\odot c.$$

(6) Since $(a\odot c)*((a\Rightarrow b)\odot(c\Rightarrow d))\leq(a*(a\Rightarrow b))\odot(c*(c\Rightarrow d))\leq b\odot d$, by (2), $(a\Rightarrow b)\odot(c\Rightarrow d)\leq(a\odot c)\Rightarrow(b\odot d)$. Similarly, $(a\to b)\odot(c\to d)\leq(a\odot c)\to(b\odot d)$.

(7) Since $(a\odot b)*(\top\odot(b\Rightarrow c))\leq(a*\top)\odot(b*(b\Rightarrow c))\leq a\odot c$, by (2), $(b\Rightarrow c)\leq(a\odot b)\Rightarrow(a\odot c)$. Similarly, $(b\to c)\leq(a\odot b)\to(a\odot c)$.

(9) It follows from:

$$\begin{aligned}&a*((a\Rightarrow b)\odot(b\to c))\leq(a\odot\top)*((a\Rightarrow b)\odot(b\to c))\\&\leq(a*(a\Rightarrow b))\odot(\top*(b\to c))\leq b\odot(b\to c)\leq c.\end{aligned}$$

(10) $(\bigwedge_{i\in\Gamma}a_i)\odot(a_i\to b_i)\leq\bigwedge_{i\in\Gamma}(a_i\odot(a_i\to b_i))\leq\bigwedge_{i\in\Gamma}b_i$.
(11) $(\bigvee_{i\in\Gamma}a_i)\odot(a_i\to b_i)\leq\bigvee_{i\in\Gamma}(a_i\odot(a_i\to b_i))\leq\bigvee_{i\in\Gamma}b_i$.
(12)

$$\begin{aligned}&c*\Big((a\to b)\odot((c\Rightarrow a)*(b\to d))\Big)\leq\Big(c*((c\Rightarrow a)*(b\to d))\Big)\odot(a\to b)\\&\leq\Big(a*(b\to d)\Big)\odot(a\to b)\leq\Big(a\odot(b\to d)\Big)\odot(a\to b)\leq b\odot(b\to d)\leq d.\end{aligned}$$

Hence $(c\Rightarrow a)*(b\to d)\leq(a\to b)\to(c\Rightarrow d)$. Similarly, $(c\to a)*(b\to d)\leq(a\to b)\to(c\to d)$.

Definition 2.8. [8,10,11] A mapping $\mathcal{F} : L^X \to L$ is called an $(L,*)$-*filter* on X if it satisfies the following conditions:

(F1) $\mathcal{F}(1_\emptyset) = \bot$ and $\mathcal{F}(1_X) = \top$, where $1_\emptyset(x) = \bot, 1_X(x) = \top$ for $x \in X$.

(F2) $\mathcal{F}(f * g) \geq \mathcal{F}(f) * \mathcal{F}(g)$, for each $f, g \in L^X$,

(F3) if $f \leq g$, $\mathcal{F}(f) \leq \mathcal{F}(g)$.

An $(L,*)$-filter is called *stratified* if

(S) $\mathcal{F}(\alpha * f) \geq \alpha * \mathcal{F}(f)$ for each $f \in L^X$ and $\alpha \in L$.

The pair $(X, \mathcal{F})$ is called an (resp. a stratified)$(L,*)$-*filter space.*

Let $(X, \mathcal{F}_1)$ and $(Y, \mathcal{F}_2)$ be two $(L,*)$-filter spaces and $\phi : X \to Y$ called an L-filter map if $\mathcal{F}_2(g) \leq \mathcal{F}_1(\phi^{\leftarrow}(g))$ for all $g \in L^Y$ where $\phi^{\leftarrow}(g) = g \circ \phi$.

Example 2.9. (1) Define a map $[x] : L^X \to L$ as $[x](f) = f(x)$. Then $[x]$ is a stratified $(L,*)$-filter on X.

(2) Define a map inf : $L^X \to L$ as $\inf(f) = \bigwedge_{x\in X} f(x)$. Then inf is a stratified $(L,*)$-filter on X.

(3)If $\mathcal{F}$ and $\mathcal{G}$ are $(L,*)$-filters on X, $\mathcal{F} \odot \mathcal{G}$ is an $(L,*)$-filter on X because

$$\begin{aligned}&(\mathcal{F} \odot \mathcal{G})(f * g) = \mathcal{F}(f * g) \odot \mathcal{G}(f * g) = (\mathcal{F}(f) * \mathcal{F}(g)) \odot (\mathcal{G}(f) * \mathcal{G}(g))\\ &\geq (\mathcal{F}(f) \odot \mathcal{G}(f)) * (\mathcal{F}(g) \odot \mathcal{G}(g)) = (\mathcal{F} \odot \mathcal{G})(f) * (\mathcal{F} \odot \mathcal{G})(g).\end{aligned}$$

Definition 2.10. [4] A map $\mathcal{I} : L^X \to L^X$ is called an interior $(L,*)$-operator on X if it satisfies

(I1) $\mathcal{I}(f) \leq f$ for each $f \in L^X$,

(I2) if $f \leq g$, then $\mathcal{I}(f) \leq \mathcal{I}(g)$,

(I3) $\mathcal{I}(f * g) \geq \mathcal{I}(f) * \mathcal{I}(g)$,

(I4) $\mathcal{I}(a * f) \geq a * \mathcal{I}(f)$ for each $a \in L$ and $f \in L^X$.

3. $(L,*,\odot)$-limit spaces

In this section, we always assume that $(L, \leq, \odot, *)$ is an ecl-premonoid.

Definition 3.1. Let $F_*(X)$ is a family of $(L,*)$-filters on X. A map lim : $F_*(X) \to L^X$ is called an $(L,*,\odot)$-*limit structure* on X if it satisfies the following conditions:

(L1) $\lim[x](x) = \top$ for all $x \in X$.

(L2) If $\mathcal{F} \leq \mathcal{G}$, then $\lim \mathcal{F}(x) \leq \lim \mathcal{G}(x)$.

(L3) $\lim \mathcal{F}(x) \odot \lim \mathcal{G}(x) \leq \lim(\mathcal{F} \odot \mathcal{G})(x)$.

The pair $(X, \lim)$ is called an $(L,*,\odot)$-*limit space.*

A map lim : $F^s_*(X) \to L^X$ is called a *stratified* $(L,*,\odot)$-*limit structure* on X where $F^s_*(X)$ is a family of stratified $(L,*)$-filters.

Let $(X, \lim_X)$ and $(Y, \lim_Y)$ be $(L,*,\odot)$-limit spaces. A map $\phi : (X, \lim_X) \to (Y, \lim_Y)$ is called *continuous* if for all $x \in X$ and $\mathcal{F} \in F_*(X)$,

$$\lim_X \mathcal{F}(x) \leq \lim_Y \phi^{\Rightarrow}(\mathcal{F})(\phi(x)).$$

We say $\lim_1$ is *finer* than $\lim_2$ (or $\lim_2$ is *coarser* than $\lim_1$) iff $\lim_1 \leq \lim_2$.

We define $\lim_\top, \lim_\perp : F_*(X) \to L^X$ as follows: for each $x \in X$,

$$\lim_\top(\mathcal{F})(x) = \begin{cases} \top, & \text{if } \mathcal{F} \geq [x], \\ \perp, & \text{otherwise.} \end{cases} \qquad \lim_\perp(\mathcal{F})(x) = \top,\ \forall \mathcal{F} \in F_*(X).$$

Then $\lim_\top$ (resp. $\lim_\perp$) is the finest (resp. coarsest) $(L,*,\odot)$ limit structure.

Remark 3.1. In above definition, a map $\lim : F^s_*(X) \to L^X$ is a SL-generalized convergence operator in Orpen's sense [10] if it satisfies (L1) and (L2). A stratified $(L,*,\wedge)$-limit structure on X is called a SL-strong limit structure in Orpen's sense [10].

Theorem 3.2. *Let* $\lim_1$ *and* $\lim_2$ *be* $(L,*,\odot)$*-limit structures on* X*. We define a map* $\lim_1 \odot_* \lim_2 : F_*(X) \to L^X$ *as follows:*

$$(\lim_1 \odot_* \lim_2)(\mathcal{F})(x) = \bigvee\{\lim_1(\mathcal{F}_1)(x) \odot \lim_2(\mathcal{F}_2)(x) \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}\}.$$

Then (1) $\lim_1 \odot_* \lim_2$ *is an* $(L,*,\odot)$*-limit structure on* X *which is coarser than* $\lim_1$ *and* $\lim_2$*. In particular, if* $\odot = *$, $\lim_1 *_* \lim_2$ *is the finest* $(L,*,*)$*-limit structure on* X *which is coarser than* $\lim_1$ *and* $\lim_2$*.*

(2) $\lim_1 \wedge \lim_2$ *is the coarsest* $(L,*,\odot)$*-limit structure on* X *which is finer than* $\lim_1$ *and* $\lim_2$*.*

Proof. (1) (L1). Since $[x] * [x] \leq [x]$, we have

$$(\lim_1 \odot_* \lim_2)([x])(x) \geq \lim_1([x])(x) \odot \lim_2([x])(x) = \top.$$

(L2) is easy. (L3)

$$\begin{aligned}
&(\lim_1 \odot_* \lim_2)(\mathcal{F})(x) \odot (\lim_1 \odot_* \lim_2)(\mathcal{G})(x) \\
&= \bigvee\{\lim_1(\mathcal{F}_1)(x) \odot \lim_2(\mathcal{F}_2)(x) \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}\} \\
&\odot \bigvee\{\lim_1(\mathcal{G}_1)(x) \odot \lim_2(\mathcal{G}_2)(x) \mid \mathcal{G}_1 * \mathcal{G}_2 \leq \mathcal{G}\} \\
&\leq \bigvee\{\lim_1(\mathcal{F}_1)(x) \odot \lim_2(\mathcal{F}_2)(x) \odot \lim_1(\mathcal{G}_1)(x) \odot \lim_2(\mathcal{G}_2)(x) \\
&\quad \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}, \mathcal{G}_1 * \mathcal{G}_2 \leq \mathcal{G}\} \\
&\leq \bigvee\{\lim_1(\mathcal{F}_1 \odot \mathcal{G}_1)(x) \odot \lim_2(\mathcal{F}_2 \odot \mathcal{G}_2)(x) \mid (\mathcal{F}_1 * \mathcal{F}_2) \odot (\mathcal{G}_1 * \mathcal{G}_2) \leq \mathcal{F} \odot \mathcal{G}\} \\
&\quad (\text{ Since } \odot \text{ dominates } *,) \\
&\leq \bigvee\{\lim_1(\mathcal{F}_1 \odot \mathcal{G}_1)(x) \odot \lim_2(\mathcal{F}_2 \odot \mathcal{G}_2)(x) \mid (\mathcal{F}_1 \odot \mathcal{G}_1) * (\mathcal{F}_2 \odot \mathcal{G}_2) \leq \mathcal{F} \odot \mathcal{G}\} \\
&\leq (\lim_1 \odot_* \lim_2)(\mathcal{F} \odot \mathcal{G})(x).
\end{aligned}$$

Since $\mathcal{F} * [x] \leq \mathcal{F}$ for each $x \in X$, we have $(\lim_1 \odot_* \lim_2)(\mathcal{F})(x) \geq \lim_i(\mathcal{F})(x) \odot \lim_j([x])(x) \geq \lim_i(\mathcal{F})(x)$ for $i \neq j \in \{1,2\}$.

If $* = \odot$ and $\lim \geq \lim_i$ for $i = 1,2$, then $\lim \geq (\lim_1 *_* \lim_2)$ from

$$\begin{aligned}
\lim(\mathcal{F})(x) &\geq \bigvee\{\lim(\mathcal{F}_1 * \mathcal{F}_2)(x) \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}\} \\
&\geq \bigvee\{\lim(\mathcal{F}_1)(x) * \lim(\mathcal{F}_2)(x) \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}\} \\
&\geq \bigvee\{\lim_1(\mathcal{F}_1)(x) * \lim_2(\mathcal{F}_2)(x) \mid \mathcal{F}_1 * \mathcal{F}_2 \leq \mathcal{F}\} \\
&= (\lim_1 *_* \lim_2)(\mathcal{F})(x).
\end{aligned}$$

(2)

$$\begin{aligned}&(\lim_1 \wedge \lim_2)(\mathcal{F}) \odot (\lim_1 \wedge \lim_2)(\mathcal{G})\\ &\leq (\lim_1(\mathcal{F}) \odot \lim_1(\mathcal{G})) \wedge (\lim_2(\mathcal{F}) \odot \lim_2(\mathcal{G}))\\ &\leq \lim_1(\mathcal{F} \odot \mathcal{G}) \wedge \lim_2(\mathcal{F} \odot \mathcal{G}) = (\lim_1 \wedge \lim_2)(\mathcal{F} \odot \mathcal{G}).\end{aligned}$$

If $\lim \leq \lim_i$ for $i = 1, 2$, then $\lim \leq (\lim_1 \wedge \lim_2)$.

Theorem 3.3. *For each $x \in X$, let $H^x : F_*(X) \to L^{L^X}$ be a map satisfying the following conditions: for $\uparrow \in \{\Rightarrow, \to\}$,*

(H1) $H^x([x])(f) \uparrow [x](f) = \top$, *for each* $f \in L^X$.

(H2) *If* $\mathcal{F} \leq \mathcal{G}$, *then* $H^x(\mathcal{F}) \geq H^x(\mathcal{G})$.

(H3) $H^x(\mathcal{F} \odot \mathcal{G}) \leq H^x(\mathcal{F}) \odot H^x(\mathcal{G})$.

We define a map $\lim^{\uparrow}_H : F(X) \to L^X$ *as follows:*

$$\lim^{\uparrow}_H(\mathcal{F})(x) = \bigwedge_{f \in L^X} (H^x(\mathcal{F})(f) \uparrow \mathcal{F}(f)).$$

Then the following properties hold.

(1) $\lim^{\uparrow}_H$ *is an* $(L, *, \odot)$*-limit structure for* $\uparrow \in \{\Rightarrow, \to\}$.

(2) $\lim^{\uparrow}_{HX}(\mathcal{F})(x) \uparrow \lim^{\uparrow}_{HY}(\psi^{\Rightarrow}(\mathcal{F})(\psi(x)) \geq \bigwedge_{g \in L^Y} \Big(H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g) \uparrow H^x_X(\mathcal{F})(\psi^{\leftarrow}(g))\Big)$.

(3) *If* $\psi : (X, H^x_X) \to (Y, H^{\psi(x)}_Y)$ *is a map such that* $H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g) \leq (H^x_X(\mathcal{F})(\psi^{\leftarrow}(g))$ *for each* $x \in X, g \in L^X, \mathcal{F} \in F_*(X)$, *then* $\psi : (X, \lim^{\Rightarrow}_{HX}) \to (Y, \lim^{\Rightarrow}_{HY})$ *is continuous.*

Proof. (L1) Since $H^x([x])(f) \uparrow [x](f) = \top$,

$$\lim^{\uparrow}_H([x])(x) = \bigwedge_{f \in L^X} (H^x([x])(f) \uparrow [x](f)) = \top.$$

(L2) If $\mathcal{F} \leq \mathcal{G}$, by (H2) an Lemma 2.7(3),

$$\lim^{\uparrow}_H(\mathcal{F})(x) = \bigwedge_{f \in L^X} (H^x(\mathcal{F})(f) \uparrow \mathcal{F}(f)) \leq \bigwedge_{f \in L^X} (H^x(\mathcal{G})(f) \uparrow \mathcal{G}(f)) = \lim^{\uparrow}_H(\mathcal{G})(x).$$

(L3) For each $\mathcal{F}, \mathcal{G} \in F(X)$,

$$\begin{aligned}&\lim^{\uparrow}_H(\mathcal{F})(x) \odot \lim^{\uparrow}_H(\mathcal{G})(x)\\ &= \Big(\bigwedge_{f \in L^X} (H^x(\mathcal{F})(f) \uparrow \mathcal{F}(f))\Big) \odot \Big(\bigwedge_{g \in L^X} (H^x(\mathcal{G})(g) \uparrow \mathcal{G}(g))\Big)\\ &\leq \bigwedge_{f \in L^X} \bigwedge_{g \in L^X} \Big((H^x(\mathcal{F})(f) \uparrow \mathcal{F}(f)) \odot (H^x(\mathcal{G})(g) \uparrow \mathcal{G}(g))\Big)\\ &\leq \bigwedge_{f \in L^X} \bigwedge_{g \in L^X} \Big((H^x(\mathcal{F})(f) \odot H^x(\mathcal{G})(g)) \uparrow (\mathcal{F}(f) \odot \mathcal{G}(g))\Big) (\text{by Lemma 7(6)})\\ &\leq \bigwedge_{f \in L^X} \Big((H^x(\mathcal{F})(f) \odot H^x(\mathcal{G})(f)) \uparrow (\mathcal{F}(f) \odot \mathcal{G}(f))\Big)\\ &\leq \bigwedge_{f \in L^X} \Big((H^x(\mathcal{F} \odot \mathcal{G})(f)) \uparrow ((\mathcal{F} \odot \mathcal{G})(f))\Big)\\ &= \lim^{\uparrow}_H(\mathcal{F} \odot \mathcal{G})(x).\end{aligned}$$

(2) For each $\mathcal{F}\in F(X)$,

$$\begin{aligned}
&\lim^{\uparrow}_{HX}(\mathcal{F})(x)\uparrow \lim^{\uparrow}_{HY}(\psi^{\Rightarrow}(\mathcal{F})(\psi(x))\\
&=\Big(\bigwedge_{f\in L^X}(H^x_X(\mathcal{F})(f)\uparrow \mathcal{F}(f))\Big)\uparrow\Big(\bigwedge_{g\in L^Y}(H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g)\uparrow \psi^{\Rightarrow}(\mathcal{F})(g))\Big)\\
&\geq \Big(\bigwedge_{g\in L^Y}(H^x_X(\mathcal{F})(\psi^{\leftarrow}(g))\uparrow \mathcal{F}(\psi^{\leftarrow}(g)))\Big)\uparrow\\
&\Big(\bigwedge_{g\in L^Y}(H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g)\uparrow \psi^{\Rightarrow}(\mathcal{F})(g))\Big)\\
&\geq \bigwedge_{g\in L^Y}\Big((H^x_X(\mathcal{F})(\psi^{\leftarrow}(g))\uparrow \mathcal{F}(\psi^{\leftarrow}(g)))\uparrow(H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g)\uparrow \mathcal{F}(\psi^{\leftarrow}(g))\Big)\\
&\text{(by Lemma 2.7(8))}\\
&\geq \bigwedge_{g\in L^Y}\Big(H^{\psi(x)}_Y(\psi^{\Rightarrow}(\mathcal{F}))(g)\uparrow H^x_X(\mathcal{F})(\psi^{\leftarrow}(g))\Big).
\end{aligned}$$

(3) For $\uparrow=\Rightarrow$, since $a\Rightarrow b=\top$ iff $a\leq b$ from Lemma 2.7(4), $\lim^{\Rightarrow}_{HX}(\mathcal{F})(x)\leq \lim^{\Rightarrow}_{HY}(\psi^{\Rightarrow}(\mathcal{F})(\psi(x))$. Hence $\psi:(X,\lim^{\Rightarrow}_{HX})\to(Y,\lim^{\Rightarrow}_{HY})$ is continuous.

Example 3.4. Let $X=\{x_1,x_2\}$ and $Y=\{y_1,y_2\}$ be sets, $(L=[0,1],*)$ an GL-monoid with $a*b=a\cdot b$ and $f,g\in[0,1]^X$ as follows:

$$f(x_1)=1,f(x_2)=0.6,\;\; g(x_1)=0.5,g(x_2)=1.$$

Define $([0,1],*)$-filters as $\mathcal{F},\mathcal{G}:[0,1]^X\to[0,1]$ as follows:

$$\mathcal{F}(h)=\begin{cases}1, & \text{if } h=1_X,\\ 0.4^n, & \text{if } f^n\leq h\not\geq f^{n-1}, n\in N\\ 0, & \text{otherwise.}\end{cases}$$

$$\mathcal{G}(h)=\begin{cases}1, & \text{if } h=1_X,\\ 0.3^n, & \text{if } g^n\leq h\not\geq g^{n-1}, n\in N\\ 0, & \text{otherwise.}\end{cases}$$

where $k^n=k^{n-1}*h$ and $h^0=1_X$ for $h\in\{f,g\}$.

(1) Let $x\odot y=x^{\frac{1}{3}}\cdot y^{\frac{1}{3}}$. Define a map $\lim:F(X)\to[0,1]^X$ as follows, for $x\in\{x_1,x_2\}$,

$$\lim(\mathcal{G})(x)=\begin{cases}1, & \text{if } \mathcal{G}\geq[x],\\ 0.5, & \text{if } \mathcal{G}\not\geq[x].\end{cases}$$

Since $\mathcal{F}(f)=0.4<[x_2](f)=0.6$, it does not satisfy the condition (L3) of Definition 1 from:

$$(0.5)^{\frac{2}{3}}=\lim(\mathcal{F})(x_2)\odot\lim(\mathcal{F})(x_2)\not\leq\lim(\mathcal{F}\odot\mathcal{F})(x_2)=0.5.$$

Hence lim is not an $([0,1],*,\odot)$-limit structure on X.

(2) Define two constant maps $H^{x_1},H^{x_2}:F(X)\to L^{L^X}$ as follows

$$H^{x_1}(\mathcal{H})=\mathcal{F},\;\; H^{x_2}(\mathcal{H})=\mathcal{G}.$$

(2-a) Give $\odot=\wedge$. Then $([0,1],*,\wedge)$ is an M-ecl-premonoid. We obtain $a\to b=1$ if $a\leq b$ and $a\to b=b$ otherwise. It satisfies the conditions (H1), (H2) and (H3) because

$$0.4^n=\mathcal{F}(h)\leq f^n(x_1)=[x_1](h)=1,\;0.3^n=\mathcal{G}(h)\leq g^n(x_2)=[x_2](h)=1$$

$$H^{x_1}(\mathcal{F}_1 \wedge \mathcal{F}_2) = \mathcal{F} = H^{x_1}(\mathcal{F}_1) \wedge H^{x_1}(\mathcal{F}_2)$$
$$H^{x_2}(\mathcal{F}_1 \wedge \mathcal{F}_2) = \mathcal{G} = H^{x_2}(\mathcal{F}_1) \wedge H^{x_2}(\mathcal{F}_2).$$

For $\uparrow = \rightarrow$, $\lim_H^{\rightarrow}$ is an $(L, *, \wedge)$-limit structure as follows:

$$\begin{aligned} \lim_H^{\rightarrow}(\mathcal{H})(x_1) &= \bigwedge_{f\in L^X}(H^{x_1}(\mathcal{H})(f) \rightarrow \mathcal{H}(f)) = \bigwedge_{n\in N}(0.4^n \rightarrow \mathcal{H}(f^n)) \\ \lim_H^{\rightarrow}(\mathcal{H})(x_2) &= \bigwedge_{n\in N}(0.3^n \rightarrow \mathcal{H}(g^n)) \end{aligned}$$

(2-b) For $*$, we obtain $a \Rightarrow b = 1$ if $a \le b$ and $a \Rightarrow b = \frac{b}{a}$ otherwise. It satisfies the conditions (H1), (H2) and (H3) in (2-a). Thus $\lim_H^{\Rightarrow}$ is an $(L, *, \wedge)$-limit structure as follows:

$$\begin{aligned} \lim_H^{\Rightarrow}(\mathcal{H})(x_1) &= \bigwedge_{f\in L^X}(H^{x_1}(\mathcal{H})(f) \Rightarrow \mathcal{H}(f)) = \bigwedge_{n\in N}(0.4^n \Rightarrow \mathcal{H}(f^n)) \\ \lim_H^{\Rightarrow}(\mathcal{H})(x_2) &= \bigwedge_{n\in N}(0.3^n \Rightarrow \mathcal{H}(g^n)). \end{aligned}$$

(2-c) Define $\psi : (X, H_X^{x_i}) \rightarrow (Y, H_Y^{\psi(x_i)})$ is a map with $\psi(x_i) = y_i$ for $i = 1, 2$ and

$$H_Y^{y_1}(\mathcal{H}) = \psi^{\Rightarrow}(\mathcal{F}),\ H_Y^{y_2}(\mathcal{H}) = \psi^{\Rightarrow}(\mathcal{G}).$$

For $\odot = \wedge, * = \cdot$, since $a \rightarrow b = \top$ iff $a \le b$ iff $a \Rightarrow b = \top$, we have, for each $x \in X, g \in L^X$,

$$\begin{aligned} &H_Y^{\psi(x_1)}(\psi^{\Rightarrow}(\mathcal{F}))(g) = \psi^{\Rightarrow}(\mathcal{F})(g) = \mathcal{F}(\psi^{\leftarrow}(g)) = H_X^{x_1}(\mathcal{F})(\psi^{\leftarrow}(g)) \\ &H_Y^{\psi(x_2)}(\psi^{\Rightarrow}(\mathcal{F}))(g) = \psi^{\Rightarrow}(\mathcal{G})(g) = \mathcal{G}(\psi^{\leftarrow}(g)) = H_X^{x_2}(\mathcal{F})(\psi^{\leftarrow}(g)) \\ &\lim_{HX}^{\uparrow}(\mathcal{F})(x) \uparrow \lim_{HY}^{\uparrow}(\psi^{\Rightarrow}(\mathcal{F})(\psi(x)) \\ &\ge \bigwedge_{g\in L^Y}\Big(H_Y^{\psi(x)}(\psi^{\Rightarrow}(\mathcal{F}))(g) \uparrow H_X^{x}(\mathcal{F})(\psi^{\leftarrow}(g))\Big) = \top. \end{aligned}$$

Hence $\psi : (X, \lim_{HX}^{\uparrow}) \rightarrow (Y, \lim_{HY}^{\uparrow})$ is continuous for each $\uparrow \in \{\Rightarrow, \rightarrow\}$.

(3) Define a constant map $H^{x_2} : F(X) \rightarrow L^{L^X}$ as follows: for all $\mathcal{H} \in F_*(X)$,

$$H^{x_2}(\mathcal{H}) = \mathcal{G}.$$

For $\odot = *$, since $H^{x_2}(\mathcal{H}_1 * \mathcal{H}_2)(g) = \mathcal{G}(g) = 0.3 \not\le H^{x_2}(\mathcal{H}_1)(g) * H^{x_2}(\mathcal{H}_2)(g) = \mathcal{G}(g) * \mathcal{G}(g) = 0.09$, it does not satisfy the condition (H3). Put $\mathcal{H}_1(g) = 0.4$. Then

$$\lim_H(\mathcal{H}_1 * \mathcal{H}_2)(x_2) = 0.3 \rightarrow (\mathcal{H}_1 * \mathcal{H}_1)(g) = \frac{8}{15},$$
$$\lim_H(\mathcal{H}_1)(x_2) * \lim_H(\mathcal{H}_2)(x_2) = (0.3 \rightarrow \mathcal{H}_1(g)) * (0.3 \rightarrow \mathcal{H}_2(g)) = 1.$$

So, $\lim_H(\mathcal{H}_1 * \mathcal{H}_2)(x_2) = \frac{8}{15} \not\ge 1 = \lim_H(\mathcal{H}_1)(x_2) * \lim_H(\mathcal{H}_2)(x_2)$. Hence $\lim_H^{\Rightarrow}$ is not an $(L, *, *)$-limit structure.

(4) Define $x \odot y = x^{\frac{1}{3}} \cdot y^{\frac{1}{3}}$ and two constant maps H^{x_1}, H^{x_2} as same in (2). We obtain $a \rightarrow b = 1$ if $a \le b^3$ and $a \rightarrow b = \frac{b^3}{a}$ otherwise. It satisfies the conditions (H1), (H2) and (H3) because

$$0.4^n = \mathcal{F}(h) \le f^n(x_1) = [x_1](h) = 1,\ 0.3^n = \mathcal{G}(h) \le g^n(x_2) = [x_2](h) = 1$$
$$H^{x_1}(\mathcal{F}_1 \odot \mathcal{F}_2)(f) = \mathcal{F}(f) \le H^{x_1}(\mathcal{F}_1) \odot H^{x_1}(\mathcal{F}_2) = \mathcal{F}(f) \odot \mathcal{F}(f) = (\mathcal{F}(f))^{\frac{2}{3}}$$
$$H^{x_2}(\mathcal{F}_1 \odot \mathcal{F}_2)(g) = \mathcal{G}(g) \le \mathcal{G}(g) \odot \mathcal{G}(g) = (\mathcal{G}(g))^{\frac{2}{3}}.$$

Hence $\lim_H^{\rightarrow}$ is an $(L,*,\odot)$-limit structure as follows:

$$\begin{aligned}\lim_H^{\rightarrow}(\mathcal{H})(x_1) &= \bigwedge_{f\in L^X}(H^{x_1}(\mathcal{H})(f)\rightarrow\mathcal{H}(f)) = \bigwedge_{n\in N}(0.4^n\rightarrow\mathcal{H}(f^n))\\ \lim_H^{\rightarrow}(\mathcal{H})(x_2) &= \bigwedge_{n\in N}(0.3^n\rightarrow\mathcal{H}(g^n)).\end{aligned}$$

Moreover, $\lim_H^{\Rightarrow}$ is an $(L,*,\odot)$-limit structure as same as in (2-b).

Example 3.5. Let $(L,*,\odot)$ be an M-ecl-premonoid. Let a map $H^x : F(X)\rightarrow L^{L^X}$ defined as $H^x(\mathcal{F}) = [x]$ for all $\mathcal{F}\in F(X)$. For $\uparrow\Longrightarrow$ since $H^x([x])(f) = [x](f)\leq[x](f)$ and

$$H^x(\mathcal{F}\odot\mathcal{G})(f) = [x](f)\leq H^x(\mathcal{F})(f)\odot H^x(\mathcal{G})(f) = [x](f)\odot[x](f),$$

it satisfies the following conditions (H1),(H2)and (H3). Thus $\lim_H^{\Rightarrow}(\mathcal{F})(x) = \bigwedge_{f\in L^X}(H^x(\mathcal{F})(f)\rightarrow\mathcal{F}(f)) = \bigwedge_{f\in L^X}([x](f)\rightarrow\mathcal{F}(f))$. Then $\lim_H^{\Rightarrow}$ is an $(L,*,\odot)$-limit structure.

Define $\psi:(X,H_X^x)\rightarrow(Y,H_Y^{\psi(x)})$ is a map with $H_Y^{\psi(x)}(\mathcal{H}) = [\psi(x)]$. Since, for each $x\in X, g\in L^X$,

$$H_Y^{\psi(x)}(\psi^{\Rightarrow}(\mathcal{F}))(g) = [\psi(x)](g) = [x](\psi^{\leftarrow}(g)) = H_X^x(\mathcal{F})(\psi^{\leftarrow}(g))$$

then $\psi:(X,\lim_{HX}^{\Rightarrow})\rightarrow(Y,\lim_{HY}^{\Rightarrow})$ is continuous.

Example 3.6. Let $(L,*,\odot)$ be an M-ecl-premonoid. Let $\mathcal{I}:L^X\rightarrow L^X$ be an $(L,*)$-interior operator. Let a map $H^x:F(X)\rightarrow L^{L^X}$ defined as $H^x(\mathcal{F})(f) = \mathcal{I}(f)(x)$ for all $\mathcal{F}\in F(X)$. For $\uparrow\Longrightarrow$, since $H^x(\mathcal{I}(-)(x))(f) = \mathcal{I}(f)(x)\leq[x](f) = f(x)$ and $H^x(\mathcal{F}\odot\mathcal{G})(f) = \mathcal{I}(f)(x)\leq H^x(\mathcal{F})(f)\odot H^x(\mathcal{G})(f) = \mathcal{I}(f)(x)\odot\mathcal{I}(f)(x)$, it satisfies the following conditions (H1),(H2)and (H3).

$$\begin{aligned}\lim_H^{\Rightarrow}(\mathcal{F})(x) &= \bigwedge_{f\in L^X}(H^x(\mathcal{F})(f)\Rightarrow\mathcal{F}(f))\\ &= \bigwedge_{f\in L^X}(\mathcal{I}(f)(x)\Rightarrow\mathcal{F}(f))\end{aligned}$$

Then $\lim_H^{\Rightarrow}$ is an $(L,*,\odot)$-limit structure.

Let $(X,\mathcal{I}_X)$ and $(Y,\mathcal{I}_Y)$ be an $(L,*)$-interior spaces. Define a surjective map $\psi:(X,H_X^x)\rightarrow(Y,H_Y^{\psi(x)})$ as $H_X^x(\mathcal{F}) = \mathcal{I}_X(-)(x)$ and $H_Y^{\psi(x)}(\mathcal{H}) = \mathcal{I}_Y(-)(\psi(x))$. Then

$$\begin{aligned}&\lim_{HX}^{\Rightarrow}(\mathcal{F})(x)\Rightarrow\lim_{HY}^{\Rightarrow}(\psi^{\Rightarrow}(\mathcal{F})(\psi(x))\\ &\geq\bigwedge_{g\in L^Y}\Big(\psi^{\leftarrow}(\mathcal{I}_Y(g))(x)\Rightarrow\mathcal{I}_X(\psi^{\leftarrow}(g))(x)\Big).\end{aligned}$$

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