J. Appl. Math. & Informatics Vol. **37***(2019), No. 3 - 4, pp. 183 - 200*
https://doi.org/10.14317/jami.2019.183

ROBUST NUMERICAL METHOD FOR SINGULARLY PERTURBED TURNING POINT PROBLEMS WITH ROBIN TYPE BOUNDARY CONDITIONS

N. GEETHA AND A. TAMILSELVAN*

Abstract. We have constructed a robust numerical method on Shishkin mesh for a class of convection diffusion type turning point problems with Robin type boundary conditions. Supremum norm is used to derive error estimates which is of order $O(N^{-1} \ln N)$. Theoretical results are verified by providing numerical examples.

AMS Mathematics Subject Classification : 65L10, 65L11, 65L12, 65L20.
Key words and phrases : Singularly perturbed turning point problems, Robin boundary conditions, Convection-Diffusion problems, Finite difference scheme, Shishkin mesh, Parameter uniform convergence.

1. Introduction

A class of differential equations with a small parameter multiplying their highest derivative terms is known as Singular Perturbation Problems(SPPs). They are widespread in nature. Typically SPPs occur in most of the branches of applied Mathematics like electrical networks, elasticity, fluid dynamics, chemical reactor theory, quantum mechanics, etc. Solving SPPs by classical numerical methods does not provide satisfactory results because of the presence of small parameter. Therefore special numerical methods have been constructed to solve SPPs for the past 40 years.

The methods to solve SPPs with Dirichlet boundary conditions are discussed in Doolan[6], O' Malley [13], Miller [15] and Roos [22]. Andreyev and Savin [3] solved a problem which has Mixed boundary condition only at the left boundary. Ansari and Hegarty [2] suggested a numerical method to solve the convection diffusion type problem with Robin boundary conditions. The authors of [14] and [20] proposed a parameter uniform numerical method using adaptively generated

Received April 11, 2017. Revised June 13, 2018. Accepted March 15, 2019. *Corresponding author.

grid which gives second order accuracy for convection and reaction diffusion problems respectively. In [16], the authors approximated the derivative of a SPP of Robin type with discontinuous convection term and source term.

Abrahammson [1] derived a priori estimate for the solution of Singularly Perturbed Turning Point Problems(SPTPPs). A sufficient condition for a uniform convergent scheme is derived for SPTPPs in [8] and [5]. Qualitative properties of SPTPPs are discussed by the authors O' Malley [13], Roos et.al [22], Wasow [24] and Watts [25]. In [12, 17, 18] and [19] the authors apply different numerical methods for SPTPPs with the conditions of Dirichlet type. In [21], E. O'Riordan and J. Quinn, proposed a robust numerical method for interior turning point problem. In [4] and [12] the authors obtained a second order convergence. For more detail one may refer [23] and the references therein. An asymptotic expansion of solution for the third order SPTPP was constructed by Jia-qi Mo et al. [11]. Parameter uniform numerical method for a third order SPTPPs is given in [9]. In [10] the authors proposed a variable mesh spline approximation method for second order SPTPPs with Robin boundary conditions.

Motivated by the works of [2, 10, 14, 16, 19, 20], we discuss a class of SPTPPs given below.

Find $u(x) \in C^1(\bar{\Omega}) \cap C^2(\Omega)$ such that

$$\begin{aligned} Lu \equiv \varepsilon u^{''}(x) + a(x)u^{'}(x) - b(x)u(x) \qquad (1)\\ = f(x), \quad \forall\, x \in \Omega = (-1,1),\\ B_1u(-1) = \beta_1 u(-1) - \varepsilon\beta_2 u'(-1) = A,\\ B_2u(1) = \gamma_1 u(1) + \varepsilon\gamma_2 u^{'}(1) = B, \qquad (2)\end{aligned}$$

where $a(x), b(x)$ and $f(x)$ are smooth functions on $\bar{\Omega}$, $0 < \varepsilon << 1$ is the parameter and

$$\begin{cases} a(0) = 0,\ a^{'}(0) < 0,\\ \quad |a(x)| \le \alpha_0,\ 0 < \beta_0 \le b(x),\ \alpha_0 < \beta_0,\ \forall\, x \in \bar{\Omega} = [-1,1],\\ |a^{'}(x)| \ge \dfrac{|a^{'}(0)|}{2}\\ \ \forall\, x \in \bar{\Omega},\ \beta_1, \beta_2 \ge 0,\ \beta_1 - \varepsilon\beta_2 > 0,\ \gamma_2 \ge 0\ \&\ \gamma_1 > 0. \end{cases} \qquad (3)$$

Assumptions (3) guarantees that the solution is unique for the problem (1)-(2) and the solution exhibits layers at both the boundaries $x = -1$ & $x = 1$ [5].

In this paper, we propose a numerical method which is shown to be robust [7, 15]. The robust numerical methods are those, whose numerical solutions U satisfy

$$||u - U|| \le CN^{-P}, P > 0$$

where u is the solution of the continuous problem, N is the number of mesh points and C is a constant independent of ε and N. That is, the numerical approximations converge for all values of ε.

The paper is organized as follows. In Section 2 minimum principle and stability results are discussed. Some analytical results are derived in Section 3. Mesh selection strategy is given in Section 4. The difference scheme, discrete minimum principle and discrete stability results are discussed in the same section. The error estimates are carried out in Section 5. Numerical examples are given in Section 6.

Notations:
Throughout the paper we use C, to denote a generic positive constant independent of N and ε. For any function $f(x) \in C^k(\bar{\Omega})$, ($k$ a non negative integer), the convergence of the numerical solution is studied using the maximum norm defined as $||f||_k = \sum_{i=0}^{k} \max_{x\in\bar{\Omega}} |f^{(i)}(x)|$.

2. Minimum Principle and Stability Result

Minimum principle and stability result for the problem $(1)-(3)$ is presented in this section.

Theorem 2.1 (Ref. [10]). **(Minimum Principle)** Let L be the differential operator defined in (1) and $w \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. If $B_1 w(-1) \geq 0,\ B_2 w(1) \geq 0$ and $Lw \leq 0\ \forall\ x \in \Omega$, then $w(x) \geq 0\ \ \forall\ \ x \in \bar{\Omega}$.

Lemma 2.2 (Ref. [10]). **(Stability Result)** If $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, then

$$|u(x)| \leq C \max\left\{|B_1 u|, |B_2 u|, ||Lu||_{\Omega}\right\}, \forall x \in \bar{\Omega}.$$

Note: As the operator L satisfies the minimum principle, the solution of the BVP (1)-(3) is unique, if it exists.

3. Derivative Estimates

In the following sections the subdomains of $\bar{\Omega} = [-1, 1]$ are denoted as $\Omega_1 = [-1, -\delta]$, $\Omega_2 = [-\delta, \delta]$ and $\Omega_3 = [\delta, 1]$, $0 < \delta \leq 1/2$. The reason for choosing $\delta = 1/2$ can be found in [5].
We derive the estimates for the solution and its derivatives in Lemma 3.1 in the interval Ω_1 and Ω_3.

Lemma 3.1 (Ref. [10]). Suppose u is the solution of (1)-(3). Then

$$\begin{aligned} ||u^{(k)}|| &\leq C\varepsilon^{-(k)} \max\{||f||, ||u||\},\ k = 1, 2 \\ ||u^{(3)}|| &\leq C\varepsilon^{-(3)} \max\{||f||, ||f'||, ||u||\}, \end{aligned}$$

$\forall x \in \Omega_1 \cup \Omega_3$, where C depends on $||a||, ||b||, ||a'||$ and $||b'||$.

Let $\beta = b(0)/a'(0)$, β_l and β_s be fixed positive constants such that $\beta_l < 1 < \beta_s$ and $\beta_l \leq |\beta| \leq \beta_s$. And we observe that $\beta < 0$.

We derive the estimates for the solution and its derivatives in Lemma 3.2 in the interval Ω_2, which contains the turning point $x = 0$.

Lemma 3.2 (Ref. [10])**.** Let u be the solution of (1)-(3). Then

$$||u^{(k)}(x)|| \leq C, \ \forall x \in \Omega_2,$$

where C depends on $||a||, ||a^{'}||, ||b||, ||b^{'}||, ||f||, ||f^{'}||$ and β.

To derive ε - uniform error estimates we require sharper bounds of the solution and its derivatives. For this we use Shishkin decomposition of the solution u as

$$u = v + w. \tag{4}$$

Here v is the solution of the problem

$$Lv = f \tag{5}$$

with boundary conditions

$$\begin{aligned} &\beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1) \\ &= \beta_1 v_0(-1) - \varepsilon\beta_2 v_0^{'}(-1) + \varepsilon(\beta_1 v_1(-1) - \varepsilon\beta_2 v_1^{'}(-1)), \end{aligned} \tag{6}$$

$$\begin{aligned} &\gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1) \\ &= \gamma_1 v_0(1) + \varepsilon\gamma_2 v_0^{'}(1) + \varepsilon(\gamma_1 v_1(1) + \varepsilon\gamma_2 v_1^{'}(1)) \end{aligned} \tag{7}$$

where $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$.
Also v_0 and v_1 are the solutions of the reduced problem:

$$av_0^{'} - bv_0 = f \text{ and } av_1^{'} - bv_1 = -v_0^{''} \tag{8}$$

and v_2 is the solution of the similar problem to that defining u

$$\begin{aligned} &Lv_2 = -v_1^{''}, \\ &\beta_1 v_2(-1) - \varepsilon\beta_2 v_2^{'}(-1) = 0, \ \gamma_1 v_2(1) + \varepsilon\gamma_2 v_2^{'}(1) = 0. \end{aligned} \tag{9}$$

The singular component w is the solution of the homogeneous problem

$$\begin{aligned} &Lw = 0, \\ &\beta_1 w(-1) - \varepsilon\beta_2 w^{'}(-1) \\ &= (\beta_1 u(-1) - \varepsilon\beta_2 u^{'}(-1)) - (\beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1)), \\ &\gamma_1 w(1) + \varepsilon\gamma_2 w^{'}(1) \\ &= (\gamma_1 u(1) + \varepsilon\gamma_2 u^{'}(1)) - (\gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1)). \end{aligned} \tag{10}$$

Lemma 3.3. For $k = 0, 1, 2, 3$ the smooth and singular components v and w and their derivatives satisfy

$$\begin{aligned} ||v^{(k)}(x)|| &\leq C(1 + \varepsilon^{2-k}), \ \forall\, x \in \Omega_1 \cup \Omega_3 \text{ and} \\ |w^{(k)}(x)| &\leq \begin{cases} C\varepsilon^{-k} e^{-\alpha(1+x)/\varepsilon}, \ \forall\, x \in \Omega_1 \\ C\varepsilon^{-k} e^{-\alpha(1-x)/\varepsilon}, \ \forall\, x \in \Omega_3 \end{cases} \end{aligned}$$

where $|a(x)| \geq \alpha > 0, \ \forall x \in \Omega_1 \cup \Omega_3$.

Proof. We prove the Lemma in the subdomain Ω_1 by adopting the approach found as in [[7], p.44].

From the assumptions that $v_0(x)$ and $v_1(x)$ are independent of ε, and $a(x), b(x)$ and $f(x)$ are smooth, one can have $|v_0^{(k)}(x)| \leq C$ and $|v_1^{(k)}(x)| \leq C$ for all $x \in \Omega_1$.

Since $v_2(x)$ is the solution of a similar problem to that of (1), then from Lemma 3.1, we have the bound

$$||v_2^{(k)}(x)|| \leq C(1 + \varepsilon^{2-k}), \text{ for } k = 0, 1, 2.$$

For $k = 3$, we have $(\varepsilon v''(x))' = (f - a(x)v'(x) + b(x)v(x))'$ which leads to the bound $||v^{(3)}(x)|| \leq C(\varepsilon^{-1})$. Therefore proof in the interval Ω_1 is thus complete.

For bounds on w and its derivatives we consider the barrier functions

$$\Psi^{\pm}(x) = C\left[\frac{e^{-\alpha(1+x)/\varepsilon} - (1 - \gamma_2\alpha/\gamma_1)e^{-\alpha(1-\delta)/\varepsilon}}{1 - (1 - \gamma_2\alpha/\gamma_1)e^{-\alpha(1-\delta)/\varepsilon}}\right] \pm w(x), \tag{11}$$

where C is a constant chosen in such a way that $B_1(\Psi^{\pm})(-1) \geq 0$, $B_2(\Psi^{\pm})(-\delta) \geq 0$. Also it can be verified that $L\Psi^{\pm}(x) \leq 0$. Applying the minimum principle, we have $\Psi^{\pm}(x) \geq 0$ and the required result follows. That is

$$|w(x)| \leq Ce^{-\alpha(1+x)/\varepsilon}, \ \forall\, x \in \Omega_1.$$

Bounds on the derivatives of w are established as in [7]. Similarly we can prove the result in the subdomain Ω_3. Hence the result. □

Theorem 3.4. For $k = 0, 1, 2, 3$ the smooth and singular components v and w and their derivatives satisfy

$$\begin{aligned} ||v^{(k)}(x)|| &\leq C(1 + \varepsilon^{2-k}), \quad \text{and} \\ |w^{(k)}(x)| &\leq C\varepsilon^{-k}(e^{-\alpha(1+x)/\varepsilon} + e^{-\alpha(1-x)/\varepsilon}), \ \forall\, x \in \bar{\Omega}. \end{aligned}$$

Proof. As the solution of the problem (1)-(3) and its derivatives are smooth (by Lemma 3.2) in Ω_2 the proof follows immediately. □

4. Discrete Problem

4.1. Mesh selection strategy. We discretize the problem (1)-(3) using classical finite difference scheme on the Shiskin mesh, $\bar{\Omega}^N$, $N \geq 4$. The domain $\bar{\Omega}$ is divided into three subintervals $\Omega_L = [-1, -1 + \tau]$, $\Omega_C = [-1 + \tau, 1 - \tau]$ and $\Omega_R = [1 - \tau, 1]$ such that $\bar{\Omega} = \Omega_L \cup \Omega_C \cup \Omega_R$, where the transition parameter $\tau = \min\left\{\frac{1}{2}, \frac{2\varepsilon \ln N}{\alpha}\right\}$.

4.2. Finite difference method for the problem (1) - (3). The subdomains Ω_L and Ω_R contains $N/4$ mesh elements and $N/2$ elements in Ω_C. The resulting difference scheme is to find $U(x_i)$ for $i = 0, 1, 2, \cdots N$, $x_i \in \bar{\Omega}^N$,

$$L^N U(x_i) := \varepsilon\delta^2 U(x_i) + a(x_i)D^* U(x_i) - b(x_i)U(x_i) \tag{12}$$

$$= f(x_i), \tag{13}$$

$$B_1^N U(x_0) = \beta_1 U(x_0) - \varepsilon\beta_2 D^+ U(x_0)$$

$$\begin{aligned} &= \beta_1 u(-1) - \varepsilon\beta_2 u^{'}(-1), \\ B_2^N U(x_N) &= \gamma_1 U(x_N) + \varepsilon\gamma_2 D^- U(x_N) \\ &= \gamma_1 u(1) + \varepsilon\gamma_2 u^{'}(1), \end{aligned} \tag{14}$$

where $D^+U(x_i) = \dfrac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}$, $D^-U(x_i) = \dfrac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}}$,
$\delta^2 U(x_i) = \dfrac{D^+U(x_i) - D^-U(x_i)}{(x_{i+1} - x_{i-1})/2}$ and $D^*U(x_i) = \begin{cases} D^+U(x_i) & \text{if } a(x_i) > 0 \\ D^-U(x_i) & \text{if } a(x_i) < 0 \end{cases}$.

5. Numerical solution and error estimates

Analogous to the continuous results stated in Theorem 2.1 and Lemma 2.2 we can prove the following results easily.

Theorem 5.1. Consider the discrete problem (13)- (14). If ψ is any mesh function defined on this mesh such that $B_1^N \psi(x_0) \geq 0,\ B_2^N \psi(x_N) \geq 0$ and $L^N \psi(x_i) \leq 0,$ for $i = 1(1)N-1$ then

$$\psi(x_i) \geq 0,\ \forall\ x_i\ \in \bar{\Omega}^N.$$

Proof. The test function $s(x_i)$ is defined as $s(x_i) = 2 + x_i$.
Then $B_1^N s(x_0) > 0, B_2^N s(x_N)$ and $L^N s(x_i) < 0$ for $i = 1(1)N-1$. Further we define

$$\xi = \max_{x_i \in \bar{\Omega}_\varepsilon^N} \left(\frac{-\psi}{s}\right)(x_i).$$

Suppose the theorem is not true. Then $\xi > 0$ and we have $(\psi + \xi s)(x_i) \geq 0$ for $x_i \in \bar{\Omega}_\varepsilon^N$. For some $i = k$, we may have $(\psi + \xi s)(x_k) = 0$
Case(i): $(\psi + \xi s)(x_k) = 0$, for $x_k = -1$. Then

$$\begin{aligned} 0 &\leq B_1^N(\psi + \xi s)(x_k) \\ &= \beta_1(\psi + \xi s)(x_k) - \varepsilon\beta_2 D^+(\psi + \xi s)(x_k) \\ &< 0, \end{aligned}$$

a contradiction.
Case(ii):$(\psi + \xi s)(x_k) = 0$, for $0 < k < N$. Then

$$\begin{aligned} 0 &\geq L^N(\psi + \xi s)(x_k) \\ &= \begin{cases} \varepsilon\delta^2(\psi + \xi s)(x_k) + a(x_k)D^+(\psi + \xi s)(x_k) - b(x_k)(\psi + \xi s)(x_k) \\ \quad \text{if } a(x_k) > 0 \\ \varepsilon\delta^2(\psi + \xi s)(x_k) + a(x_k)D^-(\psi + \xi s)(x_k) - b(x_k)(\psi + \xi s)(x_k) \\ \quad \text{if } a(x_k) < 0 \end{cases} \\ &> 0, \end{aligned}$$

a contradiction.
Case(iii): $(\psi + \xi s)(x_k) = 0$, for $x_k = 1$. Then

$$0 \leq B_2^N(\psi + \xi s)(x_k)$$

$$
\begin{aligned}
&= \gamma_1(\psi+\xi s)(x_k)+\varepsilon\gamma_2 D^-(\psi+\xi s)(x_k)\\
&< 0,
\end{aligned}
$$

a contradiction.
Therefore $\psi(x_i)\geq 0, \forall x_i\in\bar{\Omega}^N$. □

Lemma 5.2. Consider the scheme (13)- (14) to problem (1)-(3). If $\psi(x_i)$ is any mesh function then, for all $x_i\in\bar{\Omega}^N$

$$|\psi(x_i)|\leq C\max\{|B_1^N\psi(x_0)|,|B_2^N\psi(x_N)|,\max_{1\leq i\leq N-1}|L^N\psi(x_i)|\}.$$

Proof. Let $C_1 = C\max\{|B_1\psi(x_0)|,|B_2\psi(x_N)|,\max_{1\leq i\leq N-1}|L^N\psi(x_i)|\}$.
Define the mesh functions

$$W^{\pm}x_i = C_1(2+x_i)\pm\psi(x_i).$$

Then we have $B_1^N W^{\pm}(x_0)\geq 0, B_2^N W^{\pm}(x_N)\geq 0$ and $L^N W^{\pm}(x_i)\geq 0$. Using Theorem 5.1 the lemma can be proved. □

Lemma 5.3. The solution of the constant coefficient problem

$$\varepsilon\delta^2\Phi_i+\omega D^+\Phi_i = 0 \;\; 1\leq i\leq N-1, \tag{15}$$

where $\omega>0$, with the boundary conditions

$$\beta_1\Phi_0-\varepsilon\beta_2 D^+\Phi_0 = 1,\;\; \gamma_1\Phi_N+\varepsilon\gamma_2 D^-\Phi_N = 0 \tag{16}$$

on a uniform mesh or the Shishkin mesh $\bar{\Omega}^N$ satisfies

$$D^+\Phi_i\leq 0 \;\forall\; 1\leq i\leq N-1.$$

Proof. We need to consider the cases $\tau = 1/2$ and $\tau = \dfrac{2\epsilon}{\alpha}\ln N$ separately. For $\tau = 1/2$, we have

$$\Phi_i = \frac{\lambda^{N-i}+(\omega\gamma_2/\gamma_1)-1}{[\beta_1(\lambda^N+(\omega\gamma_2/\gamma_1)-1)+\beta_2\omega\lambda^{N-1}]},\;\; \lambda = 1+\omega h/\varepsilon \tag{17}$$

and therefore

$$D^+\Phi_i = -\frac{\omega\lambda^{N-i-1}}{\varepsilon[\beta_1(\lambda^N+(\omega\gamma_2/\gamma_1)-1)+\beta_2\omega\lambda^{N-1}]}\leq 0 \tag{18}$$

For the second case $\tau = \dfrac{2\epsilon}{\alpha}\ln N$, we start by noting that the solution of problem is

$$\Phi_i = \begin{cases}\Phi_{N/2}+(1-\beta_1\Phi_{N/2})\chi_i & \text{if } i\leq N/2\\ \Phi_{N/2}\varsigma_i & \text{if } i\geq N/2\end{cases} \tag{19}$$

where

$$\chi_i = \frac{\lambda^{N/2-i}-1}{\beta_1(\lambda^{N/2}-1)+\beta_2\omega\lambda^{N/2-1}},\;\; \lambda = 1+\frac{\omega h}{\varepsilon} \tag{20}$$

$$\varsigma_i \;=\; \frac{\Lambda^{N-i}+(\gamma_2\omega/\gamma_1)-1}{\Lambda^{N/2}+(\gamma_2\omega/\gamma_1)-1},\ \Lambda=1+\frac{\omega H}{\varepsilon} \tag{21}$$

and $\Phi_{N/2}$ satisfies, $(\varepsilon\delta^2+\omega D^+)\Phi_{N/2}=0$. Note that since $\Lambda^{N/2}>\Lambda^{N/2-1}$, we know from (21) that $\varsigma_{N/2+1}<1$.

Also, $\chi_{N/2-1} = \dfrac{\omega h}{\varepsilon[\beta_1(\lambda^{N/2}-1)+\beta_2\omega\lambda^{N/2-1}]} \geq 0$. From (21) and from the above results, we have

$$\Phi_{N/2}=\frac{\varepsilon N\chi_{N/2-1}}{(h/H)(\varepsilon N+\omega)(1-\varsigma_{N/2+1})+\varepsilon N\beta_1\chi_{N/2-1}}\geq 0 \tag{22}$$

Also,

$$1-\beta_1\Phi_{N/2}=\frac{(h/H)(\varepsilon N+\omega)(1-\varsigma_{N/2+1})}{(h/H)(\varepsilon N+\omega)(1-\varsigma_{N/2+1})+\varepsilon N\beta_1\chi_{N/2-1}}\geq 0 \tag{23}$$

and

$$D^+\chi_i=-\frac{\Lambda^{N/2-i-1}\omega}{\varepsilon[\beta_1(\lambda^{N/2}-1)+\beta_2\omega\lambda^{N/2-1}]}\leq 0,\ 1\leq i\leq N/2 \tag{24}$$

In addition, we note that

$$D^-\chi_{N/2}=-\frac{\Lambda^{-1}\omega}{\varepsilon[\beta_1(\lambda^{N/2}-1)+\beta_2\omega\lambda^{N/2-1}]}\leq 0. \tag{25}$$

Furthermore, applying the forward difference operator, D^+ to (21) we get

$$D^+\varsigma_i=\frac{\omega\Lambda^{N-i-1}}{\varepsilon(\Lambda^{N/2}+(\gamma_2\omega/\gamma_1)-1)}\leq 0,\ i\geq N/2 \tag{26}$$

combining the above results we have the desired result.

□

Lemma 5.4. The solution of the constant coefficient problem

$$\varepsilon\delta^2\Phi_i+\omega D^-\Phi_i=0\ 1\leq i\leq N-1, \tag{27}$$

where $\omega>0$, with the boundary conditions

$$\beta_1\Phi_0-\varepsilon\beta_2D^+\Phi_0=0,\ \gamma_1\Phi_N+\varepsilon\gamma_2D^-\Phi_N=1 \tag{28}$$

on a uniform mesh or the Shishkin mesh $\bar{\Omega}^N$ satisfies

$$D^-\Phi_i\geq 0\ \forall\ 1\leq i\leq N-1.$$

Proof. Using the technique adopted in Lemma 5.3, the present lemma can be proved. □

Analogous to the continuous case, the discrete solution U can be decomposed as

$$U=V+W,$$

where V and W are the solutions of the problems

$$L^NV=f(x_i),\ x_i\in\bar{\Omega}^N, \tag{29}$$

$$\beta_1 V(-1) - \varepsilon\beta_2 D^+ V(-1) = \beta_1 v(-1) - \varepsilon\beta_2 v^{'}(-1),$$
$$\gamma_1 V(1) + \varepsilon\gamma_2 D^- V(1) = \gamma_1 v(1) + \varepsilon\gamma_2 v^{'}(1)$$

and

$$\begin{aligned} &L^N W = 0,\ x_i \in \bar{\Omega}^N, \\ &\beta_1 W(-1) - \varepsilon\beta_2 D^+ W(-1) = \beta_1 w(-1) - \varepsilon\beta_2 w^{'}(-1), \\ &\gamma_1 W(1) + \varepsilon\gamma_2 D^- W(1) = \gamma_1 w(1) + \varepsilon\gamma_2 w^{'}(1) \end{aligned} \tag{30}$$

respectively.

We obtain separate error estimates for each component of the solution.

Lemma 5.5. The error in the smooth component of the numerical solution is bounded as

$$|(V - v)(x_i)| \le CN^{-1},\ \text{for all } x_i \in \bar{\Omega}^N,$$

where v is the solution of (5)-(8) and V is the solution of (29).

Proof. Consider the local truncation error

$$L^N(V - v) = (L - L^N)v = \varepsilon(\frac{d^2}{dx^2} - \delta^2)v + a(\frac{d}{dx} - D^*)v. \tag{31}$$

Then by (31) and Lemma 3.3, we obtain

$$\begin{aligned} &|L^N(V - v)(x_i)| \\ &\le \begin{cases} \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v^{(3)}| + \frac{a(x_i)}{2}(x_{i+1} - x_i)|v^{(2)}| & \text{if } a(x_i) > 0 \\ \frac{\varepsilon}{3}(x_{i+1} - x_{i-1})|v^{(3)}| + \frac{a(x_i)}{2}(x_i - x_{i-1})|v^{(2)}| & \text{if } a(x_i) < 0 \end{cases} \\ &\le CN^{-1} \end{aligned}$$

Note that

$$\begin{aligned} |\beta_1(V - v)(-1) - \varepsilon\beta_2 D^+(V - v)(-1)| &\le CN^{-1}, \\ |\gamma_1(V - v)(1) + \varepsilon\gamma_2 D^-(V - v)(1)| &\le CN^{-1}. \end{aligned}$$

Now applying Lemma 5.2 to the mesh functions $(V - v)(x_i)$ we can easily obtain $|(V - v)(x_i)| \le CN^{-1},\ \forall\ x_i \in \bar{\Omega}^N$. □

Lemma 5.6. The error in the singular component of the numerical solution is bounded as

$$|(W - w)(x_i)| \le CN^{-1}\ln N, \forall x_i \in \bar{\Omega}^N,$$

where w is the solution of (10) and W is the solution of (30).

Proof. Using (30) and Lemma 3.3 we have,

$$|\beta_1(W - w)(-1) - \varepsilon\beta_2 D^+(W - w)(-1)| \le CN^{-1}\ln N, \tag{32}$$
$$|\gamma_1(W - w)(1) + \varepsilon\gamma_2 D^-(W - w)(1)| \le CN^{-1}\ln N. \tag{33}$$

We first consider the uniform mesh case, when $\tau = \frac{1}{2}$, and so $\varepsilon^{-1} \leq C \ln N$ and $h = N^{-1}$. Using the standard bound for the local truncation error and Lemma 3.3 we have

$$\begin{aligned}
&|L^N(W-w)(x_i)| \\
&\leq \begin{cases} \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|w^{(3)}| + \frac{a(x_i)}{2}(x_{i+1}-x_i)|w^{(2)}| & \text{if } i \leq N/2 \\ \frac{\varepsilon}{3}(x_{i+1}-x_{i-1})|w^{(3)}| + \frac{a(x_i)}{2}(x_i-x_{i-1})|w^{(2)}| & \text{if } i > N/2 \end{cases} \\
&\leq C\varepsilon^{-2}N^{-1}(e^{-\alpha(1+x_i)/\varepsilon} + e^{-\alpha(1-x_i)/\varepsilon}) \text{ if } 1 \leq i \leq N-1
\end{aligned}$$

Consider the mesh functions

$$\Psi^{\pm}(x_i) = \begin{cases} C\frac{e^{2\sigma h/\varepsilon}}{\sigma(\alpha-\sigma)Y_1(x_{N/4})}\varepsilon^{-1}N^{-1}Y_1(x_i) \pm (W-w)(x_i), \\ \quad \text{for } 0 \leq i \leq N/4 \\ C\frac{e^{2\sigma h/\varepsilon}}{\sigma(\alpha-\sigma)(2-\tau)}\varepsilon^{-1}N^{-1}(1-x_i) \pm (W-w)(x_i), \\ \quad \text{for } N/4 \leq i \leq N/2 \\ C\frac{e^{2\sigma h/\varepsilon}}{\sigma(\alpha-\sigma)(2-\tau)}\varepsilon^{-1}N^{-1}(1+x_i) \pm (W-w)(x_i), \\ \quad \text{for } N/2 \leq i \leq 3N/4 \\ C\frac{e^{2\sigma h/\varepsilon}}{\sigma(\alpha-\sigma)Y_2(x_{3N/4})}\varepsilon^{-1}N^{-1}Y_2(x_i) \pm (W-w)(x_i), \\ \quad \text{for } 3N/4 \leq i \leq N \end{cases} \tag{34}$$

where σ is a constant with $0 < \sigma < \alpha$, Y_1 is the solution of the constant coefficient problem (15)-(16) and Y_2 is the solution of the constant coefficient problem (27)-(28) with $\omega = \sigma$. Using Lemma 3.3 we can choose C large enough such that $L^N\Psi^{\pm} \leq 0$, and also $B_1^N\Psi^{\pm}(x_0) \geq 0$ and $B_2^N\Psi^{\pm}(x_N) \geq 0$. Then by the discrete minimum principle we conclude that $\Psi^{\pm}(x_i) \geq 0$ and so for all $x_i \in \bar{\Omega}^N$, $|(W-w)(x_i)| \leq CN^{-1}\ln N$.

We now consider the case $\tau = \frac{2\varepsilon}{\alpha}\ln N$. The subdomains Ω_L and Ω_R are generated with the mesh spacing $4\tau/N$ and the subdomain Ω_c is generated with the mesh spacing $2\tau/N$. We give separate proofs for fine and coarse mesh subintervals.

A different argument is used to bound $|(W-w))(x_i)|$ in each of the subintervals. Actually $\Omega_C = [-1+\tau, 0] \cup [0, 1-\tau]$, has no boundary layer, both W and w are small, and by the triangle inequality we have

$$|(W-w)(x_i)| \leq |W(x_i)| + |w(x_i)|. \tag{35}$$

But here we consider only the subinterval $[-1+\tau, 0]$ for our discussion as one can obtain a similar estimate in the same way for the subinterval $[0, 1-\tau]$. Using Lemma 3.3 we have,

$$|w(x_i)| \leq CN^{-1}. \tag{36}$$

The bound for $|W(x_i)|$ is established by considering the following mesh functions

$$\Psi^{\pm}(x_i) = CN^{-1}(1-x_i) \pm W(x_i),$$

where C is a constant chosen in such a way that $\beta_1\Psi_0^{\pm} - \varepsilon\beta_2\Psi_0^{\pm} \geq 0$ and $\gamma_1\Psi_N^{\pm} + \varepsilon\gamma_2 D^-\Psi_N^{\pm} \geq 0$. Also $L^N(\Psi_i^{\pm})(x_i) \leq 0$. Thus, applying Theorem 5.1, we have $\Psi_i^{\pm} \geq 0$ and

$$|W(x_i)| \leq CN^{-1} \tag{37}$$

Combining the equations (36) and (37) we have,

$$|(W-w)(x_i)| \leq CN^{-1} \; \forall \; x_i \in [-1+\tau, 0] \tag{38}$$

Now we have to prove the result for $x_i \in [-1, -1+\tau]$.

The proof follows on similar lines to the case $\tau = 1/2$, except that we use the discrete minimum principle on $[-1, -1+\tau]$. We will also need to use the bound $W(x_{N/4}) \leq CN^{-1}$.

From Lemma 3.3, we have in this case

$$|L^N(W-w)(x_i)| \leq C\tau\varepsilon^{-2}N^{-1}e^{-\alpha(1+x_i)/\varepsilon} \; \forall \; 0 \leq i \leq N/4.$$

Analogously to the earlier case, the mesh functions are defined as

$$\Psi^{\pm}(x_i) = \frac{Ce^{2\sigma h/\varepsilon}}{\sigma(\alpha-\sigma)}\tau\varepsilon^{-1}N^{-1}Z_i + CN^{-1} \pm (W-w)(x_i),$$

where σ is a constant with $0 < \sigma < \alpha$ and Z_i is the solution of the constant coefficient problem

$$(\varepsilon\delta^2 + \sigma D^+)(Z_i) = 0, \;\; \beta_1 Z_0 - \varepsilon\beta_2 D^+ Z_0 = 1, \;\; \gamma_1 Z_{N/4} + \varepsilon\gamma_2 D^- Z_{N/4} = 0.$$

$$\text{Thus } Z_i = \frac{\lambda^{N/4-i} + (\gamma_2\sigma/\gamma_1) - 1}{\lambda^{N/4} + (\gamma_2\sigma/\gamma_1) - 1}, \;\; \lambda = 1 + \frac{\sigma h}{\varepsilon}.$$

Note that

$$D^+Z_i = \frac{\sigma\lambda^{N/4-i-1}}{\varepsilon[\lambda^{N/4} + (\gamma_2\sigma/\gamma_1) - 1]} \leq 0.$$

Now, $\beta_1\Psi_0^{\pm} - \varepsilon\beta_2 D^+\Psi_0^{\pm} \geq 0, \gamma_1\Psi_{N/4}^{\pm} + \varepsilon\gamma_2 D^-\Psi_{N/4}^{\pm} \geq 0$, $L^N(\Psi_i^{\pm}) \leq 0$; therefore, applying Theorem 5.1, we have $\Psi^{\pm}(x_i) \geq 0, \forall x_i \in [0, -1+\tau)$.

Hence

$$|(W-w)(x_i)| \leq CN^{-1}\ln N \;, 0 \leq i \leq N/4 \tag{39}$$

Combining the estimates as given in (38) and (39), we obtain

$$|(W-w)(x_i)| \leq CN^{-1}\ln N, \; \forall \; 0 \leq i \leq N/2. \tag{40}$$

A similar estimate as that of (40) can be obtained for the interval [0,1], that is for $N/2 \leq i \leq N$. □

Theorem 5.7. If u is the solution of the problem $(1)-(3)$ and U is the corresponding numerical solution using the method outlined in (13)-(14), then we have

$$\sup_{0<\varepsilon\leq 1} ||U-u||_{\bar{\Omega}^N} \leq CN^{-1}\ln N \ \forall\ N\geq 4,$$

where the constant C is independent of ε and N.

Proof. Applying Lemma 5.5 and Lemma 5.6 to $U-u=V-v+W-w$, the theorem gets proved. □

6. Numerical Results

To illustrate the numerical method, we present two examples. The following examples have a turning point at $x=1/2$. The double mesh principle given as in [6] is used to calculate the error and order of convergence of the numerical method. The numerical results are presented for various values of the perturbation parameter $\varepsilon\in\{2^{-20},2^{-19},...,2^{-1}\}$.

Let U^{2N} be the piecewise linear interpolants of the numerical solution U^N on the mesh Ω^{2N}, where N, $2N$ are the number of mesh points. Define the fitted mesh differences as D_ε^N which denote, the numerical solutions obtained using N and $2N$ mesh intervals

$$D_\varepsilon^N=\left\{\ \max_{x_i\in\bar{\Omega}^N}|U^N(x_i)-U^{2N}(x_i)|\ \right\},\ \text{and}\ D^N=\max_{\varepsilon} D_\varepsilon^N.$$

Further, we calculate the order of convergence as

$$p^N=log_2\left(\frac{D^N}{D^{2N}}\right).$$

Example 6.1. Consider the following singularly perturbed turning point problem

$$\varepsilon u^{''}(x)-2(2x-1)u^{'}(x)-4u(x)=0,\ x\in(0,1)$$

$$u(0)-\varepsilon u^{'}(0)=1,\ u(1)+\varepsilon u^{'}(1)=1$$

The computed maximum pointwise error D^N and the computed rate of convergence p^N for the Example 6.1 are given in Table 1. In Figure 1, the graph of the numerical solution of Example 6.1 is plotted. The loglog plot of the maximum pointwise errors for the solution component U of the Example 6.1 is given in Figure 3.

Example 6.2. Consider the following singularly perturbed turning point problem

$$\varepsilon u^{''}(x)-2(2x-1)u^{'}(x)-4u(x)=-\exp(x),\ x\in(0,1)$$

$$u(0)-\varepsilon u^{'}(0)=1,\ u(1)+\varepsilon u^{'}(1)=1$$

The computed maximum pointwise error D^N and the computed rate of convergence p^N for Example 6.2 are given in Table 2. In Figure 2, the graph of the numerical solution of Example 6.2 is plotted. The loglog plot of the maximum pointwise errors for the solution component U of the Example 6.2 is given in Figure 4.

TABLE 1. *Values of D^N, p^N for the solution component u for Example (6.1)*

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	1.5066e-2	7.9260e-3	4.0681e-3	2.0612e-3	1.0375e-3
2^{-2}	2.5816e-2	1.4128e-2	7.4079e-3	3.7954e-3	1.9213e-3
2^{-3}	2.1268e-2	1.3633e-2	8.3390e-3	4.9104e-3	2.8118e-3
2^{-4}	2.1686e-2	1.3574e-2	8.1941e-3	4.7941e-3	2.7369e-3
2^{-5}	2.2666e-2	1.3850e-2	8.1977e-3	4.7603e-3	2.7063e-3
2^{-6}	2.3271e-2	1.4220e-2	8.3416e-3	4.7755e-3	2.7020e-3
2^{-7}	2.3583e-2	1.4441e-2	8.4807e-3	4.8431e-3	2.7148e-3
2^{-8}	2.3729e-2	1.4553e-2	8.5638e-3	4.8984e-3	2.7450e-3
2^{-9}	2.3797e-2	1.4604e-2	8.6045e-3	4.9315e-3	2.7685e-3
2^{-10}	2.3829e-2	1.4627e-2	8.6223e-3	4.9473e-3	2.7826e-3
2^{-11}	2.3844e-2	1.4637e-2	8.6299e-3	4.9539e-3	2.7891e-3
2^{-12}	2.3852e-2	1.4642e-2	8.6332e-3	4.9565e-3	2.7918e-3
2^{-13}	2.3856e-2	1.4644e-2	8.6347e-3	4.9576e-3	2.7927e-3
2^{-14}	2.3858e-2	1.4646e-2	8.6354e-3	4.9580e-3	2.7931e-3
2^{-15}	2.3858e-2	1.4646e-2	8.6357e-3	4.9583e-3	2.7932e-3
2^{-16}	2.3859e-2	1.4646e-2	8.6359e-3	4.9584e-3	2.7933e-3
2^{-17}	2.3859e-2	1.4647e-2	8.6360e-3	4.9584e-3	2.7933e-3
2^{-18}	2.3859e-2	1.4647e-2	8.6360e-3	4.9584e-3	2.7934e-3
2^{-19}	2.3859e-2	1.4647e-2	8.6361e-3	4.9585e-3	2.7934e-3
2^{-20}	2.3859e-2	1.4647e-2	8.6361e-3	4.9585e-3	2.7934e-3
D^N	2.5816e-2	1.4647e-2	8.6361e-3	4.9585e-3	2.8118e-3
p^N	8.1769e-1	7.6212e-1	8.0048e-1	8.1842e-1	-

Table 2. *Values of D^N, p^N for the solution component u for Example (6.2)*

	Number of mesh points N				
ε	64	128	256	512	1024
2^{-1}	4.9981e-3	2.4980e-3	1.2484e-3	6.2405e-4	3.1198e-4
2^{-2}	8.7075e-3	4.4185e-3	2.2254e-3	1.1167e-3	5.5937e-4
2^{-3}	1.6243e-2	8.5205e-3	4.3658e-3	2.2100e-3	1.1119e-3
2^{-4}	1.5099e-2	9.2525e-3	5.4713e-3	3.1497e-3	1.7774e-3
2^{-5}	1.6126e-2	9.7767e-3	5.7340e-3	3.2844e-3	1.8477e-3
2^{-6}	1.6874e-2	1.0181e-2	5.9299e-3	3.3777e-3	1.8936e-3
2^{-7}	1.7322e-2	1.0463e-2	6.0790e-3	3.4478e-3	1.9253e-3
2^{-8}	1.7552e-2	1.0630e-2	6.1840e-3	3.5039e-3	1.9512e-3
2^{-9}	1.7660e-2	1.0713e-2	6.2458e-3	3.5448e-3	1.9736e-3
2^{-10}	1.7709e-2	1.0752e-2	6.2762e-3	3.5689e-3	1.9905e-3
2^{-11}	1.7732e-2	1.0769e-2	6.2896e-3	3.5805e-3	2.0006e-3
2^{-12}	1.7743e-2	1.0777e-2	6.2954e-3	3.5854e-3	2.0053e-3
2^{-13}	1.7748e-2	1.0781e-2	6.2980e-3	3.5873e-3	2.0072e-3
2^{-14}	1.7751e-2	1.0783e-2	6.2992e-3	3.5882e-3	2.0079e-3
2^{-15}	1.7752e-2	1.0783e-2	6.2997e-3	3.5885e-3	2.0082e-3
2^{-16}	1.7753e-2	1.0784e-2	6.3000e-3	3.5887e-3	2.0083e-3
2^{-17}	1.7753e-2	1.0784e-2	6.3001e-3	3.5888e-3	2.0083e-3
2^{-18}	1.7754e-2	1.0784e-2	6.3002e-3	3.5888e-3	2.0084e-3
2^{-19}	1.7754e-2	1.0784e-2	6.3002e-3	3.5889e-3	2.0084e-3
2^{-20}	1.7754e-2	1.0784e-2	6.3002e-3	3.5889e-3	2.0084e-3
D^N	1.7754e-2	1.0784e-2	6.3002e-3	3.5889e-3	2.0084e-3
p^N	7.1919e-1	7.7545e-1	8.1188e-1	8.3750e-1	-

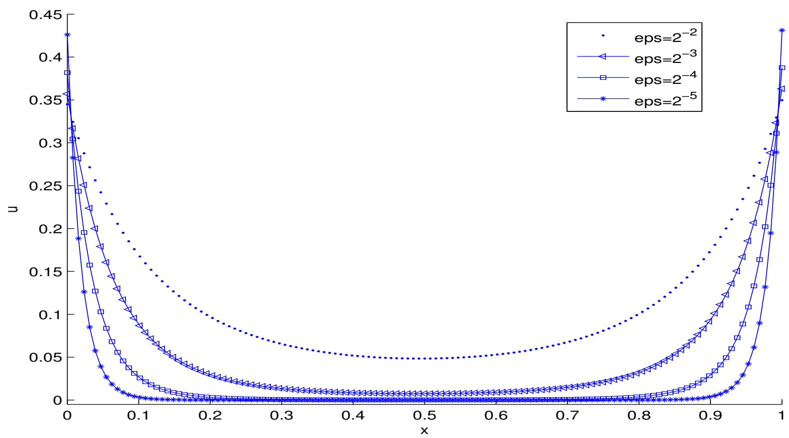

FIGURE 1. Numerical solution graph of Example 6.1 for various values of $\varepsilon(eps)$ and $N = 64$.

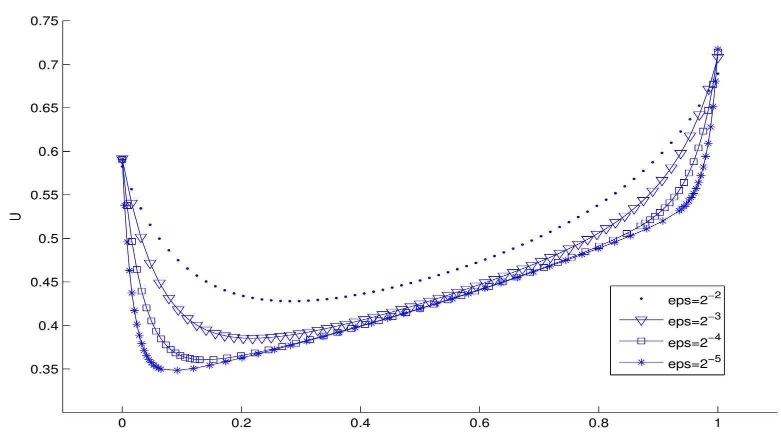

FIGURE 2. Numerical solution graph of Example 6.1 for various values of $\varepsilon(eps)$ and $N = 64$..

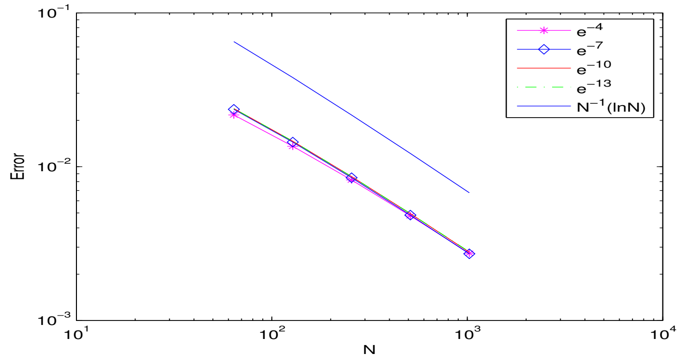

FIGURE 3. Maximum pointwise errors as a function of N and ε for the solution U for Example 6.1

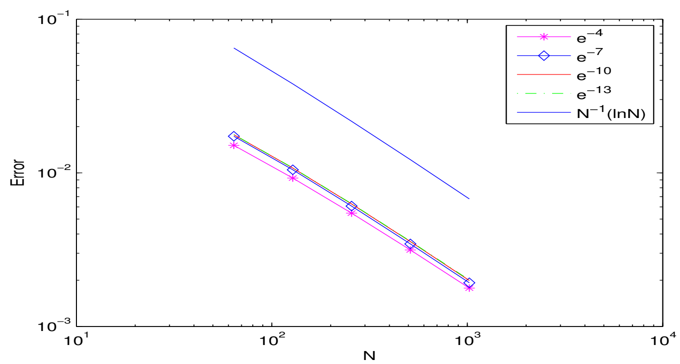

FIGURE 4. Maximum pointwise errors as a function of N and ε for the solution U for Example 6.2

7. Conclusion

A parameter uniform numerical method is suggested to solve a second order singularly perturbed differential equations with a turning point exhibiting boundary layers. An appropriate piecewise uniform mesh (Shishkin mesh) is considered and a classical finite difference schemes is applied on this mesh. An error estimate is derived by using supremum norm and it is of order $O(N^{-1}(\ln N))$. The computed maximum pointwise errors for the solution and rate of convergence were presented in the tables 1 and 2 for the test problems 6.1 and 6.2

respectively. Clearly the Tables 1-2 and Figures 3-4 illustrate that the numerical method is parameter uniform.

References

1. L.R. Abrahamsson, *A priori estimates for solutions of singular perturbations with a turning point*, Studies in Applied Mathematics **56** (1977), 51-69.
2. Ali.R. Ansari, Alan.F. Hegarty, *Numerical solution of a convection diffusion problem with Robin boundary conditions*, Journal of Computational and Applied Mathematics **156** (2003), 221-238.
3. V.B. Andreyev, I.A. Savin, *The computation of boundary flow with uniform accuracy with respect to a small parameter*, Comp. Math. Phys. **36** (1997), 1697-1692.
4. S. Becher, H.G. Roos, *Richardson extrapolation for a singularly perturbed turning point problem with exponential boundary layers*, Journal of Computational and Applied Mathematics, http://dx.doi.org/10.1016/j.cam.2015.05.022.
5. A.E. Berger, H. Han and R.B. Kellogg, *A priori estimates and analysis of a numerical method for a turning point problem*, Mathematics of Computation **42** (1984), 465-492.
6. E.P. Doolan, J.J.H. Miller, W.H.A. Schildres, *Uniform numerical methods for problems with initial and boundary layers*, Boole Press, Dublin, 1980.
7. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E.O' Riordan, G.I. Shishkin, *Robust computational techniques for boundary layers*, Chapman and Hall/CRC Press, Boca Raton, 2000.
8. P.A. Farrell, *Sufficient conditions for the uniform convergence of a difference scheme for a singularly perturbed turning point problem*, SIAM J. on Numerical Analysis **25** (1988), 618-643.
9. N. Geetha, A. Tamilselvan, *Parameter uniform numerical method for third order singularly perturbed turning point problems exhibiting boundary layers*, International Journal of Applied and Computational mathematics, DOI 10.1007/s40819-015-0064-4.
10. N. Geetha, A. Tamilselvan, *Variable Mesh Spline Approximation Method for Solving Second Order Singularly Perturbed Turning Point Problems with Robin Boundary Conditions.*,International Journal of Applied and Computational Mathematics, DOI 10.1007/s40819-016-0140-4, 2016.
11. Jia-qi Mo, Zhao-hui Wen, *A class of boundary value problems for third order differential equation with a turning point*, Applied Mathematics and Mechanics **31** (2010), 1027-1032.
12. M.K. Kadalbajoo, K.C. Patidar, *Variable mesh spline approximation method for solving singularly perturbed turning point problems having boundary layer(s)*, Computer Mathematics with Applications **42** (2001), 1439-1453.
13. R.E.O' Malley, *Introduction to singular perturbations*, Academic Press, New York, 1974.
14. J. Mohapatra, S. Natesan, *Parameter uniform numerical methods for singularly perturbed mixed boundary value problems using grid equidistribution*, J. Appl. Math. Comput. **37** (2011), 247-265.
15. J.J.H. Miller, E.O'Riordan, G.I. Shishkin, *Fitted numerical methods for singularly perturbed problems. Error Estimates in the minimum norm for linear problems in one and two dimensions*, Revised Edition, world scientific publishing Co. Pvt. Ltd., Singapore, 2012.
16. R. Mythili Priyadharshini, N. Ramanujam, *Approximation of derivative for a singularly perturbed second-order ordinary differential equation of Robin type with discontinuous convection coefficient and source term*, Numer. Math. Theor. Meth. Appl. **2** (2009), 100-118.
17. S. Natesan, N. Ramanujam, *A computational method for solving singularly perturbed turning point problems exhibiting twin boundary layers*, Applied Mathematics and Computation **93** (1998), 259-275.
18. S. Natesan, N. Ramanujam, *Initial-Value Technique for Singularly Perturbed Turning Point Problems Exhibiting Twin Boundary Layers*, Journal of Optimization Theory and Applications **99** (1998), 37-52.

19. S. Natesan, J. Jayakumar, J. Vigo-Aguiar, *Parameter uniform numerical method for singularly perturbed turning point problems exhibiting boundary layers*, Journal Computational and Applied Mathematics **158** (2003), 121-134.
20. Pratibhamoy Das, Srinivasan Natesan, *Higher order parameter uniform convergent schemes for robin type reaction-diffusion problems using adaptively generated grid*, International Journal of Computational Methods **9** (2012), 125-152.
21. E. O'Riordan, J. Quinn, *A Singularly perturbed convection diffusion turning point problem with an interior layer*, Comput. Methods. Appl. Math. **12** (2012), 206-220.
22. H.G. Roos, M. Stynes, L. Tobiska, *Numerical methods for singularly perturbed differential equations-convection-diffusion and flow problems*, Springer, 2006.
23. Kapil K. Sharma, Pratima Rai, Kailash C. Patidar, *A review on singularly perturbed differential equations with turning points and interiror layers*, Applied Mathematics and Computation **219** (2013), 10575-10609.
24. W. Wasow, *Linear turning point theory*, Springer Verlag, New York, 1984.
25. A.M. Watts, *A Singular perturbation problem with a turning point*, Bull. Austral. Math. Soc. **5** (1971), 61-73.

N. Geetha received Ph.D. from Bharathidasan University. Her area of interests are Differential Equations and Numerical Analysis.

Department of Mathematics,School of Mathematical Sciences, Bharathidasan University, Tiruchirappalli - 620024, India.
e-mail: nhgeetha@gmail.com

A. Tamilselvan is Chair, Professor & Head, Department of Mathematics, School of Mathematical Sciences, Bharathidasan University. His area of interests are Singular Perturbation Problems and Numerical Analysis.

Department of Mathematics, School of Mathematical Sciences, Bharathidasan University, Tiruchirappalli - 620024, India.
e-mail: mathats@bdu.ac.in