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PEBBLING ON THE MIDDLE GRAPH OF A COMPLETE BINARY TREE

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Abstract. Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles at an adjacent vertex. The t-pebbling number, $f_t(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f_t(G)$ pebbles, t pebbles can be moved to any specified vertex by a sequence of pebbling moves. A graph G has the $2t$-pebbling property if for any distribution with more than $2f_t(G) - q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using the sequence of pebbling moves, to put $2t$ pebbles on any vertex. In this paper, we determine the t-pebbling number for the middle graph of a complete binary tree $M(B_h)$ and we show that the middle graph of a complete binary tree $M(B_h)$ satisfies the $2t$-pebbling property.

Key words and phrases : t-pebbling number, $2t$-pebbling property, middle graph, complete binary tree.

1. Introduction

Pebbling in graphs was first considered by Chung [1]. Consider a connected graph with fixed number of pebbles distributed on its vertices. A *pebbling move* consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The *pebbling number of a vertex v* in a graph G is the smallest number $f(G, v)$ such that for every placement of $f(G, v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. The *t-pebbling number of v* in G is the smallest number $f_t(G, v)$ such that from every placement of $f_t(G, v)$ pebbles, it is possible to move t pebbles to v. Then the *pebbling number* of G and the *t-pebbling number* of G are the smallest numbers, $f(G)$ and $f_t(G)$, such that from any distribution of $f(G)$ pebbles or $f_t(G)$ pebbles, respectively, it is possible to move one or t pebbles, respectively,

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to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ and $f_t(G)$ are the maximum values of $f(G,v)$ and $f_t(G,v)$ over all vertices v.

Given a pebbling of G, let p be the number of pebbles, q be the number of vertices with at least one pebble, we say that G satisfies the 2- *pebbling property* if it is possible to move two pebbes to any specified target vertex whenever p and q satisfy the inequality $p + q > 2f(G)$. Further a graph G satisfies the *$2t$-pebbling property* if $2t$ pebbles can be moved to a specified vertex whenever p and q satisfy the inequality $p + q > 2f_t(G)$. Lourdusamy et al. [4], [5], [6], [7] have proved that the star graph, the complete graph, the complete multi-partite graph, the n-cube, the cycle and the wheel graph have the $2t$-pebbling property.

The Cartesian product of graphs G and H is denoted by $G \times H$. The following well-known conjecture first appeared in [1].

Conjecture 1.1. (Graham [1]) *For any connected graphs G and H, $f(G \times H) \leq f(G).f(H)$.*

Further Lourdusamy [4] extended this conjecture as follows.

Conjecture 1.2. (Lourdusamy [4]) *For any connected graphs G and H, $f_t(G \times H) \leq f(G).f_t(H)$.*

Lourdusamy et al.[4], [5], [6], [7] proved that if G is a fan graph, a wheel graph, a complete graph, a star graph, a path, a complete multi-partite graph and H has the $2t$-pebbling property, then Conjecture 1.2 holds.

The purpose of this paper is to find the t-pebbling number for the middle graph of a complete binary tree $M(B_h)$ and prove that the middle graph of a complete binary tree $M(B_h)$ satisfies the $2t$-pebbling property. In other words, the Conjecture 1.2 is true when G is a fan graph, a wheel graph, a complete graph, a star graph, a path, a complete multi-partite graph and H is the middle graph of a complete binary tree.

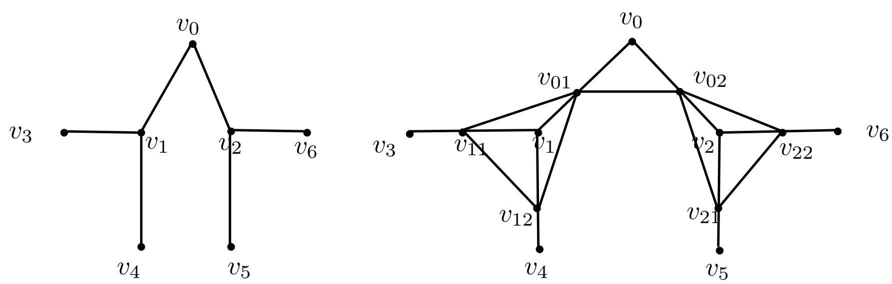
We now introduce some definitions and notations which will be useful for the subsequent sections. For graph theoretic terminologies we refer to [2].

Definition 1.3. A *complete binary tree*, denoted by B_h is a binary tree of height h, with 2^k vertices at a distance k from the root. Each vertex of B_h has two children, except for the set of 2^h vertices that are distance h away from the root, each of which has no children.

Definition 1.4. The *middle graph* $M(G)$ of a graph G is the graph obtained from G by inserting a new vertex into every edge of G and by joining the edges of those pair of these new vertices which lie on the adjacent edges of G.

Let $H \subseteq V(G)$. Let $< H >$ be the induced subgraph of the graph G, induced by the vertices in the set H. Label the root vertex of the complete binary tree of height h by v_0. For each level i, $1 \leq i \leq h$ label the vertices of the graph from the left to the right by $v_{2^i-1}, v_{2^i}, v_{2^i+1}, \ldots, v_{2^{i+1}-2}$.

Now we create the middle graph of the complete binary tree of height h. First the edges $v_0v_1, v_0v_2, v_1v_3, v_1v_4, \ldots, v_{2^h-2}v_{2^{h+1}-2}$ are subdivided by introducing the new vertices $v_{01}, v_{02}, v_{11}, v_{12}, \ldots, v_{2^{h-2}2}$ respectively. Then joining the edges of those pairs of these new vertices which lie on the adjacent edges of the graph. Let us denote the middle graph of a complete binary tree of height h by $M(B_h)$.



B_2 $M(B_2)$

Figure 1.1.

The reader can easily view that $M(B_h)$ has 2^{h-i} copies of $M(B_i)$, where $1 \le i \le h-1$. In this paper, $M(B_i)^{(j)}$ $j = 1, 2, \ldots$ represents the j^{th} occurance of $M(B_i)$ counted from left. For instance in Figure 1.1 we have two $M(B_1)$.

With regard to the pebbling number of middle graphs, we find the following theorems in [3] and [9].

Theorem 1.5. ([3]) *Let P_n be the path with n, $(n \ge 2)$ vertices. Then $f(M(P_n)) = 2^n + n - 2$.*

Theorem 1.6. ([3]) *Let K_n be the Complete graph with n, $(n \ge 2)$ vertices. Then $f(M(K_n)) = \frac{n(n+1)}{2}$.*

Theorem 1.7. ([3]) *Let $K_{1,n}$ be the star graph with $n+1$, $(n \ge 2)$ vertices. Then $f(M(K_{1,n})) = 3n + 3$.*

Theorem 1.8. ([9]) *Let C_n be the cycle with n vertices. Then*

$$f(M(C_n)) = \begin{cases} 2^{n+1} + 2n - 2, & n \ge 2, \quad n \text{ is even} \\ \left\lfloor \frac{2^{n+3}}{3} \right\rfloor + 2n, & n \text{ is odd} \end{cases}$$

Theorem 1.9. ([9]) *Let F_n be the fan graph with n, $(n \ge 4)$ vertices. Then $f(M(F_n)) = 3n - 1$.*

In Section 2, we compute the t-pebbling number for $M(B_h)$. In Section 3, we prove that $M(B_h)$ satisfies the $2t$-pebbling property.

2. The t-pebbling number for $M(B_h)$

Remark 2.1. A distribution of pebbles on the vertices of the graph G is a function $p : V(G) \to N \cup \{0\}$. Let $p(v)$ denote the number of pebbles on the vertex v and $p(A)$ denote the number of pebbles on the vertices of the set $A \subseteq V(G)$. Let v be a target vertex in the graph G. If $p(v) = 1$ or $p(u) \geq 2$, where $uv \in E(G)$, then we can move a pebble to v easily. So we always assume that $p(v) = 0$ and $p(u) \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Lemma 2.1. *Suppose $4m + 1$ pebbles are distributed on the subgraph $H =< v_0, v_{01}, v_{02}, v_2 >$ of the graph $M(B_1)$, then m pebbles can be moved to v_{01}.*

Proof. The proof is by induction on m. Let D be any distribution of 5 pebbles on the vertices of H. By Remark 2.1, $p(v_0) \leq 1$ and $p(v_{02}) \leq 1$. Then we have at least four pebbles on the path $P : v_2, v_{02}, v_{01}$ and hence we can move a pebble to v_{01}.

Assume the lemma is true for $2 \leq m' < m$. Now consider any distribution of $4m + 1$ pebbles on the vertices of H. First assume that $p(v_{01}) = x$, where $1 \leq x \leq m-1$. The remaining number of pebbles on the vertices of $< H - \{v_{01}\} >$ is at least $4m+1-x \geq 4(m-x)+1$. Thus we can move $m-x$ additional pebbles to the vertex v_{01}. Assume $p(v_{01}) = 0$. Clearly we can move one pebble to v_{01} at a cost of at most four pebbles and hence we have at least $4(m-1)+1$ remaining pebbles on the vertices of H. So, we can move $m - 1$ additional pebbles to the vertex v_{01} by induction. □

Lemma 2.2. *Suppose $8m + 5$ pebbles are distributed on the subgraph $H =< M(B_1)^{(2)} \cup \{v_{02}, v_{01}, v_0\} >$ of the graph $M(B_2)$, then m pebbles can be moved to v_{01}.*

Proof. The proof is by induction on m. Let D be any distribution of 13 pebbles on the vertices of H. Clearly $p(v_{01}) = 0$ and $p(v_0) \leq 1$ by Remark 2.1. Using 12 pebbles on the vertices of $< H - \{v_0\} >$ we can move a pebble to v_{01}, since $< H - \{v_0\} >$ is isomorphic to $M(K_{1,3})$ and $f(M(K_{1,3})) = 12$.

Assume the lemma is true for $2 \leq m' < m$. Now consider any distribution of $8m + 5$ pebbles on the vertices of H. First assume that $p(v_{01}) = x$, where $1 \leq x \leq m-1$. The remaining number of pebbles on the vertices of $< H - \{v_{01}\} >$ is at least $8m + 5 - x \geq 8(m - x) + 5$. Thus we can move $m - x$ additional pebbles to the vertex v_{01}. Assume $p(v_{01}) = 0$. Clearly, we can move one pebble to v_{01} at a cost of at most eight pebbles hence we have at least $8(m - 1) + 5$ remaining pebbles on the vertices of H. So, we can move $m - 1$ additional pebbles to the vertex v_{01} by induction. □

Lemma 2.3. *Suppose $16m + 17$ pebbles are distributed on the subgraph $H =< M(B_2)^{(2)} \cup \{v_{02}, v_{01}, v_0\} >$ of the graph $M(B_3)$, then m pebbles can be moved to v_{01}.*

Proof. The proof is by induction on m. Let D be any distribution of 33 pebbles on the vertices of H. Clearly $p(v_{01}) = 0, p(v_0) \leq 1$ and $p(v_{02}) \leq 1$ by Remark 2.1. Assume $p(v_{02}) = 1$. Then $C_1 \cup \{v_2\}$ or $C_2 \cup \{v_2\}$ has at least 16 pebbles, where $C_1 =< \{v_{21}\} \cup M(B_1)^{(3)} >$ and $C_2 =< \{v_{22}\} \cup M(B_1)^{(4)} >$. Thus we can move a pebble to v_{02} by Lemma 2.2 and hence we reach the target. Therefore assume that $p(v_{02}) = 0$. If $p(v_2) \geq 4$, then we can move a pebble to the target easily. If $p(v_2)$ =2 or 3, we can move a pebble to v_{02}. Then either C_1 or C_2 has at least 15 pebbles. Thus we can move another pebble to v_{02} by Lemma 2.2 and hence we reach the target. Assume $p(v_2) \leq 1$. Then either C_1 or C_2 has at least 16 pebbles. Without loss of generality, assume that $p(C_1) \geq 16$. If $p(C_1) \geq 21$, then we can move two pebbles to v_{02} by Lemma 2.2. Therefore $16 \leq p(C_1) \leq 20$. Suppose $p(v_2) = 1$. If $p(C_1) = 20$, then $p(C_1 \cup \{v_2\}) = 21$. Now we can move two pebbles to v_{02} by Lemma 2.2. Therefore assume that $p(C_1) \leq 19$. Then there are at least 12 pebbles distributed on C_2. If $p(C_2) = 12$, then $p(C_2 \cup \{v_2\}) = 13$. Otherwise $p(C_2) \geq 13$. Thus in either cases, we can move one pebble from C_1 and another pebble from C_2 to v_{02} by Lemma 2.2. Now assume $p(v_2) = 0$. Then one pebble can be moved to v_{02} from C_1 and also C_2 contains at least 12 pebbles. Since $p(v_2) = 0$, using 12 pebbles we can move another pebble to v_{02} from C_2 and hence we are done.

Assume the lemma is true for $2 \leq m' < m$. Now consider any distribution of $16m + 17$ pebbles. First assume that $p(v_{01}) = x$, where $1 \leq x \leq m - 1$. The remaining number of pebbles on the vertices of $< H - \{v_{01}\} >$ is at least $16m + 17 - x \geq 16(m - x) + 17$. Thus we can move $m - x$ additional pebbles to v_{01}. Assume $p(v_{01}) = 0$. Clearly, we can move one pebble to v_{01} at a cost of at most sixteen pebbles and hence we have at least $16(m - 1) + 17$ remaining pebbles on the vertices of H. So, we can move $m - 1$ additional pebbles to the vertex v_{01} by induction. $\square$

Lemma 2.4. *Suppose* $m2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$ *pebbles are distributed on the subgraph* $H =< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ *of the graph* $M(B_h)$*, then* m *pebbles can be moved to* v_{01}*.*

Proof. The proof is by induction on h, where the cases $h = 1, 2$, and 3 follow from Lemma 2.1, Lemma 2.2 and Lemma 2.3 respectively. Assume the lemma for $4 \leq h' < h$. We now prove the lemma for height h. Let D be any distribution of $m2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$ pebbles on H. We prove that m pebbles can be moved to v_{01}.

We prove this by induction on m. For $m = 1$, $2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$ pebbles are distributed on the vertices of H. Clearly $p(v_{01}) = 0$, $p(v_0) \leq 1$ and $p(v_{02}) \leq 1$ by Remark 2.1. Assume $p(v_{02}) = 1$. Then either $C_1 \cup \{v_2\}$ or $C_2 \cup \{v_2\}$ has at least $2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$ pebbles, where $C_1 =< \{v_{21}\} \cup M(B_{h-2})^{(3)} >$ and $C_2 =< \{v_{22}\} \cup M(B_{h-2})^{(4)} >$. Then we can move a pebble to v_{02} by induction and hence we reach the target. Therefore assume that $p(v_{02}) = 0$. If $p(v_2) \geq 4$, then we can move a pebble to the target easily. If $p(v_2) = 2$ or 3, then we can move a pebble to v_{02}. Also either C_1 or C_2 has at least $2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$ pebbles. Then we can move a pebble to v_{02} by induction and hence we are done. Assume $p(v_2) \leq 1$.

Case 1 : Let $p(v_2) = 1$.

Then $p(C_j) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$ for some $j = 1, 2$. Without loss of generality, assume that $p(C_1) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$.

Suppose $p(C_1) \geq 2.2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$, then we can move two pebbles to v_{02} and hence we are done. If $p(C_1) = 2.2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$, then we can reach the target easily. Therefore we assume that $p(C_1) \leq 2.2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 2$. Then $p(C_2) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$. If $p(C_2) = 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$, then $p(C_2 \cup \{v_2\}) = 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$ and hence we are done. Otherwise $p(C_2) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$. Then we can move one pebble from C_2 and another from C_1 and hence we are done.

Case 2 : Let $p(v_2) = 0$.

Then $p(C_j) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$ for some $j = 1, 2$. Without loss of generality, assume that $p(C_1) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$. Suppose $p(C_1) \geq 2.2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1)$, then we can move two pebbles to v_{02} and hence we are done. If $p(C_1) \leq 2.2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$, then $p(C_2) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$. Since $p(v_2) = 0$, we can move a pebble to v_{02} using $2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1$ pebbles on C_2 and another pebble to v_{02} from C_1. Hence we can move a pebble to v_{01}.

Assume the lemma is true for $2 \leq m' < m$. Now consider any distribution of $m2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$ pebbles on the vertices of H. First assume that $p(v_{01}) = x$, where $1 \leq x \leq m-1$. The remaining number of pebbles on the vertices of $< G - \{v_{01}\} >$ is at least

$$m2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1) - x \geq (m-x)2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1).$$

Thus we can move $m-x$ additional pebbles to the vertex v_{01}. Assume $p(v_{01}) = 0$. Clearly, we can move one pebble to v_{01} at a cost of at most 2^{h+1} pebbles and hence we have

$$m2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1) - 2^{h+1} \geq (m-1)2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$$

pebbles on the vertices of G. So, we can move $m - 1$ additional pebbles to the vertex v_{01} by induction. □

In [3], we find that $f(M(P_3)) = 9$. Since $M(B_1)$ is isomorphic to $M(P_3)$, we conclude that $f(M(B_1)) = 9$.

Theorem 2.5. [3] *For the middle graph of a complete binary tree of height one,* $f(M(B_1)) = 9$.

Theorem 2.6. *For the middle graph of a complete binary tree of height two,* $f(M(B_2)) = 41$.

Proof. Placing 31 pebbles on v_3, 3 pebbles each on v_4 and v_5 and one pebble each on v_0, v_1, v_2 we cannot reach v_6. Thus $f(M(B_2)) \geq 41$.

Now we prove that $f(M(B_2)) \leq 41$. Let D be any distribution of 41 pebbles on the vertices of $M(B_2)$.

Case 1: Let v_3 be the target vertex.
Clearly $p(v_3) = 0$ and $p(v_{11}) \leq 1$ by Remark 2.1. If $p(M(B_1)^{(1)}) \geq 9$ then we are done. If $5 \leq p(M(B_1)^{(1)}) \leq 8$ then we can move one pebble to v_{11} by Lemma 2.1. Also the minimum number of pebbles distributed on the vertices of $< M(B_1)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is 33. By using Lemma 2.2, we can move at least three pebbles to v_{01} and hence we reach the target. Suppose $0 \leq p(M(B_1)^{(1)}) \leq 4$. Then the minimum number of pebbles distributed on the vertices of $< M(B_1)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is 37. By using Lemma 2.2, we can move four pebbles to v_{01} and hence we are done.

Case 2 : Let v be the target vertex other than the pendant vertices.
Without loss of generality, assume that $v \in < M(B_1)^{(1)} \cup \{v_0, v_{01}\} >$. Since $< M(B_1)^{(1)} \cup \{v_0, v_{01}\} >$ is isomorphic to $M(K_{1,3})$, we are done if $p(< M(B_1)^{(1)} \cup \{v_0, v_{01}\} >) \geq 12$. Otherwise there are at least 30 pebbles distributed on the vertices of $< M(B_1)^{(2)} \cup \{v_{02}\} >$. By using Lemma 2.2, we can reach the target. □

Theorem 2.7. *For the middle graph of a complete binary tree of height three,* $f(M(B_3)) = 161$.

Proof. Placing 127 pebbles on v_7, seven pebbles each on v_9 and v_{11}, three pebbles each on v_8, v_{10}, v_{12} and v_{13} and one pebble each on $v_0, v_1, v_2, v_3, v_4, v_5$ and v_6, we cannot move a pebble to v_{14}. Thus $f(M(B_3)) \geq 161$.

Now we prove that $f(M(B_3)) \leq 161$. Let D be any distribution of 161 pebbles on the vertices of $M(B_3)$.

Case 1 : Let v_7 be the target vertex.
Clearly $p(v_7) = 0$ and $p(v_{31}) \leq 1$ by Remark 2.1. If $p(M(B_1)^{(1)}) \geq 9$ or $p(M(B_2)^{(1)}) \geq 41$, then we are done. Therefore assume $p(M(B_1)^{(1)}) \leq 8$ and $p(M(B_2)^{(1)}) \leq 40$. Then the minimum number of pebbles distributed on the vertices of $< M(B_2)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is 121. Thus we can move at least six pebbles to v_{01} by Lemma 2.2. If $13 \leq p(< M(B_1)^{(1)} \cup \{v_1, v_{11}, v_{12}\} >) \leq 40$, then we can move at least one pebble to v_{11} and hence we reach the target. So assume $p(< M(B_1)^{(1)} \cup \{v_1, v_{11}, v_{12}\} >) \leq 12$. Suppose $5 \leq p(M(B_1)^{(1)}) \leq 8$. Then by Lemma 2.1 we can move at least one pebble to v_{31} and hence we are done. Therefore assume $p(M(B_1)^{(1)}) \leq 4$. Now the minimum number of pebbles distributed on the vertices of $< M(B_2)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is 145. Thus we can

move eight pebbles to v_{01} by Lemma 2.3 and hence we are done.

Case 2 : Let v_0 be the target vertex.
Clearly $p(v_0) = 0$, $p(v_{01}) \leq 1$ and $p(v_{02}) \leq 1$ by Remark 2.1. Then $p(< M(B_2)^{(i)} \cup \{v_0, v_{01}, v_{02}\} >) \geq 80$ for some $j = 1, 2$. By using Lemma 2.3, we can easily reach the target.

Case 3 : Let v be the target vertex, where $v \in \{v_{01}, v_{02}\}$.
Without loss of generality, let v_{01} be the target vertex. Clearly $p(v_{01})$ =0. If $p(< M(B_2)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \geq 33$, then by Lemma 2.3 we are done. Suppose $p(< M(B_2)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \leq 32$. Then the minimum number of pebbles distributed on the vertices of $M(B_2)^{(1)}$ is 129. Hence any one of $< M(B_1)^{(i)} \cup \{v_{1i}, v_1, v_{01}\} >$, where $i = 1, 2$ contains at least 64 pebbles and thus we reach the target by Lemma 2.2.

Case 4 : Let v be the target vertex, other than v_{01}, v_{02}, v_0 and the pendant vertices.
Without loss of generality, assume that $v \in M(B_2)^{(1)}$. If $p(M(B_2)^{(1)}) \geq 41$, then we are done. Therefore assume $p(M(B_2)^{(1)}) \leq 40$. Then the minimum number of pebbles distributed on the vertices of $< M(B_2)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is 121. By using Lemma 2.3, we can move at least six pebbles to v_{01} and hence we are done. □

Theorem 2.8. *For the middle graph of a complete binary tree of height h, $f(M(B_h)) = 2^h(2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1)$.*

Proof. The proof is by induction on h, where the cases $h = 1, 2$, and 3 follow from Theorem 2.5, Theorem 2.6 and Theorem 2.7 respectively. Assume the theorem is true for $4 \leq h' < h$. Let D be any distribution of $2^h(2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1)$ pebbles.

Case 1: Let v_{2^h-1} be the target vertex.

Clearly, $p(v_{2^h-1}) = 0$ and $p(v_{(2^{h-1}-1)1}) \leq 1$ by Remark 2.1. If $p(M(B_{h-1})^{(1)}) \geq f(M(B_{h-1}))$, then we are done. Assume $p(M(B_{h-1})^{(1)}) \leq f(M(B_{h-1})) - 1$. If $p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \geq (2^h)2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1)$,

then by Lemma 2.4, we can move 2^h pebbles to v_{01} and hence we are done. Therefore assume

$$p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \leq (2^h)2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1) - 1.$$

If $p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \leq f(M(B_h)) - f(M(B_{h-1}))$, then $M(B_{h-1})^{(1)}$ contains at least $f(M(B_{h-1}))$ pebbles and hence we are done. So, assume

$$p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \geq f(M(B_h)) - f(M(B_{h-1})) + 1.$$

Since

$$f(M(B_h)) - f(M(B_{h-1})) + 1 \geq (2^h - 2^{h-2})2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i \ (2^{h-i} - 1),$$

by Lemma 2.4 we can move at least $2^h - 2^{h-2}$ pebbles to v_{01}.

Let x be the number of pebbles on v_{01} after the sequence of pebbling moves on $< M(B_{h-1})^{(2)} \cup \{v_0, v_{02}\} >$. Clearly, $2^h - 2^{h-2} \leq x \leq 2^h - 1$. Suppose $v_{(2^{h-1}-1)1}$ is already occupied, then we are done. So, assume $p(v_{(2^{h-1}-1)1}) = 0$.

Suppose any one of $< M(B_{i-1})^{(2)} \cup \{v_{2^{h-i}-1}, v_{(2^{h-i}-1)1}, v_{(2^{h-i}-1)2} >$, where $i = 2, 3, \ldots, h-1$ contains at least $m_i 2^{i+1} + 2^{i-1} + \sum_{j=0}^{i-2} 2^j (2^{i-j} - 1)$ pebbles, where $m_i = 2^i - \left\lfloor \frac{x}{2^{h-i}} \right\rfloor$, then by Lemma 2.4 we can move m_i pebbles to $v_{(2^{h-i}-1)1}$ and hence we are done. Otherwise,

$$p(< M(B_{i-1})^{(2)} \cup \{v_{2^{h-i}-1}, v_{(2^{h-i}-1)1}, v_{(2^{h-i}-1)2} >) \leq m_i 2^{i+1} + 2^{i-1} + \sum_{j=0}^{i-2} 2^j (2^{i-j} - 1) - 1$$

for every $i = 2, 3, \ldots, h-1$. Now the minimum number of pebbbles distributed on the vertices of $M(B_1)^{(1)}$ is

$$f(M(B_h)) - \{[(x+1)2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i (2^{h-i} - 1) - 1] + \sum_{i=2}^{h-1} [m_i 2^{i+1} + 2^{i-1} + \sum_{j=0}^{i-2} 2^j (2^{i-j} - 1) - 1]\}$$

$$= 2^{2h+1} - (x+1)2^{h+1} + 2^h + 2^h - h - 2 + h - 1 - m_2 2^3 - m_3 2^4 - \ldots - m_{h-1} 2^h$$

$$\geq 5, \text{ since } h \geq 4.$$

Thus we can move a pebble to $v_{(2^{h-1}-1)1}$ by Lemma 2.1 and hence we are done.

Case 2: Let v_0 be the target vertex.

Clearly $p(v_0) = 0$ and any of $< M(B_{h-1})^{(i)} \cup \{v_{01}, v_{02}\} >$, where $i = 1, 2$ contains at least $2.2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i (2^{h-i} - 1)$ pebbles. Then by Lemma 2.4, we are done.

Case 3: Let v_{0i}, $i = 1, 2$ be the target vertex.

Without loss of generality, assume that v_{01} be the target vertex. Clearly $p(v_{01}) = 0$. Suppose

$$p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \geq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1),$$

then by Lemma 2.4 we are done. Assume

$$p(< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >) \leq 2^h + 2^{h-2} + \sum_{i=0}^{h-3} 2^i(2^{h-i-1} - 1) - 1.$$

Then there exists an i such that

$$p(< M(B_{h-2})^{(i)} \cup \{v_1, v_{1i}, v_{01}\} >) \geq 2^{h-1} + 2^{h-3} + \sum_{i=0}^{h-4} 2^i(2^{h-i-2} - 1),$$

where $i = 1, 2$. Thus by Lemma 2.4 we can move a pebble to the target.

Case 4: Let v be the target vertex other than v_0, v_{01}, v_{02} and the pendant vertices.

Without loss of generality, assume that $v \in M(B_{h-1})^{(1)}$. Suppose $p(M(B_{h-1})^{(1)}) \geq f(M(B_{h-1}))$, then we are done. Therefore assume that $p(M(B_{h-1})^{(1)}) \leq f(M(B_{h-1})) - 1$. Then there are at least $f(M(B_h)) - (f(M(B_{h-1})) - 1)$ pebbles on $< M(B_{h-1})^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$. Since

$$f(M(B_h)) - (f(M(B_{h-1}) - 1) \geq (2^h - 2^{h-2})2^{h+1} + 2^{h-1} + \sum_{i=0}^{h-2} 2^i(2^{h-i} - 1),$$

we can move at least $2^h - 2^{h-2}$ pebbles to v_{01}. Since $2^h - 2^{h-2} \geq 2^{h-1}$ and $d(v_{01}, v) \leq h - 1$, we are done. □

Theorem 2.9. *For the middle graph of a complete binary tree of height h, $f_t(M(B_h)) = 2^h(t2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1)$.*

Proof. The proof is by induction on t. For $t = 1$, the theorem follows from Theorem 2.8. Assume the theorem is true for $2 \leq t' < t$. Let D be any distribution of $2^h(t2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1)$ pebbles on the vertices of the graph $M(B_h)$.

Let v be any target vertex. Suppose $p(v) = 0$. We can move a pebble to v at a cost of at most 2^{2h+1} pebbles, since $M(B_h)$ contains at least 2^{2h+2} pebbles and diameter of $M(B_h)$ is $2h + 1$. Now the minimum number of pebbles distributed on the vertices of $M(B_h)$ is

$$\begin{aligned} &2^h(t2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1) - 2^{2h+1} \\ &\qquad = 2^h((t-1)2^{h+1} + 1) - 1 + \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1). \end{aligned}$$

Thus we can move $t - 1$ additional pebbles to v by induction. Suppose $p(v) = x$, where $1 \leq x \leq t - 1$. Then the minimum number of pebbles distributed on the vertices of $< M(B_h) - \{v\} >$ is

$$2^h(t2^{h+1}+1)-1+\textstyle\sum_{i=0}^{h-2}2^{i+1}(2^{h-i}-1)-x \geq 2^h((t-x)2^{h+1}+1)-1+\textstyle\sum_{i=0}^{h-2}2^{i+1}(2^{h-i}-1).$$

Thus we can move $t-x$ additional pebbles to v by induction. Thus $f_t(M(B_h)) = 2^h(t2^{h+1}+1)-1+\sum_{i=0}^{h-2}2^{i+1}(2^{h-i}-1)$. □

3. The 2t-pebbling property

In this section, we prove that the middle graph of a complete binary tree $M(B_h)$ satisfies the 2t-pebbling property.

Remark 3.1. Consider the graph G with n vertices and $2f(G)-q+1$ pebbles on it and we choose a target vertex v from G. If $p(v)=1$, then the number of pebbles remaining in G is $2f(G)-q \geq f(G)$, since $f(G) \geq n$ and $q \leq n$, and hence we can move the second pebble to v. Let us assume that $p(v)=0$. If $p(u) \geq 2$ where $uv \in E(G)$, we move a pebble to v from u. Then the graph G has at least $2f(G)-q+1-2$ pebbles, since $f(G) \geq n$ and $q \leq n-1$, and hence we can move the second pebble to v. So, we always assume that $p(v)=0$ and $p(u) \leq 1$ for all $uv \in E(G)$, when v is the target vertex.

Theorem 3.1. *The graph $M(B_2)$ satisfies the 2-pebbling property.*

Proof. The graph $M(B_2)$ has at least $2f(M(B_2))-q+1 \geq 83-q$ pebbles on it.

Case 1 : Let $q \leq 10$.

Let v be any target vertex. Clearly $p(v)=0$ and $p(u) \leq 1$ for all $uv \in E(M(B_2))$ by Remark 3.1. We can move one pebble to v at a cost of at most 2^5 pebbles, since $83-q \geq 41$ and $d(v,v_i) \leq 5$, for all $v_i \in M(B_2)$. Then the minimum number of pebbles distributed on the vertices of $M(B_2)$ is $83-10-2^5=41$. Hence we can move an additional pebble to v by Theorem 2.9.

Case 2 : Let $q=11$.

Subcase 2.1 : Let v_3 be the target vertex.

Clearly $p(v_3)=0$ and $p(v_{11}) \leq 1$ by Remark 3.1. Let $p(v_{11})=1$. Suppose $p(v_{12}) \geq 2$ or $p(v_{01}) \geq 2$ or $p(v_1) \geq 2$ then we can move a pebble to v_3. Then the graph $M(B_2)$ has at least $83-q-3=69 \geq 41$ pebbles and hence by Theorem 2.9, we are done. Therefore assume $p(v_4) \leq 4$. Clearly, at least 65 pebbles are distributed on $< M(B_1)^{(2)} \cup \{v_0, v_{02}\} >$ and so by Lemma 2.2, we can move two pebbles to v_3.

Let $p(v_{11})=0$. If $p(M(B_1)^{(1)}) \geq 9$, then one pebble can be moved to the target. Then the graph $M(B_2)$ has at least $83-q-9=63 \geq 41$ pebbles and hence we are done by Theorem 2.9. Suppose $4 \leq p(M(B_1)^{(1)}) \leq 8$. Since

$p(v_{11}) = 0$ and $q = 11$, we can move a pebble to v_{11}. Also the minimum number of pebbles distributed on the vertices of $< M(B_1)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ is $83 - q - 8 = 64 \geq 8(7) + 5$. Then by Lemma 2.2, we can move seven pebbles to v_{01} and hence we are done. Assume $p(M(B_1)^{(1)}) \leq 3$. Then the graph $< M(B_1)^{(2)} \cup \{v_0, v_{01}, v_{02}\} >$ contains at least $83-q-3 = 69 \geq 8(8)+5$ pebbles. Thus we can move eight pebbles to v_{01} by Lemma 2.2 and hence we are done.

Subcase 2.2 : Let v_0 be the target vertex.

If $p(< M(B_1)^{(1)} \cup \{v_0, v_{01}\} >) \geq 12$, then we can move a pebble to v_0, since $< M(B_1)^{(1)} \cup \{v_0, v_{01}\} >$ is isomorphic to $M(K_{1,3})$ and $f(M(K_{1,3})) = 12$. Then the minimum number of pebbles distributed on the vertices of $M(B_2)$ is $83 - q - 12 \geq 41$. Hence we can move another pebble to v_0. Assume $p(< M(B_1)^{(1)} \cup \{v_0, v_{01}\} >) \leq 11$. Then $< M(B_1)^{(2)} \cup \{v_0, v_{02}\} >$ has at least $83 - q - 11 = 61 \geq 8(7) + 5$ pebbles and hence by Lemma 2.2, we are done.

Subcase 2.3 : Let v be any target vertex other than the root vertex and the pendant vertices.

Without loss of generality, assume that $v \in M(B_1)^{(1)}$. Suppose there exists a pendant vertex in $M(B_1)^{(1)}$ with at least four pebbles, then we can move one pebble to v. Then the graph $M(B_2)$ has at least 41 pebbles and we can move an additional pebble to v by Theorem 2.9. Suppose $p(v_3) + p(v_4) \geq 5$, then one pebble can be moved to v, since $q = 11$. Also the graph $M(B_2)$ has at least 41 pebbles and hence we are done. So, assume $p(v_3) + p(v_4) \leq 4$. Now the minimum number of pebbles distributed on the vertices of $< M(B_1)^{(2)} \cup v_{\{}02\} >$ is $83 - q - 4 - 5 = 63 \geq 8(7) + 5$, since $q = 11$. Thus by Lemma 2.2, we can move seven pebbles to v_{01} and hence we are done.

Case 3 : Let $q = 12$.

Let v be any target vertex. Clearly $p(u) \geq 2$ for some $u \in M(B_2)$. We can easily move a pebble to v at a cost of at most six pebbles, since the diameter of $M(B_2)$ is five. Then the minimum number of pebbles distributed on the vertices of $M(B_2)$ is $83 - q - 6 \geq 41$. Hence we can move an additional pebble to v by Theorem 2.9. □

Theorem 3.2. *The graph $M(B_h)$, $h \geq 3$ satisfies the 2-pebbling property.*

Proof. Let D be any distribution with at least $2(f(M(B_h))) - q + 1$ pebbles on $M(B_h)$. Since $q \leq 2^{h+2} - 3$, it is easy to see that,

$$2(f(M(B_h))) - q + 1 = 2^{2h+2} + 2^{h+1} - 2 + 2\sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1) - 2^{h+2} + 4$$

$$= f_2(M(B_h)) + k,$$

where $k = \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1) + 3 - 3(2h)$. Since $k \geq 0$, we conclude that $2(f(M(B_h))) - q + 1$ exceeds $f_2(M(B_h))$. And hence we can move two pebbles to any target vertex by Theorem 2.9. □

Theorem 3.3. *The graph $M(B_h)$, $h \leq 2$ satisfies the $2t$-pebbling property.*

Proof. The proof is by induction on t. For $t = 1$, it follows from Theorem 3.2. Assume the theorem is true for $2 \leq t' < t$. Let D be any distribution with at least $2(f_t(M(B_h))) - q + 1$ pebbles on $M(B_h)$. Let v be any target vertex and suppose $p(v) = 0$. We can easily move two pebbles to v at a cost of at most 2^{2h+2} pebbles, since $2(f_t(M(B_h))) - q + 1 \geq f_2(M(B_h))$ and the diameter of $M(B_h)$ is $2h + 1$. Then the minimum number of pebbles distributed on the vertices of $M(B_h)$ is $2(f_t(M(B_h))) - q + 1 - 2^{2h+2}$. Since

$$2(f_t(M(B_h))) - q + 1 - 2^{2h+2} = 2(f_{t-1}(M(B_h))) - q + 1,$$

we can move $2(t - 1)$ additional pebbles to v by induction. Suppose $p(v) = x$, where $1 \leq x \leq 2t - 1$. The remaining number of pebbles on the vertices of $< M(B_h) - \{v\} >$ is $2(f_t(M(B_h))) - q + 1 - x$. Since $q \leq 2^{h+2} - 3$, it follows that,

$$2(f_t(M(B_h))) - q + 1 - x \geq f_{(2t-x)}(M(B_h)).$$

Hence we can move $2t - x$ additional pebbles to v by Theorem 2.9. □

Theorem 3.4. *The graph $M(B_h)$, $h \geq 3$ satisfies the $2t$-pebbling property.*

Proof. Let D be any distribution with at least $2(f_t(M(B_h))) - q + 1$ pebbles on $M(B_h)$. Since $q \leq 2^{h+2} - 3$, it is easy to see that,

$$2(f_t(M(B_h))) - q + 1 = t2^{2h+2} + 2^{h+1} - 2 + 2\sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1) - 2^{h+2} + 4$$

$$= f_{2t}(M(B_h)) + k,$$

where $k = \sum_{i=0}^{h-2} 2^{i+1}(2^{h-i} - 1) + 3 - 3(2h)$. Since $k \geq 0$, we conclude that $2(f_t(M(B_h))) - q + 1$ exceeds $f_{2t}(M(B_h))$. And hence we can move $2t$ pebbles to any target vertex by Theorem 2.9. □

4. The t-pebbling Conjecture

Lourdusamy et al. [4], [5], [6], [7] proved that if G is a fan graph, a wheel graph, a complete graph, a complete multipartite graph, a path and H has the $2t-$pebbling property then Conjencture 1.2 holds.

Since $M(B_h)$ has the $2t-$ pebbling property we conclude that conjecture 1.2 holds if G is a fan graph, a wheel graph, a complete graph, a complete multipartite graph, a path and H is the middle graph of a complete binary tree.

Conjecture 4.1. *$f_t(G \times H) \leq f(G).f_t(H)$, where G and H are middle graphs of a complete binary tree.*

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