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A NOTE ON THE SOLUTION EQUIVALENCE OF GENERAL MINIMUM VARIANCE AND MINIMAX DISPARITY PROBLEMS FOR OWA OPERATOR†

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Abstract. This note provides the solution equivalence of general minimum variance and minimax disparity problems for OWA operator. This result generalize a main theorem of Liu [International Journal of Approximate Reasoning, 48 (2008) 598-627.]

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1. Introduction and preliminaries

Liu [6, Theorem 2] considered a general convex OWA operator optimization problem with given orness lenel:

$$\begin{aligned} \text{Minimize} \quad & V_W = \sum_{i=1}^{n} F(w_i) \\ \text{subject to} \quad & orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 < \alpha < 1, \\ & w_1 + \cdots + w_n = 1, 0 \le w_i, i = 1, \cdots, n. \end{aligned} \tag{1}$$

where F is a strictly convex function on $[0,1]$, and it is at least two order differentiable.

When $F(x) = x \ln x$, (1) becomes the maximum entropy OWA operator problem that was discussed in [2,3,8,10]. $F(x) = x^2$ in (1) corresponds to another discussed minimum variance OWA operator problem [4,6]. More generally, when $F(x) = x^p, p > 1$, (1) becomes the OWA problem of Rényi entropy [9], which

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includes the maximum entropy and minimum variance OWA problem as special cases.

Liu [7] solved problem (1) analytically using the Kuhn-Tucker second-order sufficiency conditions for optimality [1, p.58]. The condition of the second differentiability of F is essential to use the Kuhn-Tucker second-order sufficiency conditions for optimality in proving the general OWA operator optimization problem. Indeed, I do not need the second differentiability of F. In the following session, we give a simple new proof for the problem (1) assuming continuous first differentiability of F instead of the second differentiability of F.

Recently, Hong [5] considered a general minimax disparity ordered weighted averaging (OWA) operator problem with given orness level:

$$\begin{aligned}
\text{Minimize} \quad & \max_{i \in \{1,\cdots,n-1\}} |F'(w_i) - F'(w_{i+1})| \\
\text{subject to} \quad & orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 \le \alpha \le 1, \\
& w_1 + \cdots + w_n = 1, 0 \le w_i, i = 1, \cdots, n.
\end{aligned} \tag{2}$$

where F is a strictly convex function on $[0, \infty)$ and $F'(x)$ is continuous.

The main point is that the optimal solutions of the two problems (1) and (2) are same. This result generalizes a main theorem of Liu [7, Theorem 13].

2. The solution equivalence of two models

Recently, Hong [5] proved the following result:

Theorem 2.1. *Assume that F is strictly convex and $F'(x)$ is continuous. The optimal solution for problem (2) with given orness level $0 < \alpha < 1$ is the weighting function*

$$w_i^* = \max\left\{(F')^{-1}(a^* i + b^*),\ 0\right\}$$

where a^*, b^* is determined by the constraints:

$$\begin{cases} \sum_{i \in H} \frac{n-i}{n-1} (F')^{-1}(a^* i + b^*) = \alpha \\ \sum_{i \in H} (F')^{-1}(a^* i + b^*) = 1, \end{cases}$$

and $H = \left\{ i | (F')^{-1}(a^* i + b^*) > 0 \right\}$.

We now prove the following general convex OWA operator optimization problem with given orness level:

$$\begin{aligned}
\text{Minimize} \quad & V_W = \sum_{i=1}^{n} F(w_i) \\
\text{subject to} \quad & orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha,\ 0 < \alpha < 1, \\
& w_1 + \cdots + w_n = 1, 0 \le w_i, i = 1, \cdots, n.
\end{aligned} \tag{3}$$

where F is strictly convex on $[0,1]$ and F' is continuous on $(0,1)$.

It is noted that F is strictly convex if and only if F' is strictly increasing. If $F'(x)$ is strictly increasing, then there are two possible cases:

$$F'(0^+) = -\infty, F'(0^+) < \infty,$$

where $lim_{x\to 0^+} = F'(0^+)$. In case of $F'(0^+) < \infty$, we define $F'(0^+) = F'(0)$.

Theorem 2.2. *The optimal solutions of the two problems (2) and (3) are the same. That is, they both have the form as*

$$w_i^* = \begin{cases} (F')^{-1}(a^*i + b^*), & \text{if } (F')^{-1}(a^*i + b^*) > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

where a^*, b^* is determined by the constraints:

$$\begin{cases} \sum_{i\in H} \frac{n-i}{n-1}(F')^{-1}(a^*i + b^*) = \alpha \\ \sum_{i\in H}(F')^{-1}(a^*i + b^*) = 1, \end{cases}$$

and $H = \{i|(F')^{-1}(a^*i + b^*) > 0\}$.

Proof. Let $w_i^* = max\{(F')^{-1}(a^*i + b^*), 0\}$ such that

$$\sum i w_i^* \quad = \quad n - (n-1)\alpha \quad \left(\Leftrightarrow \sum_{i=1}^{n} \frac{n-i}{n-1} w_i^* = \alpha \right) \tag{4}$$

$$\sum w_i^* \quad = \quad 1, \tag{5}$$

and let $w_i, i = 1, \cdots, n$ be a weighting vector such that

$$\sum_{i=1}^{n} i w_i = n - (n-1)\alpha, \tag{6}$$

$$\sum_{i=1}^{n} w_i = 1,\ 0 \le w_i,\ i = 1, \cdots, n. \tag{7}$$

We put $w_i = w_i^* + \beta_i,\ i = 1, \cdots, n$. Then, noting that

$$w_i = \beta_i,\ i \notin H, \tag{8}$$

we have, from (5) and (7),

$$\sum_{i\notin H} w_i + \sum_{i\in H} \beta_i = \sum_{i=1}^{n} \beta_i = 0, \tag{9}$$

since $1 = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} w_i^* + \sum_{i=1}^{n} \beta_i = 1 + \sum_{i=1}^{n} \beta_i$. We also have, from (4) and (6)

$$\sum_{i\notin H} i w_i + \sum_{i\in H} i\beta_i = \sum_{i=1}^{n} i\beta_i = 0, \tag{10}$$

since $\sum_{i=1}^{n} iw_i = \sum_{i=1}^{n} i(w_i^* + \beta_i) = \sum_{i=1}^{n} iw_i^* + \sum_{i=1}^{n} i\beta_i$. We now show that

$$\sum_{i=1}^{n} F(w_i) \geq \sum_{i=1}^{n} F(w_i^*).$$

It is because, since $F(x) - F(x_0) \geq F'(x_0)(x - x_0)$ (the equality holds if and only if $x = x_0$) we have that

$$\begin{aligned} \sum_{i=1}^{n} F(w_i) - \sum_{i=1}^{n} F(w_i^*) &= \sum_{i=1}^{n} F(w_i^* + \beta_i) - \sum_{i=1}^{n} F(w_i^*) \\ &\geq \sum_{i=1}^{n} F'(w_i^*)\beta_i = D \end{aligned} \tag{11}$$

Case 1) $F'(0^+) = -\infty$

We assume that $F'(0_+) = -\infty$. Then $(F')^{-1}(x) > 0$ and hence $H = \{1, 2, \cdots, n\}$. Then, from (9) and (10),

$$\begin{aligned} D &= \sum_{i \in H} \beta_i (a^* i + b^*) \\ &= a^* \sum_{i}^{n} i\beta_i + b^* \sum_{i}^{n} \beta_i \\ &= 0. \end{aligned}$$

Case 2) $F'(0^+) < \infty$

If $F'(0^+) < \infty$, then $F'(w_i^*) = max\{a^* i + b^*,\ F'(0)\}$. If $i \in H$, then $F'(w_i^*) = a^* i + b^* > F'(0)$. If $i \notin H$, then $(F')^{-1}(a^* i + b^*) \leq 0$, and hence $a^* i + b^* \leq F'(0) = F'(w_i^*)$. Then

$$\begin{aligned} D &= \sum_{i \in H} \beta_i (a^* i + b^*) + \sum_{i \notin H} \beta_i F'(0) \\ &= a^* \sum_{i \in H} i\beta_i + b^* \sum_{i \in H} \beta_i + \sum_{i \notin H} \beta_i F'(0) \\ &= a^* \left(-\sum_{i \notin H} iw_i \right) + b^* \left(-\sum_{i \notin H} w_i \right) + \sum_{i \notin H} w_i F'(0) \\ &= \sum_{i \notin H} w_i \left(F'(0) - a^* i - b^* \right) \\ &\geq 0, \end{aligned} \tag{12}$$

where the third equality comes from (8), (9) and (10). The equality in (11) and (12) holds if and only if $\beta_i = 0, i = 1, 2, \cdots, n$ if and only if $w_i = w_i^*, i = 1, 2, \cdots, n$. This proves the uniqueness of W^*. □

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