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SOME RELATIONSHIPS BETWEEN (p, q)-EULER POLYNOMIAL OF THE SECOND KIND AND (p, q)-OTHERS POLYNOMIALS†

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Abstract. We use the definition of Euler polynomials of the second kind with (p, q)-numbers to identify some identities and properties of these polynomials. We also investigate some relationships between (p, q)-Euler polynomials of the second kind, (p, q)-Bernoulli polynomials, and (p, q)-tangent polynomials by using the properties of (p, q)-exponential function.

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1. Introduction

For any $n \in \mathbb{C}$, the q-number is defined by

$$[n]_q = \frac{1 - q^n}{1 - q} = \sum_{0 \leq i \leq n} q^i = 1 + q + q^2 + \cdots + q^{n-1}.$$

An intensive and somewhat surprising interest in q-numbers appeared in many areas of mathematics and applications including q-difference equations, special functions, q-combinatorics, q-integrable systems, variational q-calculus, q-series, and so on.

In [1-6], R. Chakrabarti and R. Jagannathan, G. Brodimas et al., and M. Arik et al. introduced the (p, q)-number to unify various forms of q-oscillator algebras.

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For any $n \in \mathbb{C}$, the (p,q)-number is defined by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}. \tag{1.1}$$

We can note that the (p,q)-number reduces to a q-number when $p = 1$. In particular, $\lim_{q\to 1}[n]_{p,q} = n$ with $p = 1$(see [1-2,10]). In [5], R. Chakrabarti and R. Jagannathan studied (p,q)-differentiation, (p,q)-integration, and the (p,q)-exponential with the introduction of the (p,q)-numbers. Some applications of (p,q)-hypergeometric series in the context of two-parameter quantum groups could be found in 2003, and many mathematicians have studied (p,q)-Stirling numbers, (p,q)-Bernoulli polynomials, and others polynomials by using the (p,q)-numbers (see [1-17]).

Definition 1.1. We define the (p,q)-derivative operator as

$$D_{p,q}f(x) = \frac{f(px) - f(qx)}{(p-q)x}, \qquad x \neq 0, \tag{1.2}$$

and provided that f is differentiable at 0, $D_{p,q}f(0) = f'(0)$. The following properties of (p,q)-derivative operator are immediate.

Theorem 1.2. For the operator $D_{p,q}$ the following hold:

(i) Derivative of a product

$$\begin{aligned} D_{p,q}(f(x)g(x)) &= f(px)D_{p,q}g(x) + g(qx)D_{p,q}f(x) \\ &= g(px)D_{p,q}f(x) + f(qx)D_{p,q}g(x). \end{aligned} \tag{1.3}$$

(ii) Derivative of a ratio

$$\begin{aligned} D_{p,q}\left(\frac{f(x)}{g(x)}\right) &= \frac{g(qx)D_{p,q}f(x) - f(qx)D_{p,q}g(x)}{g(px)g(qx)} \\ &= \frac{g(px)D_{p,q}f(x) - f(px)D_{p,q}g(x)}{g(px)g(qx)}. \end{aligned} \tag{1.4}$$

In [6], R. B. Corcino found (p,q)-extension of binomials coefficients to establish various properties that are related to horizontal, triangular, and vertical functions.

Definition 1.3. The (p,q)-analogue of $(x+a)^n$ is defined by

(i) $$(x+a)^n_{p,q} = \begin{cases} 1 & \text{if } n = 0 \\ (x+a)(px+aq)\cdots(p^{n-2}x+aq^{n-2})(p^{n-1}x+aq^{n-1}) & \text{if } n \neq 0 \end{cases}$$

(ii) $$(x+a)^n_{p,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{k}{2}} q^{\binom{n-k}{2}} x^k a^{n-k}, \tag{1.5}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}$ is (p,q)-Gauss-Binomial coefficient.

Definition 1.4. Let z be any complex number with $|z| < 1$. Two forms of (p,q)-exponential functions are defined by

$$\begin{aligned} e_{p,q}(z) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}, \\ e_{p^{-1},q^{-1}}(z) &= \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{z^n}{[n]_{p^{-1},q^{-1}}!} = \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{z^n}{[n]_{p,q}!}. \end{aligned} \tag{1.6}$$

These forms are connected by the following relationship:

$$e_{p,q}(z) e_{p^{-1},q^{-1}}(-z) = 1. \tag{1.7}$$

In 1961, L.Calitz introduced the Euler numbers and polynomials of the second kind and found some properties thereof.

Definition 1.5. The classical Euler numbers, $\widetilde{E}_n$, and the classical Euler polynomials, $\widetilde{E}_n(x)$, of the second kind are defined by means of the following functions:

$$\sum_{n=0}^{\infty} \widetilde{E}_n \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}}, \quad \sum_{n=0}^{\infty} \widetilde{E}_n(x) \frac{t^n}{n!} = \frac{2}{e^t + e^{-t}} e^{tx}. \tag{1.8}$$

These numbers and polynomials are related to the coefficients of the series for function $\frac{1}{\cosh t}$.

Theorem 1.6. For any positive integer n, we have

(i) For any positive integer m(=odd),

$$\widetilde{E}_n(x) = m^n \sum_{i=0}^{m-1} (-1)^i \widetilde{E}_n \left(\frac{2i + x + 1 - m}{m} \right) \text{ for } n \geq 0, \tag{1.9}$$

(ii) $$\widetilde{E}_l(x+y) = \sum_{n=0}^{l} \binom{l}{n} \widetilde{E}_n(x) y^{l-n},$$

(iii) $$\widetilde{E}_n(x) = (-1)^n \widetilde{E}_n(-x).$$

Since these numbers and polynomials' discovery, many mathematicians have extensively studied these numbers and polynomials of the second kind and expanded several properties thereof (see [1-2,9,11,14,16]). R. P. Agarwal and C. S. Ryoo used q-numbers to define the q-extension of Euler polynomials of the second kind and investigate their properties. In [2], the q-extension of Euler polynomials of the second kind maintained the properties of Euler polynomials of the second kind, that is, an alternative sum of polynomials with even coefficients despite containing q-numbers.

Definition 1.7. Let n be any non-negative integer. For $|q| < 1, x \in \mathbb{C}$, q-Euler polynomials of the second kind are defined by

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,q}(x) \frac{t^n}{[n]_q!} = \frac{2}{e_q(t) + e_q(-t)} e_q(tx). \tag{1.10}$$

Theorem 1.8. For $|q| < 1$, we have

$$\begin{aligned}
\text{(i)} \quad & 2\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} x^l \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^k + \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_q (-1)^{n-l} \right) \widetilde{\mathcal{E}}_{l,q}(x), \\
\text{(ii)} \quad & 2\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q q^{\binom{n-l}{2}} x^l \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1+q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1-q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x), \\
\text{(iii)} \quad & 2\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q (-1)^{n-l} q^{\binom{n-l}{2}} x^l \\
&= \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_q \left(\prod_{k=0}^{n-l-1} (1-q^k) + (-1)^{n-l} \prod_{k=0}^{n-l-1} (1+q^k) \right) \widetilde{\mathcal{E}}_{l,q}(x).
\end{aligned} \tag{1.11}$$

In this paper, the main aim is to extend q-Euler polynomials of the second kind and study some properties of these polynomials by using (p,q)-numbers. We construct an important lemma to find the special properties of (p,q)-Euler polynomials of the second kind. Our paper is organized as follows: In Section 2, we define (p,q)-Euler polynomials of the second kind and find some properties thereof. In Section 3, we consider some properties of (p,q)-Euler polynomials of the second kind and establish some relationships between (p,q)-Euler polynomials of the second kind and (p,q)-other polynomials.

2. Some basic properties of the (p,q)-Euler polynomials of the second kind

In this section, we define Euler numbers of the second kind and Euler polynomials of the second kind with (p,q)-numbers. Using the generating function of these polynomials of the second kind, we find some basic properties and identities.

Definition 2.1. Let n be nonnegative integers and $|q/p|<1$. Then we define (p,q)-Euler polynomials of the second kind by

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}=\frac{2}{e_{p,q}(t)+e_{p,q}(-t)}e_{p,q}(tx). \tag{2.1}$$

Substituting $x=0$ in Definiton 6, above can be simplified as follows:

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(0)\frac{t^n}{[n]_{p,q}!}=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}\frac{t^n}{[n]_{p,q}!}=\frac{2}{e_{p,q}(t)+e_{p,q}(-t)}=\frac{1}{\cosh_{p,q}(t)}, \tag{2.2}$$

where $\widetilde{\mathcal{E}}_{n,p,q}$ are (p,q)-Euler numbers of the second kind. If $q\to 1$ and $p=1$, we can obtain the classical Euler numbers of the second kind.

Theorem 2.2. For $|q|<|p|$, we get

$$\begin{aligned}
&\text{(i)} \qquad \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{k}{2}}(1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,p,q}(x)=2p^{\binom{n}{2}}x^n,\\
&\text{(ii)} \qquad \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{k}{2}}(1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,p,q}=\begin{cases} 2 & \text{if } n=0\\ 0 & \text{if } n\neq 0.\end{cases}
\end{aligned} \tag{2.3}$$

Proof. (i) For $e_{p,q}(t)\neq e_{p,q}(-t)$, we can turn Definition 6 into

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}(e_{p,q}(t)+e_{p,q}(-t))=2e_{p,q}(tx). \tag{2.4}$$

Using Cauchy's product, we can transform the above equation (2.4) as

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}(e_{p,q}(t)+e_{p,q}(-t))\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}p^{\binom{k}{2}}(1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!}\\
&=2\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{2.5}$$

The required relation now follows at once.

(ii) Using the same method as (i), we can obtain the required result, so we omit the proof. □

Theorem 2.3. Let $|q/p|<1$. Then, the following holds:

$$\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_{p,q}(1+(-1)^k)p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-k,p,q}(x)=2\sum_{k=0}^{[\frac{\hat{n}}{2}]}\begin{bmatrix} n\\ n-2k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-2k,p,q}(x), \tag{2.6}$$

where $[\hat{n}]$ is the greatest integer not to exceed n.

Proof. The left-hand side in the Theorem 2.2.(i) can be changed to

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}2p^{\binom{n}{2}}\frac{t^{2n}}{[2n]_{p,q}!}\\
&=2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}2n-k\\ k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}(x)\right)\frac{t^{2n-k}}{[2n-k]_{p,q}!} \qquad (2.7)\\
&=2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{\hat{n}}{2}]}\begin{bmatrix}n\\ n-2k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-2k,p,q}(x)\right)\frac{t^{n}}{[n]_{p,q}!},
\end{aligned}$$

where $[\hat{n}]$ is the greatest integer not to exceed n.
Therefore, we obtain the required relation at once. □

Example 2.3. Using Mathematica, the first six (p,q)-Euler polynomials are:

$$\begin{aligned}
\widetilde{\mathcal{E}}_{0,p,q}(x)&=1,\\
\widetilde{\mathcal{E}}_{1,p,q}(x)&=x,\\
\widetilde{\mathcal{E}}_{2,p,q}(x)&=-1+px^2,\\
\widetilde{\mathcal{E}}_{3,p,q}(x)&=x(-p^2-pq-q^2+p^3x^2),\\
\widetilde{\mathcal{E}}_{4,p,q}(x)&=q^4-p^5x^2+p^6x^4+p^4(1-qx^2)\\
&\quad+p^2q^2(2-qx^2)+p^3(q-2q^2x^2)+p(-1+q^3-q^4x^2),\\
\widetilde{\mathcal{E}}_{5,p,q}(x)&=-\frac{p(p^5-q^5)x}{p-q}+p^{10}x^5\\
&\quad-(p^2+q^2)(p^4+p^3q+p^2q^2+pq^3+q^4)x(-p^2-pq-q^2+p^3x^2).
\end{aligned} \qquad (2.8)$$

Corollary 2.4. From Theorem 2.3, we can see

$$\sum_{k=0}^{n}\begin{bmatrix}n\\ k\end{bmatrix}_{p,q}p^{\binom{k}{2}}(1+(-1)^k)\widetilde{\mathcal{E}}_{n-k,p,q}=2\sum_{k=0}^{[\frac{\hat{n}}{2}]}\begin{bmatrix}n\\ n-2k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-2k,p,q}, \qquad (2.9)$$

where $[\hat{n}]$ is a greatest integer not to exceed n.

Corollary 2.5. From Theorem 2.2, Theorem 2.3, and Corollary 2.4, one holds

$$\begin{aligned}
&\text{(i)}\qquad 2\sum_{k=0}^{[\frac{\hat{n}}{2}]}\begin{bmatrix}n\\ n-2k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-2k,p,q}(x)=p^{\binom{n}{2}}x^n,\\
&\text{(ii)}\qquad 2\sum_{k=0}^{[\frac{\hat{n}}{2}]}\begin{bmatrix}n\\ n-2k\end{bmatrix}_{p,q}p^{\binom{k}{2}}\widetilde{\mathcal{E}}_{n-2k,p,q}=\begin{cases}1 & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases},
\end{aligned} \qquad (2.10)$$

where $[\hat{n}]$ is a greatest integer not to exceed n.

Theorem 2.6. Let $|q/p|<1$. Then we have

$$\widetilde{\mathcal{E}}_{n,p,q}(x)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}x^{n-k}. \tag{2.11}$$

Proof. From the generating function of (p,q)-Euler polynomials of the second kind, we can obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}&=\frac{2}{e_{p,q}(t)+e_{p,q}(-t)}e_{p,q}(tx)\\&=\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}\frac{t^n}{[n]_{p,q}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}x^{n-k}\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{2.12}$$

The required relation now follows at once. □

Theorem 2.7. Let $|q/p|<1$. One has

$$\widetilde{\mathcal{E}}_{n,p,q}=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(-1)^{n-k}q^{\binom{n-k}{2}}x^{n-k}\widetilde{\mathcal{E}}_{k,p,q}(x). \tag{2.13}$$

Proof. Since $e_{p,q}(tx)e_{p^{-1},q^{-1}(-tx)}=1$, we can find

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}\frac{t^n}{[n]_{p,q}!}&=\frac{2}{e_{p,q}(t)+e_{p,q}(-t)}e_{p,q}(tx)e_{p^{-1},q^{-1}}(-tx)\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}(-1)^{n-k}q^{\binom{n-k}{2}}x^{n-k}\widetilde{\mathcal{E}}_{k,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!},\end{aligned} \tag{2.14}$$

which gives the required result immediately. □

Theorem 2.8. Let n,k be nonnegative integers. Then the follwing holds:

$$D_{p,q}\widetilde{\mathcal{E}}_{n,p,q}(x)=[n+1]_{p,q}\widetilde{\mathcal{E}}_{n,p,q}(px). \tag{2.15}$$

Proof. Considering (p,q)-derivative of x^{n-k} in Theorem 2.6, we find

$$\begin{aligned}D_{p,q}\widetilde{\mathcal{E}}_{n,p,q}(x)&=\sum_{k=0}^{n-1}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}[n-k]_{p,q}p^{\binom{n-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}x^{n-k-1}\\&=[n+1]_{p,q}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n+1-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}(x)^{n-k}\\&=[n+1]_{p,q}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}(px)^{n-k}\widetilde{\mathcal{E}}_{k,p,q}.\end{aligned} \tag{2.16}$$

Using Theorem 2.6 again, the required relation now follows. □

Theorem 2.9. For $|q/p| < 1$, the follwing holds:

$$\int_0^1 \widetilde{\mathcal{E}}_{n,p,q}(x)d_{p,q}x = \frac{\widetilde{\mathcal{E}}_{n+1,p,q}\left(\frac{1}{p}\right) - \widetilde{\mathcal{E}}_{n+1,p,q}}{[n+1]_{p,q}}, \tag{2.17}$$

where $\widetilde{\mathcal{E}}_{n,p,q}$ are (p,q)-Euler numbers of the second kind.

Proof. Applying (p,q)-integral in Theorem 2.6, we get

$$\begin{aligned}\int_0^1 \widetilde{\mathcal{E}}_{n,p,q}(x)d_{p,q}x &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \widetilde{\mathcal{E}}_{k,p,q} \int_0^1 x^{n-k} d_{p,q}x \\ &= \frac{1}{[n+1]_{p,q}} \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k \end{bmatrix}_{p,q} p^{\binom{n-k}{2}} \widetilde{\mathcal{E}}_{k,p,q} x^{n-k+1} \Big|_0^1 \\ &= \frac{1}{[n+1]_{p,q}} \left(\widetilde{\mathcal{E}}_{n+1,p,q}\left(\frac{1}{p}\right) - \widetilde{\mathcal{E}}_{n+1,p,q}(0) \right).\end{aligned} \tag{2.18}$$

Therefore, we complete the proof of Theorem 2.9. □

Corollary 2.10. From Theorem 2.9, one has

$$\int_a^b \widetilde{\mathcal{E}}_{n,p,q}(x)d_{p,q}x = \frac{\widetilde{\mathcal{E}}_{n+1,p,q}\left(p^{-1}b\right) - \widetilde{\mathcal{E}}_{n+1,p,q}\left(p^{-1}a\right)}{[n+1]_{p,q}}. \tag{2.19}$$

Theorem 2.11. For $|q/p| < 1$, we derive

$$\begin{aligned} &\text{(i)} \qquad \widetilde{\mathcal{E}}_{n,p,q}(x) = (-1)^n \widetilde{\mathcal{E}}_{n,p,q}(-x), \\ &\text{(ii)} \qquad p^{\binom{n}{2}} \widetilde{\mathcal{E}}_{n,p^{-1},q^{-1}}(x) = (-1)^1 \widetilde{\mathcal{E}}_{n,1,\frac{p}{q}}(-x). \end{aligned} \tag{2.20}$$

Proof. (i) Replacing x, t with $-x, -t$, respectively, in (p,q)-Euler polynomials of the second kind, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}(-x) \frac{(-t)^n}{[n]_{p,q}!} &= \frac{2}{e_{p,q}(-t) + e_{p,q}(t)} e_{p,q}(tx) \\ &= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!},\end{aligned} \tag{2.21}$$

which on comparing the coefficients of both sides immediately gives the required relation.

(ii) Setting $p = 1$ and $q = p/q$ in generating function of (p,q)-Euler polynomials of the second kind, we have

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,1,\frac{p}{q}}(-x) \frac{(-t)^n}{[n]_{1,\frac{p}{q}}!} = \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \widetilde{\mathcal{E}}_{n,1,\frac{p}{q}}(-x) \frac{t^n}{[n]_{p,q}!}. \tag{2.22}$$

From a property of (p, q)-numbers, we can note that

$$\begin{aligned}
&\text{(i)} \qquad [n]_{1,\frac{p}{q}}! = \frac{[n]_{p,q}!}{q^{\binom{n}{q}}} \\
&\text{(ii)} \qquad e_{1,\frac{p}{q}}(t) = e_{p^{-1},q^{-1}}(t) \\
&\text{(iii)} \qquad [n]_{p^{-1},q^{-1}}! = \frac{[n]_{p,q}!}{p^{\binom{n}{2}} q^{\binom{n}{2}}}.
\end{aligned} \tag{2.23}$$

Applying the above properties, (2.23), we derive

$$\begin{aligned}
\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,1,\frac{p}{q}}(-x) \frac{(-t)^n}{[n]_{1,\frac{p}{q}}!} &= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p^{-1},q^{-1}}(x) \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \\
&= \sum_{n=0}^{\infty} p^{\binom{n}{2}} q^{\binom{n}{2}} \widetilde{\mathcal{E}}_{n,p^{-1},q^{-1}}(x) \frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{2.24}$$

Therefore, we complete the proof of Theorem 2.11. □

3. Some relationships between (p, q)-Euler polynomials of the second kind and (p, q)-other polynomials

In this section, we find symmetric properties of (p, q)-Euler polynomials of the second kind. Using the horizontal generating function for (p, q)-binomial coefficient, we also investigate some relations among (p, q)-Euler polynomials of the second kind, (p, q)-Bernoulli polynomials and (p, q)-tangent polynomials.

Theorem 3.1. For $a, b \neq 0$, we have

$$\begin{aligned}
&\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{b}{a}\right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,p,q}(\frac{a}{b}x) \widetilde{\mathcal{E}}_{k,p,q}(\frac{b}{a}y) \\
&= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{a}{b}\right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,p,q}(\frac{b}{a}x) \widetilde{\mathcal{E}}_{k,p,q}(\frac{a}{b}y).
\end{aligned} \tag{3.1}$$

Proof. Suppose that

$$A = \frac{4e_{p,q}(tx)e_{p,q}(ty)}{\left(e_{p,q}(\frac{b}{a}t) + e_{p,q}(-\frac{b}{a}t)\right)\left(e_{p,q}(\frac{a}{b}t) + e_{p,q}(-\frac{a}{b}t)\right)}. \tag{3.2}$$

The form A turns into

$$\begin{aligned}
A &= \frac{2e_{p,q}(tx)}{\left(e_{p,q}(\frac{b}{a}t) + e_{p,q}(-\frac{b}{a}t)\right)} \frac{2e_{p,q}(ty)}{\left(e_{p,q}(\frac{a}{b}t) + e_{p,q}(-\frac{a}{b}t)\right)} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{b}{a}\right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,p,q}(\frac{a}{b}x) \widetilde{\mathcal{E}}_{k,p,q}(\frac{b}{a}y) \right) \frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{3.3}$$

The form A can also be transformed as

$$\begin{aligned} A &= \frac{2e_{p,q}(tx)}{\left(e_{p,q}(\frac{a}{b}t)+e_{p,q}(-\frac{a}{b}t)\right)} \frac{2e_{p,q}(ty)}{\left(e_{p,q}(\frac{b}{a}t)+e_{p,q}(-\frac{b}{a}t)\right)} \\ &= \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}\left(\frac{b}{a}x\right) \frac{\left(\frac{a}{b}t\right)^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}\left(\frac{a}{b}y\right) \frac{\left(\frac{b}{a}t\right)^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(\frac{a}{b}\right)^{n-2k} \widetilde{\mathcal{E}}_{n-k,p,q}\left(\frac{b}{a}x\right) \widetilde{\mathcal{E}}_{k,p,q}\left(\frac{a}{b}y\right) \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{3.4}$$

Comparing the coefficients of both sides in (3.3) and (3.4), we can find the required result. □

Corollary 3.2. Putting $p=1$ and $q \to 1$, the following holds:

$$\sum_{k=0}^{n} \binom{n}{k} \left(\frac{b}{a}\right)^{n-2k} \widetilde{E}_{n-k}(\frac{a}{b}x) \widetilde{E}_{k}(\frac{b}{a}y) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{b}\right)^{n-2k} \widetilde{E}_{n-k}(\frac{b}{a}x) \widetilde{E}_{k}(\frac{a}{b}y). \tag{3.5}$$

Lemma 3.3. Let r be a nonnegative integer. Then the following relations hold

$$\begin{aligned} &\text{(i)} \quad e_{p,q}(t)e_{p^{-1},q^{-1}}(t) = \sum_{n=0}^{\infty} \prod_{r=0}^{n-1} (p^r+q^r) \frac{t^n}{[n]_{p,q}!}, \\ &\text{(ii)} \quad e_{p,q}(t)e_{p^{-1},q^{-1}}(-t) = \sum_{n=0}^{\infty} \prod_{r=0}^{n-1} (p^r-q^r) \frac{t^n}{[n]_{p,q}!}, \\ &\text{(iii)} \quad e_{p,q}(-t)e_{p^{-1},q^{-1}}(-t) = \sum_{n=0}^{\infty} (-1)^n \prod_{r=0}^{n-1} (p^r+q^r) \frac{t^n}{[n]_{p,q}!}, \\ &\text{(iv)} \quad e_{p,q}(-t)e_{p^{-1},q^{-1}}(t) = \sum_{n=0}^{\infty} (-1)^n \prod_{r=0}^{n-1} (p^r-q^r) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{3.6}$$

Proof. (i) From a property of (p,q)-numbers we can note that

$$_{p^{-1},q^{-1}}! = \frac{[n]_{p,q}!}{p^{\binom{n}{2}}q^{\binom{n}{2}}}. \tag{3.7}$$

Using the above property, we can get

$$\begin{aligned} e_{p,q}(t)e_{p^{-1},q^{-1}}(t) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \\ &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \\ &= \sum_{n=0}^{\infty} \prod_{r=0}^{n-1} (p^r+q^r) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \tag{3.8}$$

(ii) Multiplying $e_{p^{-1},q^{-1}}(-t)$ with $e_{p,q}(t)$, we have

$$\begin{aligned}
e_{p,q}(t)e_{p^{-1},q^{-1}}(-t) &= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{(-t)^n}{[n]_{p^{-1},q^{-1}}!} \\
&= \sum_{n=0}^{\infty} p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{(-t)^n}{[n]_{p^{-1},q^{-1}}!} \qquad (3.9) \\
&= \sum_{n=0}^{\infty} \prod_{r=0}^{n-1} (p^r - q^r) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

(iii) Multiplying $e_{p^{-1},q^{-1}}(-t)$ with $e_{p,q}(-t)$, we have

$$\begin{aligned}
e_{p,q}(-t)e_{p^{-1},q^{-1}}(-t) &= \sum_{n=0}^{\infty} (-1)^n p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} (-1)^n p^{-\binom{n}{2}} \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \\
&= \sum_{n=0}^{\infty} (-1)^n p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} (-1)^n q^{\binom{n}{2}} \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \qquad (3.10) \\
&= \sum_{n=0}^{\infty} (-1)^n \prod_{r=0}^{n-1} (p^r + q^r) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

(iv) Multiplying $e_{p^{-1},q^{-1}}(t)$ with $e_{p,q}(-t)$, we have

$$\begin{aligned}
e_{p,q}(-t)e_{p^{-1},q^{-1}}(t) &= \sum_{n=0}^{\infty} (-1)^n p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} p^{-\binom{n}{2}} \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \\
&= \sum_{n=0}^{\infty} (-1)^n p^{\binom{n}{2}} \frac{t^n}{[n]_{p,q}!} \sum_{n=0}^{\infty} q^{\binom{n}{2}} \frac{t^n}{[n]_{p^{-1},q^{-1}}!} \qquad (3.11) \\
&= \sum_{n=0}^{\infty} (-1)^n \prod_{r=0}^{n-1} (p^r - q^r) \frac{t^n}{[n]_{p,q}!}.
\end{aligned}$$

Hence, we can find the following results and finish the proof of Lemma 1. □

Theorem 3.4. Let r be a nonnegative integer. Then we obtain

$$2 \prod_{r=0}^{n-1} (q^r + x^r p^r) = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \left(\prod_{r=0}^{l-1} (p^r + q^r) + (-1)^l \prod_{r=0}^{l-1} (p^r - q^r) \right) \widetilde{\mathcal{E}}_{n-l,p,q}(x). \qquad (3.12)$$

Proof. From generating function of (p,q)-Euler polynomials of the second kind, we can find

$$\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{2e_{p^{-1},q^{-1}}(t)}{e_{p^{-1},q^{-1}}(t)(e_{p,q}(t) + e_{p,q}(-t))} e_{p,q}(tx). \qquad (3.13)$$

If $e_{p,q}(t)e_{p^{-1},q^{-1}}(t)+e_{p,q}(-t)e_{p^{-1},q^{-1}}(t)\neq 0$, then we have

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\left(e_{p,q}(t)e_{p^{-1},q^{-1}}(t)+e_{p,q}(-t)e_{p^{-1},q^{-1}}(t)\right)=2e_{p^{-1},q^{-1}}(t)e_{p,q}(tx). \tag{3.14}$$

Using Lemma 3.3 (i), (iv) on the above equations, the left-hand side can transform to

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}\right)\sum_{n=0}^{\infty}\left(\prod_{r=0}^{n-1}(p^r+q^r)+(-1)^n\prod_{r=0}^{n-1}(p^r-q^r)\frac{t^n}{[n]_{p,q}!}\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}\left(\prod_{r=0}^{l-1}(p^r+q^r)+(-1)^l\prod_{r=0}^{l-1}(p^r-q^r)\right)\widetilde{\mathcal{E}}_{n-l,p,q}(x)\right)\frac{t^n}{[n]_{p,q}!},\end{aligned} \tag{3.15}$$

and the right-hand side is transformed as

$$\begin{aligned}2e_{p^{-1},q^{-1}}(t)e_{p,q}(tx)&=2\sum_{n=0}^{\infty}p^{-\binom{n}{2}}\frac{t^n}{[n]_{p^{-1},q^{-1}}!}\sum_{n=0}^{\infty}p^{\binom{n}{2}}x^n\frac{t^n}{[n]_{p,q}!}\\&=2\sum_{n=0}^{\infty}\left(\sum_{l=0}^{\infty}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}q^lp^{\binom{n-l}{2}}x^{n-l}\right)\frac{t^n}{[n]_{p,q}!}\\&=2\sum_{n=0}^{\infty}\left(\prod_{r=0}^{n-1}(q^r+xp^r)\right)\frac{t^n}{[n]_{p,q}!}.\end{aligned} \tag{3.16}$$

Therefore, we finish the proof of the required result. □

Corollary 3.5. When $x=0$ in Theorem 3.4, we can see

$$2q^{\binom{n}{2}}=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}\left(\prod_{r=0}^{l-1}(p^r+q^r)+(-1)^l\prod_{r=0}^{l-1}(p^r-q^r)\right)\widetilde{\mathcal{E}}_{n-l,p,q}, \tag{3.17}$$

where $\widetilde{\mathcal{E}}_{n,p,q}$ are (p,q)-Euler numbers of the second kind.

Theorem 3.6. For a nonnegative integer r, we have

$$2(-1)^n\prod_{r=0}^{n-1}(q^r-xp^r)=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_{p,q}\left(\prod_{r=0}^{l-1}(p^r-q^r)+(-1)^l\prod_{r=0}^{l-1}(p^r-q^r)\right)\widetilde{\mathcal{E}}_{n-l,p,q}(x). \tag{3.18}$$

Proof. We consider that

$$\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!}=\frac{2e_{p^{-1},q^{-1}}(-t)}{e_{p^{-1},q^{-1}}(-t)(e_{p,q}(t)+e_{p,q}(-t))}e_{p,q}(tx). \tag{3.19}$$

We omit this proof since we can use a similiar pattern as Theorem 3.4 to obtain the following result. □

Corollary 3.7. Setting $x=0$ in Theorem 3.6, we can see

$$2(-1)^n q^{\binom{n}{2}} = \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \left(\prod_{r=0}^{l-1}(p^r-q^r) + (-1)^l \prod_{r=0}^{l-1}(p^r-q^r) \right) \widetilde{\mathcal{E}}_{n-l,p,q}, \quad (3.20)$$

where $\widetilde{\mathcal{E}}_{n,p,q}$ are (p,q)-Euler numbers of the second kind.

Now we refer to (p,q)-Euler polynomials, (p,q)-Bernoulli polynomials and (p,q)-tangent polynomials. Combining these polynomials on (p,q)-Euler polynomials of the second kind, we investigate some identities.

Definition 3.8. (p,q)-Euler polynomials, (p,q)-Bernoulli polynomials and (p,q)-tangent polynomials, respectively, are defined as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{E}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \frac{[2]_{p,q}}{e_{p,q}(t)+1} e_{p,q}(tx), \\ \sum_{n=0}^{\infty} \mathcal{B}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \frac{t}{e_{p,q}(t)-1} e_{p,q}(tx), \\ \sum_{n=0}^{\infty} \mathcal{T}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} &= \frac{2}{e_{p,q}(2t)-1} e_{p,q}(tx). \end{aligned} \quad (3.21)$$

Theorem 3.9. Let m be nonnegative integer with $m>0$. Then we have

$$\begin{aligned} &\widetilde{\mathcal{E}}_{n,p,q}(x) \\ &= \frac{1}{[2]_{p,q}} \sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \left(\sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q} \frac{p^{\binom{n-l-k}{2}} \widetilde{\mathcal{E}}_{k,p,q}}{m^{n-k}} + \frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{m^l} \right) \mathcal{E}_{l,p,q}(mx), \end{aligned} \quad (3.22)$$

where $\mathcal{E}_{n,p,q}(x)$ are (p,q)-Euler polynomials.

Proof. From the definition of (p,q)-Euler polynomials of the second kind we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty} \widetilde{\mathcal{E}}_{n,p,q}(x) \frac{t^n}{[n]_{p,q}!} = \frac{[2]_{p,q}}{e_{p,q}(\frac{t}{m})+1} e_{p,q}(tx) \frac{e_{p,q}(\frac{t}{m})+1}{[2]_{p,q}} \frac{2}{e_{p,q}(t)+e_{p,q}(-t)} \\ &= \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \sum_{k=0}^{n-l} \begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q} \frac{p^{\binom{n-l-k}{2}}}{m^{n-k}} \widetilde{\mathcal{E}}_{k,p,q} \mathcal{E}_{l,p,q}(mx) \right) \frac{t^n}{[n]_{p,q}!} \\ &+ \frac{1}{[2]_{p,q}} \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \begin{bmatrix} n \\ l \end{bmatrix}_{p,q} \frac{1}{m^l} \mathcal{E}_{l,p,q}(mx) \widetilde{\mathcal{E}}_{n-l,p,q} \right) \frac{t^n}{[n]_{p,q}!}. \end{aligned} \quad (3.23)$$

Therefore, we complete the proof of Theorem 3.9. □

Corollary 3.10. Setting $p=1$ and $q\to 1$ in the Theorem 14, we find

$$\widetilde{E}_n(x)=\frac{1}{2}\sum_{l=0}^{n}\binom{n}{l}\left(\sum_{k=0}^{n-k}\binom{n-l}{k}\frac{\widetilde{E}_k}{m^{n-l}}-\frac{\widetilde{E}_{n-l}}{m^l}\right)E_l(mx), \tag{3.24}$$

where $\widetilde{E}_n(x)$ is classical Euler polynomials of the second kind, and $E_n(x)$ is classical Euler polynomials.

Theorem 3.11. Let l,k be nonnegative integers. Then we have

$$\begin{aligned}&\widetilde{\mathcal{E}}_{n-1,p,q}(x)[n]_{p,q}\\&=[n-1]_{p,q}\sum_{l=0}^{n}\begin{bmatrix}n-1\\l-1\end{bmatrix}_{p,q}\left(\sum_{k=0}^{n-1}\begin{bmatrix}n-l-1\\k-1\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}}{[k]_{p,q}m^{n-k}}\frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{[l]_{p,q}m^l}\right)\mathcal{B}_{l,p,q}(mx).\end{aligned} \tag{3.25}$$

Proof. From Definition 2.1, we can obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}}\\&=\frac{t}{e_{p,q}(\frac{t}{m})-1}e_{p,q}(\frac{t}{m}mx)\frac{e_{p,q}(\frac{t}{m})-1}{t}\frac{2}{e_{p,q}(t)+e_{p,q}(-t)}\\&=\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n-1\\l\end{bmatrix}_{p,q}\sum_{k=0}^{n-l}\begin{bmatrix}n-l-1\\k-1\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}}{[k]_{p,q}m^{n-k}}\mathcal{B}_{l,p,q}(mx)\right)\frac{t^n}{[n-1]_{p,q}!}\\&-\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix}n-1\\l-1\end{bmatrix}_{p,q}\frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{[l]_{p,q}m^l}\mathcal{B}_{l,p,q}(mx)\right)\frac{t^{n-1}}{[n-1]_{p,q}!}.\end{aligned} \tag{3.26}$$

The above equation is transformed to

$$\begin{aligned}&\frac{[n]_{p,q}}{[n-1]_{p,q}}\widetilde{\mathcal{E}}_{n-1,p,q}(x)\\&=\sum_{l=0}^{n}\begin{bmatrix}n-1\\l-1\end{bmatrix}_{p,q}\left(\sum_{k=0}^{n-l}\begin{bmatrix}n-l-1\\k-1\end{bmatrix}_{p,q}\frac{p^{\binom{n-l-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}}{[k]_{p,q}m^{n-k}}-\frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{[l]_{p,q}m^l}\right)\mathcal{B}_{l,p,q}(mx),\end{aligned} \tag{3.27}$$

and we immediately find the result of Theorem 3.11. □

Corollary 3.12. Setting $p=1$ and $q\to 1$, we have

$$n\widetilde{E}_{n-1}(x)=(n-1)\sum_{l=0}^{n}\binom{n-1}{l-1}\left(\sum_{k=0}^{n-1}\binom{n-l-1}{k-1}\frac{\widetilde{E}_k}{km^{n-k}}-\frac{\widetilde{E}_{n-l}}{lm^l}\right)B_l(mx), \tag{3.28}$$

where $\widetilde{E}_n(x)$ is classical Euler polynomials of the second kind, $\widetilde{E}$ is classical Euler numbers of the second kind, and $B_n(x)$ is classical Bernoulli polynomials.

Theorem 3.13. For nonnegative integers, l and k, we investigate

$$\widetilde{\mathcal{E}}_{n,p,q}(x) = \frac{1}{[2]_{p,q}}\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q}\frac{2^{n-k-l}p^{\binom{n-l-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}}{m^{n-k}} + \frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{m^{l}}\right)\mathcal{T}_{l,p,q}(\frac{mx}{2}). \tag{3.29}$$

Proof. Using generating function of (p,q)-Euler polynomials of the second kind, we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}\widetilde{\mathcal{E}}_{n,p,q}(x)\frac{t^n}{[n]_{p,q}!} \\
&= \frac{[2]_{p,q}}{e_{p,q}(\frac{2t}{m})+1}e_{p,q}(\frac{2t}{m}\frac{my}{2})\frac{e_{p,q}(\frac{2t}{m})+1}{[2]_{p,q}}\frac{2}{e_{p,q}(t)+e_{p,q}(-t)} \\
&= \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left(\mathcal{T}_{n,p,q}(\frac{mx}{2})\frac{t^n}{m^n[n]_{p,q}!}\right)\sum_{n=0}^{\infty}\left(p^{\binom{n}{2}}(\frac{2}{m})^n\frac{t^n}{[n]_{p,q}!}\right)\sum_{n=0}^{\infty}\left(\widetilde{\mathcal{E}}_{n,p,q}\frac{t^n}{[n]_{p,q}!}\right) \\
&+ \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left(\mathcal{T}_{n,p,q}(\frac{mx}{2})\frac{t^n}{m^n[n]_{p,q}!}\right)\sum_{n=0}^{\infty}\left(\widetilde{\mathcal{E}}_{n,p,q}\frac{t^n}{[n]_{p,q}!}\right) \\
&= \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_{p,q}\frac{2^{n-k-l}p^{\binom{n-l-k}{2}}\widetilde{\mathcal{E}}_{k,p,q}}{m^{n-k}}\mathcal{T}_{l,p,q}(\frac{mx}{2})\right)\frac{t^n}{[n]_{p,q}!} \\
&+ \frac{1}{[2]_{p,q}}\sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\begin{bmatrix} n \\ l \end{bmatrix}_{p,q}\frac{\widetilde{\mathcal{E}}_{n-l,p,q}}{m^l}\mathcal{T}_{l,p,q}(\frac{mx}{2})\right)\frac{t^n}{[n]_{p,q}!}.
\end{aligned} \tag{3.30}$$

Hence, we finish the proof of Theorem 3.13. □

Corollary 3.14. Putting $p=1$ in Theorem 3.13, the following relation holds:

$$\widetilde{\mathcal{E}}_{n,q}(x) = \frac{1}{[2]_{q}}\sum_{n=0}^{\infty}\begin{bmatrix} n \\ l \end{bmatrix}_{q}\left(\sum_{k=0}^{n-l}\begin{bmatrix} n-l \\ k \end{bmatrix}_{q}\frac{2^{n-k-l}\widetilde{\mathcal{E}}_{k,q}}{m^{n-k}} + \frac{\widetilde{\mathcal{E}}_{n-l,q}}{m^{l}}\right)\mathcal{T}_{l,q}(\frac{mx}{2}), \tag{3.31}$$

where $\mathcal{T}_{n,q}(x)$ is q-tangent polynomials.

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