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THE SUM OF SOME STRING OF CONSECUTIVE WITH A DIFFERENCE OF $2k$

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ABSTRACT. This study is about the number expressed and the number not expressed in terms of the sum of consecutive natural numbers with a difference of $2k$. Since it is difficult to generalize in cases of onsecutive positive integers with a difference of $2k$, the table of cases of $4, 6, 8, 10$, and 12 was examined to find the normality and to prove the hypothesis through the results. Generalized guesswork through tables was made to establish and prove the hypothesis of the number of possible and impossible numbers that are to all consecutive natural numbers with a difference of $2k$. Finally, it was possible to verify the possibility and impossibility of the sum of consecutive cases of 124 and 2010. It is expected to be investigated the sum of consecutive natural numbers with a difference of $2k + 1$, as a future research task.

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1. Introduction and Preliminaries

It is known that the sum of consecutive natural numbers, i.e. the sum of the natural numbers with difference 1, and the number of natural numbers with difference 2.[1][2] For odd numbers, it is always expressed as the sum of consecutive natural numbers, such as $k + (k + 1) = 2k + 1$. For example, it is also known that odd 21 can be expressed as the longest sum of lengths in terms of how it is expressed as the sum of consecutive natural numbers, such as $10 + 11 = 6 + 7 + 8 = 1 + 2 + 3 + 4 + 5 + 6$.[3][5]

Remark 1.1. The power of 2 cannot be expressed as the sum of consecutive natural numbers.

Remark 1.2. The number other than the number power of 2 can all be expressed as $2^k(2a + 1)$(k is zero or natural, a is natural) except for 1.

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Remark 1.3. All non-square numbers are represented by the sum of consecutive natural numbers.

Remark 1.4. Prime numbers cannot be expressed as the sum of the natural numbers with difference 2

Remark 1.5. Composite numbers can be expressed as the sum of the natural numbers with difference 2

Suppose a prime number p is expressed as the sum of even or odd numbers of consecutive $n+1$ natural numbers. in other words, $p = a + (a+2) + (a+4) + \cdots + (a+2n) = (n+1)(a+n)$ then $n+1$ and $a+n$ are both natural numbers of two or more, so these two multiples are synthetic. This is a contradiction because it cannot be the same as p. The Composite numbers qr (q and r are natural number more than 2) is the sum of $n+1$ consecutive even or odd numbers starting with a, i.e. $qr = a + (a+2) + (a+4) + \cdots + (a+2n) = (n+1)(a+n)$. $n+a \geq n+1$, $n+1 \geq 2$, $n+a \geq 2$ because $a \geq 1$. Since $n+1 = r$, $n+a = q$, you can see that there is natural number a and n, then all of them become natural number. Also, in relation to The number of partitions of n into an odd number of consecutive parts is equal to the number of odd divisors of n.[4]

Remark 1.6. The number of partitions of n into an odd number of consecutive parts is equal to the number of odd divisors of n less than $\sqrt{2n}$, while the number of partitions into an even number of consecutive parts is equal to the number of odd divisors greater than $\sqrt{2n}$.

Proof. see[4]. □

The researcher seeks to explore the sum of consecutive natural numbers and seek generalization when the differences are even numbers greater than 2.

2. ELEMENTARY PROPERTIES FOR EVEN NUMBER $2, 4, 6, 8, 10, 12$.

To generalize it and find features, we classified numbers into two groups: numbers that can, and cannot be expressed when difference are $4, 6, 8, 10$ and 12. Then we found out that there are some similarities between the results. Furthermore, we made a list of numbers up to 10000000 that cannot be expressed when $4, 6, 8, 10$ and 12, and factorized it by using the computer program. Analyzing it, we set up the theorems and proved them. Following table is the results when $d = 2, 4, 6, 8, 10, 12$.

d=2k	Numbers that cannot be expressed
2	1 and p
4	1, p, and pq ($p \leq q < 2p-1$)
6	1, p, and pq($p \leq q < 3p-2$)
8	1, p, and pq($p \leq q < 4p-3$)
10	1, p, pq($p \leq q < 5p-4$), 8, and 27
12	1, p, pq($p \leq q < 6p-5$), 8, 12, and 27

Here, p and q are prime numbers. The proofs when $d = 2, 4, 6, 8, 10, 12$ are similar, so we will mention when $d = 12$ only.

Theorem 2.1. *When $d = 12$, p(p is a prime number), pq(p and q are prime numbers that satisfy $p \leq q$),8,12 and 27 cannot be expressed.*

Proof. As the equation that we mentioned, $N = (n+1)(a_0+6n)$ $(n \geq 1, a_0 \geq 1$ and $N > 12)$. When N is a prime number, either $n+1$ and a_0+6n must be 1, but it is impossible because $n \geq 1$ and $a_0 \geq 1$. Therefore, a prime number cannot be expressed. When N is the square of a prime number, $N = (n+1)(a_0+6n) = p^2$. As both $n+1$ and a_0+6n are bigger than 1, the equation $n+1 = a_0+6n = p$ is satisfied. It means, $a_0+6n = 1$ have to be true, but it repugnant with $n \geq 1$ and $a_0 \geq 1$. Therefore, the square of a prime number cannot be expressed. When N is the product of two different prime numbers, $N = (n+1)(a_0+6n) = pq$ (p and q are prime numbers that satisfy $p < q$). As both $n+1$ and a_0+6n are bigger than 1 and a_0+6n is bigger than $n+1$, the equations $n+1 = p$ and $a_0+6n = q$ are satisfied. As $n = p-1$ and $a_0 = q-6n = q-6p+6 \geq 1$, it can be expressed when $q+5 \geq 6p$. Therefore, when $p < q < 6p-5$, it cannot be expressed. When N is the product of three prime numbers, $N = (n+1)(a_0+6n) = pqr$ (p, q and r are prime numbers that satisfy $p \leq q \leq r$)

(1) When $p \geq 5$, let's put $n = p-1$ and $a_0 = qr-6p+6$. Then, $n = p-1 \geq 1$ and $a_0 = qr-6p+6 \geq pq-6p+6 = p(q-6)+6 \geq 1$. That is, $n \geq 1$ and $a_0 \geq 1$. Therefore, it can be expressed with any p, q and r.

(2) When $p = 3$.

1) Let's put $n+1 = p = 3$ and $a_0 = qr-6q+6$. Then, the equation $qr = a_0+6p-6 = a_0+12 \geq 13$ (Because $a_0 \geq 1$) have to be satisfied. Therefore, not to be expressed, $qr < 13$.

2) Let's put $n+1 = q$ and $a_0 = pr-6q+6$. Then, as $a_0 = 3r-6q+6 \geq 1, 3r \geq 6q-5$ is satisfied. That is, $r \geq \frac{6q-5}{3}$. Therefore, not to be expressed, r must be smaller than $\frac{6q-5}{3}$. That is, not to be expressed, two inequalities: $qr < 13$ and have to be satisfied. We know that $3 \leq q \leq r$ is true, so satisfying (p, q) is only $(3, 3)$. Therefore, every natural number except 27 can be expressed.

(3) When $p = 2$.

1) Let's put $n+1 = p = 2$ and $a_0 = qr-6p+6$. Then, $qr = a_0+6p-6$, $qr-6 = a_0 \geq 1$ have to be satisfied. That is, $qr \geq 7$. Therefore, not to be expressed, $qr < 7$

2) Let's put $n+1 = q$, and $a_0 = pr-6q+6$. Then, as $a_0 = 2r-6q+6 \geq 1$, $2r \geq 6p-5$. That is, $r \geq \frac{6q-5}{2}$. Therefore, not to be expressed Therefore, not to be expressed, two inequalities: $qr < 7$ and have to be satisfied.

We know that $2 \leq q \leq r$ is true, so satisfying (p,q) are $(2, 2)$ and $(2, 3)$. That is, every natural number except 8 and 12 can be expressed. However, 8 and 12 are smaller than 14, so they are ignorable. Therefore, p(p is a prime number), pq(p and q are prime numbers that satisfy $p \leq q < 6p-5$), 8, 12 and 27 cannot be expressed when $d = 12$. □

Theorem 2.2. *All of the natural number except p(p is a prime number), pq(p and q are prime numbers that satisfy $p \leq q < 6p-5$), 8, 12 and 27 can be expressed when $d = 12$.*

Proof. Any natural number except p(p is a prime number), pq(p and q are prime numbers that satisfy $p \leq q < 6p-5$), and pqr(p, q and r are prime numbers that satisfy $p \leq q \leq r$) can be expressed by $abcd$(a, b, c and d are natural numbers that satisfy $1 < a \leq b \leq c \leq d$). Let's put $N = abcd(1 < a \leq b \leq c \leq d)$. Then, if we put $n+1 = a, a_0 = bcd - 6a + 6 \geq 8 - 12 + 6 = 2$ is satisfied when $a = 2$ because b, c and $d \geq 2$.

And, when $a \geq 3$, because $bcd \geq 9a$, $a_0 = bcd - 6a + 6 \geq 3a + 6 \geq 15$ is satisfied. That is, $a_0 \geq 1$ and $n \geq 1$ are satisfied. Therefore, all of the product of four natural numbers except 1 can be expressed. As a result, every natural number except p(p is a prime number), pq(p and q are prime numbers that satisfy $p \leq q < 6p-5$), 8, 12 and 27 can be expressed when $d = 12$. ☐

From the result, we assumed that prime numbers cannot be expressed regardless of d. And we supposed all composite number pq(p is the smallest prime factor and $p \leq q$) satisfy $pq < kp - k + 1$. Also, in the process of proving a composite number can be expressed, there were something in common. For example, we proved that when $p = n+1$ and $q = a_0 + nk$ are satisfy, $N = pq(1 < p < q)$ is expressed. Similarly, when we examined in case of $N = pqr(1 < p < q < r)$, and $N = pqrs(1 < p < q < r < s)$, we proved that when $p = n+1$ and $q = a_0 + nk$ are satisfy, $N = pqr(1 < p < q < r)$ is expressed and when $p = n+1$, and $qrs = a_0 + nk$ are satisfy, $N = pqrs(1 < p < q < r < s)$is expressed. From these common features, we inferred that the smallest prime factor p satisfies $p = n+1$ is the basic condition, regardless of, which composite number should satisfy to be expressed. Synthesizing the assumption, we formed the following theorems.

3. MAIN RESULT

Theorem 3.1. 1*, prime number, and composite N(p is the smallest prime factor and $p^2 \leq N < kp^2 - kp + p$) cannot be expressed as the sum of consecutive that a common difference is* $2k$*.*

Theorem 3.2. *All natural number except* 1*, prime number, and composite N(p is the smallest prime factor and $p^2 \leq N < kp^2 - kp + p$) can be expressed as the sum of consecutive that a common difference is* $2k$*.*

To examine the above theorems we checked a random numbers' possibility of expression as an arithmetic sum when $d = 124$ and 2010, and we ensured the theorems.

Now, let's prove it.

Proof of theorem 3.1, 3.2. First of all, 'A natural number N can be expressed' means 'A natural number a_0, n which satisfies $N = (a_0 + nk)(n+1)$ exists'.

(1) When $N = 1$, If 1 can be expressed, it has to satisfy $1 = N = (n + 1)(a_0 + nk) \geq 2(k + 1) \geq 4$. But this is definitely false. Therefore, it cannot be expressed.

(2) When N is a prime number, If N can be expressed, it has to satisfy $N = (n + 1)(a_0 + nk)$. However a_0, n is a natural number meaning $a_0 + nk$, $n+1$. So N should be a composite number. But this is definitely false. Therefore, prime number cannot be expressed.

(3) When N is composite number, It can be expressed ($p_1, p_2, p_3, , p_m$ are prime number, $e_1, e_2, e_3, \cdots, e_m$, m are natural number that satisfy $e_1 + e_2 + e_3 + \cdots + e_m \geq 2$) Without loss of generosity, let's put $p_1 < p_2 < p_3 < \cdots < p_m$. First, let's prove that 'some composite number N can be expressed.' and 'N can be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1 - 1)$' is the necessary and sufficient condition.

'some composite number N can be expressed.', 'N can be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1) - 1$'.

This proposition is self-evident.

Let's suppose N cannot be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1) - 1$ but has to be expressed because of the necessary condition. So let q which is a divisor of N and satisfies $n = q - 1$, $a_0 = \frac{N}{p_1} - k(p_1) - 1$. But q cannot be 1 and p_1 so $q > p_1$ (because p_1 is the smallest prime factor). Also, $\frac{N}{p} < \frac{N}{p_1}$, $kq < -kp_1$. So it satisfies $1a_0 = \frac{N}{q} - k(q-1) = \frac{N}{q} - kq + k < \frac{N}{p_1} - kp + k = \frac{N}{p_1} - k(p-1)$ and $p_1 \geq 2$. And this means $p_1 - 1 \geq 1$, $\frac{N}{p_1} - k(p-1) > 1$. In other words N can be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1 - 1)$. But it is definitely false by the above supposition. So the supposition is false. Therefore, when N can be expressed, N can also be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1 - 1)$ Now, suppose N can be expressed. From the above proposition, N can also be expressed when $n = p_1 - 1$, $a_0 = \frac{N}{p_1} - k(p_1 - 1)$. So $n = p_1 - 1 \geq 1$,$a_0 = \frac{N}{p_1} - k(p_1 - 1) \geq 1$. In other words $n = p_1 - 1 \geq 1$, $N \geq kp_1^2 - kp_1 + p_1$ (By multiplying p_1 at the both sides). So the condition that N which cannot be expressed should satisfy is $p_1^2 \leq N = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_m^{e_m} < kp_1^2 - kp_1 + p_1$ (p_1 is the smallest prime factor) because $N = p_1^{e_1} p_2^{e_2} p_3^{e_3} p_m^{e_m} \geq p_1^{e_1 + e_2 + e_3 + + e_m} \geq p_1^2$. Therefore, 1, prime number, and composite number N ($p_1^2 \leq N < kp_1^2 - kp_1 + p_1$ and N is composite number that p_1 is the smallest prime factor of it) cannot be expressed as an arithmetic sum that a common difference is $2k$. □

Theorem 3.3. *The number of partitions of N into an odd number of consecutive parts with a difference of $2k$ is equal to the number of odd divisors of N less than $\sqrt{2N}$.*

Proof of theorem 3.3. Suppdse N is the sum of an odd number of consecutive parts with a difference of $2k$. Then the middle part is an integer and is the average of the parts. Suppose the middle part is a, and the number of parts is $2n + 1$. The partition of N is $N = (a - 2nk) + \cdots + a + \cdots + (a + 2nk)$

and $N = (2n+1)a$. So $d = 2n+1$ is an odd divisor of N and its codivisor is $d' = a$. Note that $a - 2n > a - 2nk \geq 1$, that is,$2a - (2n+1) > 0$, $d < 2d'$, $d < \frac{2N}{d}$, and $d^2 < 2N$. Conversely, suppose d is an odd divisor of N with $d^2 < 2N$, and codivisor d'. Then $d < 2d'$, and if we write $2n+1 = d$, $a = d'$ then $N = (a-2nk) + \cdots + a + \cdots + (a+2nk)$ is a partition of N into $2n+1$ consecutive parts with a difference of $2k$. $\square$

4. CONCLUSION

Finally, we studied about natural numbers that can be expressed by consecutive that a common difference is general even number. We got a result that every natural number except 1, prime number, composite number N ($p_1^2 \leq N < kp_1^2 - kp_1 + p_1$ and N is composite number that p_1 is the smallest prime factor of it)can be expressed by the sum of consecutive that a common difference is general even number. As a future task, we have to find the necessary and sufficient condition when a common difference is general odd number.

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