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COMMON FIXED POINTS FOR WEAKENED COMPATIBLE MAPPINGS SATISFYING THE GENERALIZED ϕ-WEAK CONTRACTION CONDITION

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Abstract. In this paper, we prove some common fixed point theorems for pairs of weakened compatible mappings (subcompatible and occasionally weakly compatible mappings) satisfying a generalized ϕ-weak contraction condition involving various combinations of the metric functions. In fact, our results improve the results of Jain et al. [6]. Also we provide an example for validity of our results.

1. Introduction and Preliminaries

The Banach Contraction Principle is a milestone in metric fixed point theory. It is most widely applied fixed point results in many branches of mathematics. The generalizations of Banach Contraction Principle gives new direction to researchers in the field of fixed point theory.

In 1969, Boyd and Wong [4] replaced the constant in Banach contraction principle by a control function and they obtain the fixed point.

In 1997, Alber and Gueree-Delabriere [2] introduced the concept of weak contraction and in 2001, Rhoades [10] has shown that the results of Alber and Gueree-Delabriere [2] are also valid in complete metric spaces.

In 1996, Jungck [7] introduced the notion of weakly compatible mappings.

Definition 1.1. Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called *weakly compatible* if the mappings commute at their coincidence point.

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In 2008, Al-Thagafi and Shahzad [11] weakened the concept of weakly compatible mappings by giving the new concept of occasionally weakly compatible mappings.

Definition 1.2. Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called *occasionally weakly compatible* if there exists a point $x \in X$, which is a coincidence point of f and g at which f and g commute, that is, $fgx = gfx$ for some $x \in C(f, g)$, where $C(f, g)$ is the set of coincidence points of f and g.

In 2011, Bouhadjera and Godet-Thobie [3] generalized the notion of occasionally weakly compatible mappings to subcompatible mappings in metric spaces as follows:

Definition 1.3. Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called *subcompatible* if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ and which satisfy $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.

Remark 1.4. Occasionally weakly compatible maps are subcompatible but converse need not be true.

In 1998, Pant [9] defined the concept of continuity by defining the concept of reciprocally continuous mappings as follows:

Definition 1.5. Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are called *reciprocally continuous* if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

If f and g are both continuous, then maps are obvious reciprocally continuous but the converse is not true.

Definition 1.6 ([1])**.** Let f and g be two mappings of a metric space (X, d) into itself. Then f and g are said to satisfy *property* $(E.A)$ if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 2013, Murthy and Prasad [8] introduced a new type of inequality having cubic terms that generalized the results of Alber and Gueree-Delabriere [2].

In 2017, Jain et al. [5] generalized the result of Murthy and Prasad [8] for the pair of mappings. Also, in 2018, Jain et al. [6] generalized the result of Jain et al. [5] for two pairs of compatible mappings.

In this paper, we improve the result of Jain et al. [6] for two pairs of subcompatible and occasionally weakly compatible mappings satisfying the generalized ϕ-weak contractive condition involving various combination of the metric function.

2. Main Results

Now we give the following theorem for pairs of subcompatible mappings and reciprocally continuous maps as follows:

Theorem 2.1. *Let A, B, S and T be four mappings of a complete metric space (X, d) into itself satisfying the following:*

$$S(X) \subset B(X), \quad T(X) \subset A(X); \tag{C1}$$

$$\begin{aligned}&[1 + pd(Ax, By)]d^2(Sx, Ty)\\ &\leq p \max\{1/2[d^2(Ax, Sx)d(By, Ty) + d(Ax, Sx)d^2(By, Ty)],\\ &\qquad d(Ax, Sx)d(Ax, Ty)d(By, Sx), d(Ax, Ty)d(By, Sx)d(By, Ty)\}\\ &\quad + m(Ax, By) - \phi(m(Ax, By)),\end{aligned} \tag{C2}$$

where

$$\begin{aligned}m(Ax, By) = \max\{&d^2(Ax, By), d(Ax, Sx)d(By, Ty), d(Ax, Ty)d(By, Sx),\\ &1/2[d(Ax, Sx)d(Ax, Ty) + d(By, Sx)d(By, Ty)]\},\end{aligned}$$

$p\,(\geq 0)$ is a real number and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ iff $t = 0$ and $\phi(t) > 0$ for each $t > 0$.

Assume that the pairs A, S and B, T are subcompatible and reciprocally continuous. Then A, B, S and T have unique common fixed point.

Proof. Since the pairs A, S and B, T are subcompatible and reciprocally continuous, there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ and which satisfy $\lim_{n\to\infty} d(ASx_n, SAx_n) = d(At, St) = 0$ and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z$ for some $z \in X$.

Also $\lim_{n\to\infty} d(BTy_n, TBy_n) = d(Bz, Tz) = 0$. Therefore, $At = St$ and $Bz = Tz$, that is, t is a coincidence point of A and S, and z is a coincidence point of B and T.

Now, we claim that $t = z$. On putting $x = x_n$ and $y = y_n$ in $(C2)$, we get

$$\begin{aligned}&[1 + pd(Ax_n, By_n)]d^2(Sx_n, Ty_n)\\ &\leq p \max\{1/2[d^2(Ax_n, Sx_n)d(By_n, Ty_n) + d(Ax_n, Sx_n)d^2(By_n, Ty_n)],\end{aligned}$$

$$d(Ax_n, Sx_n)d(Ax_n, Ty_n)d(By_n, Sx_n),$$
$$d(Ax_n, Ty_n)d(By_n, Sx_n)d(By_n, Ty_n)\}$$
$$+ m(Ax_n, By_n) - \phi(m(Ax_n, By_n)),$$

where

$$\begin{aligned} m(Ax_n, By_n) = \max\{&d^2(Ax_n, By_n), d(Ax_n, Sx_n)d(By_n, Ty_n), \\ &d(Ax_n, Ty_n)d(By_n, Sx_n), 1/2[d(Ax_n, Sx_n)d(Ax_n, Ty_n) \\ &+ d(By_n, Sx_n)d(By_n, Ty_n)]\}, \end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned} &[1 + pd(t, z)]d^2(t, z) \\ &\le p \max\{1/2[d^2(t, t)d(z, z) + d(t, t)d^2(z, z)], \\ &\quad d(t, t)d(t, z)d(z, t), d(t, z)d(z, t)d(z, z)\} \\ &\quad + m(t, z) - \phi(m(t, z)), \end{aligned}$$

where

$$\begin{aligned} m(t, z) &= \max\{d^2(t, z), d(t, t)d(z, z), d(t, z)d(z, t), \\ &\qquad 1/2[d(t, t)d(t, z) + d(z, t)d(z, z)]\} \\ &= d^2(t, z), \end{aligned}$$

which implies that

$$[1 + pd(t, z)]d^2(t, z) \le p \max\{1/2[0 + 0], 00\} + d^2(t, z) - \phi(d^2(t, z)).$$

Thus we get $d^2(t, z) = 0$ and hence $t = z$.

Next, we claim that $At = t$. Putting $x = t$ and $y = y_n$ in $(C2)$, we get

$$\begin{aligned} &[1 + pd(At, By_n)]d^2(St, Ty_n) \\ &\le p \max\{1/2[d^2(At, St)d(By_n, Ty_n) + d(At, St)d^2(By_n, Ty_n)], \\ &\quad d(At, St)d(At, Ty_n)d(By_n, St), d(At, Ty_n)d(By_n, St)d(By_n, Ty_n)\} \\ &\quad + m(At, By_n) - \phi(m(At, By_n)), \end{aligned}$$

where

$$\begin{aligned} &m(At, By_n) \\ &= \max\{d^2(At, By_n), d(At, St)d(By_n, Ty_n), d(At, Ty_n)d(By_n, St), \\ &\quad 1/2[d(At, St)d(At, Ty_n) + d(By_n, St)d(By_n, Ty_n)]\}, \end{aligned}$$

letting $n \to \infty$, we get

$$\begin{aligned}
&[1+pd(At,t)]d^2(At,t)\\
&\le p\max\{1/2[d^2(At,St)d(t,t)+d(At,St)d^2(t,t)],\\
&\qquad d(At,St)d(At,t)d(t,At), d(At,t)d(t,At)d(t,t)\}\\
&\quad + m(At,t)-\phi(m(At,t)),
\end{aligned}$$

where

$$\begin{aligned}
m(At,t) &= \max\{d^2(At,t), d(At,St)d(t,t), d(At,t)d(t,At),\\
&\qquad 1/2[d(At,St)d(At,t)+d(t,At)d(t,t)]\}\\
&= d^2(At,t),
\end{aligned}$$

which implies that

$$[1+pd(At,t)]d^2(At,t) \le p\max\{1/2[0+0],0,0\}+d^2(At,t)-\phi(d^2(At,t)).$$

This gives $d^2(At,t)=0$ and hence $At=t$. Therefore, we have $At=St=t$.

Next, we claim that $Bt=t$. On putting $x=t$ and $y=t$ in $(C2)$, we get

$$\begin{aligned}
&[1+pd(At,Bt)]d^2(St,Tt)\\
&\le p\max\{1/2[d^2(At,St)d(Bt,Tt)+d(At,St)d^2(Bt,Tt)],\\
&\qquad d(At,St)d(At,Tt)d(Bt,St), d(At,Tt)d(Bt,St)d(Bt,Tt)\}\\
&\quad + m(At,Bt)-\phi(m(At,Bt)),
\end{aligned}$$

where

$$\begin{aligned}
m(At,Bt) &= \max\{d^2(At,Bt), d(At,St)d(Bt,Tt), d(At,Tt)d(Bt,St),\\
&\qquad 1/2[d(At,St)d(At,Tt)+d(Bt,St)d(Bt,Tt)]\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&[1+pd(t,Bt)]d^2(t,Bt)\\
&\le p\max\{1/2[d^2(t,St)d(Bt,Tt)+d(t,Bt)d^2(Bt,Tt)],\\
&\qquad d(t,t)d(t,Bt)d(Bt,t), d(t,Bt)d(Bt,t)d(Bt,Tt)\}\\
&\quad + m(t,Bt)-\phi(m(t,Bt)),
\end{aligned}$$

where

$$\begin{aligned}
m(t,Bt) &= \max\{d^2(t,Bt), d(At,St)d(Bt,Tt), d(t,Bt)d(Bt,t),\\
&\qquad 1/2[d(At,St)d(t,Bt)+d(Bt,t)d(Bt,Tt)]\}\\
&= d^2(t,Bt).
\end{aligned}$$

Thus

$$[1+pd(t,Bt)]d^2(t,Bt) \le p\max\{1/2[0+0],0,0\}+d^2(t,Bt)-\phi(d^2(t,Bt)).$$

Thus we get $Bt = t$ and hence $Bt = Tt = t$. Therefore $t = At = St = Bt = Tt$, so t is a common fixed of A, B, S and T.

Uniqueness easily follows. This completes the proof. □

Next we prove the following theorem using property $(E.A)$ as follows:

Theorem 2.2. *Let A, B, S and T be four mappings of a complete metric space (X, d) into itself satisfying the condition $(C2)$ and the following condition:*
$(C3)$ *Assume that either*

(a) *$SX \subset BX$ and the pair A, S satisfy property $(E.A)$ and $A(X)$ is a closed subset of X; or*

(b) *$TX \subset AX$ and the pair B, T satisfy property $(E.A)$ and $B(X)$ is a closed subset of X.*

Then $C(T, B) \neq \emptyset$ and $C(A, S) \neq \emptyset$.

Proof. Suppose that (a) holds. Since the pair A, S satisfy property $(E.A)$ so there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ for some $z \in X$. Since $SX \subset BX$ so there exists a sequence $\{y_n\}$ in X such that $Sx_n = By_n$. Hence $\lim_{n\to\infty} By_n = z$.

First we claim that $\lim_{n\to\infty} Ty_n = z$. Putting $x = x_n$ and $y = y_n$ in $(C2)$, we get

$$\begin{aligned}
&[1 + pd(Ax_n, By_n)]d^2(Sx_n, Ty_n) \\
&\leq p \max\{1/2[d^2(Ax_n, Sx_n)d(By_n, Ty_n) + d(Ax_n, Sx_n)d^2(By_n, Ty_n)], \\
&\qquad d(Ax_n, Sx_n)d(Ax_n, Ty_n)d(By_n, Sx_n), \\
&\qquad d(Ax_n, Ty_n)d(By_n, Sx_n)d(By_n, Ty_n)\} \\
&\quad + m(Ax_n, By_n) - \phi(m(Ax_n, By_n)),
\end{aligned}$$

where

$$\begin{aligned}
m(Ax_n, By_n) = \max\{&d^2(Ax_n, By_n), d(Ax_n, Sx_n)d(By_n, Ty_n), \\
&d(Ax_n, Ty_n)d(By_n, Sx_n), 1/2[d(Ax_n, Sx_n)d(Ax_n, Ty_n) \\
&+ d(By_n, Sx_n)d(By_n, Ty_n)]\},
\end{aligned}$$

letting $n \to \infty$, we obtain

$$\begin{aligned}
&[1 + pd(z, z)]d^2(z, Ty_n) \\
&\leq p \max\{1/2[d^2(z, z)d(z, Ty_n) + d(z, z)d^2(z, Ty_n)], \\
&\qquad d(z, z)d(z, Ty_n)d(z, z), d(z, Ty_n)d(z, z)d(z, Ty_n)\} \\
&\quad + m(z, z) - \phi(m(z, z)),
\end{aligned}$$

where

$$\begin{aligned} m(z,z) &= \max\{d^2(z,z), d(z,z)d(z,Ty_n), d(z,Ty_n)d(z,z), \\ &\qquad 1/2[d(z,z)d(z,Ty_n) + d(z,z)d(z,Ty_n)]\} \\ &= 0, \end{aligned}$$

this gives that $d^2(z,Ty_n) \leq 0$ and hence $\lim_{n\to\infty} Ty_n = z$. Since $A(X)$ is a closed subset of X we have $z = Av$ for some $v \in X$.

Next, we claim that $Sv = z$. Putting $x = v$ and $y = y_n$ in $(C2)$, we get

$$\begin{aligned} &[1 + pd(Av, By_n)]d^2(Sv, Ty_n) \\ &\leq p\max\{1/2[d^2(Av,Sv)d(By_n,Ty_n) + d(Av,Sv)d^2(By_n,Ty_n)], \\ &\qquad d(Av,Sv)d(Av,Ty_n)d(By_n,Sv), d(Av,Ty_n)d(By_n,Sv)d(By_n,Ty_n)\} \\ &\quad + m(Av,By_n) - \phi(m(Av,By_n)), \end{aligned}$$

where

$$\begin{aligned} m(Av,By_n) = \max\{&d^2(Av,By_n), d(Av,Sv)d(By_n,Ty_n), \\ &d(Av,Ty_n)d(By_n,Sv), 1/2[d(Av,Sv)d(Av,Ty_n) \\ &+ d(By_n,Sv)d(By_n,Ty_n)]\}, \end{aligned}$$

letting $n \to \infty$, we have

$$\begin{aligned} &[1 + pd(z,z)]d^2(Sv,z) \\ &\leq p\max\{1/2[d^2(z,Sv)d(z,z) + d(z,Sv)d^2(z,z)], \\ &\qquad d(z,Sv)d(z,z)d(z,Sv), d(z,z)d(z,Sv)d(z,z)\} \\ &\quad + m(z,z) - \phi(m(z,z), \end{aligned}$$

where

$$\begin{aligned} m(z,z) &= \max\{d^2(z,z), d(z,Sv)d(z,z), d(z,z)d(z,Sv), \\ &\qquad 1/2[d(z,Sv)d(z,z) + d(z,Sv)d(z,z)]\} \\ &= 0. \end{aligned}$$

Thus

$$d^2(Sv,z) \leq p\max\{1/2[0+0], 0, 0\} + 0 - \phi(0).$$

Hence $Sv = z$ and we have $Av = Sv = z$. Therefore $C(A,S) \neq \emptyset$.

Since $SX \subset BX$ and $z \in S(X)$ so, there exists a point $u \in X$ such that $z = Bu$.

We claim that $z = Tu$. Putting $x = v$ and $y = u$ in $(C2)$, we get

$$\begin{aligned}
&[1 + pd(Av, Bu)]d^2(Sv, Tu) \\
&\le p \max\{1/2[d^2(Av, Sv)d(Bu, Tu) + d(Av, Sv)d^2(Bu, Tu)], \\
&\qquad d(Av, Sv)d(Av, Tu)d(Bu, Sv), d(Av, Tu)d(Bu, Sv)d(Bu, Tu)\} \\
&\quad + m(Av, Bu) - \phi(m(Av, Bu)),
\end{aligned}$$

where

$$\begin{aligned}
m(Av, Bu) = \max\{&d^2(Av, Bu), d(Av, Sv)d(Bu, Tu), d(Av, Tu)d(Bu, Sv), \\
&1/2[d(Av, Sv)d(Av, Tu) + d(Bu, Sv)d(Bu, Tu)]\},
\end{aligned}$$

which implies that

$$\begin{aligned}
&[1 + pd(z, z)]d^2(z, Tu) \\
&\le p \max\{1/2[d^2(z, z)d(z, Tu) + d(z, z)d^2(z, Tu)], \\
&\qquad d(z, z)d(z, Tu)d(z, z), d(z, Tu)d(z, z)d(z, Tu)\} \\
&\quad + m(z, z) - \phi(m(z, z)),
\end{aligned}$$

where

$$\begin{aligned}
m(z, z) &= \max\{d^2(z, z), d(z, z)d(z, Tu), d(z, Tu)d(z, z), \\
&\qquad 1/2[d(z, z)d(z, Tu) + d(z, z)d(z, Tu)]\} \\
&= 0.
\end{aligned}$$

Thus

$$d^2(z, Tu) \le p \max\{1/2[0 + 0], 0, 0\} + 0 - \phi(0).$$

Hence $Tu = z$ and we have $Bu = Tu = z$. Therefore $C(B, T) \neq \emptyset$.

Similarly, we can prove for (b). □

Finally, we prove the following theorem for two pairs of occasionally weakly compatible mappings as follows:

Theorem 2.3. *Let A, B, S and T be four mappings of a complete metric space (X, d) into itself satisfying the conditions $(C2)$ and $(C3)$.*

Assume that the pairs A, S and B, T are occasionally weakly compatible. Then A, B, S and T have a unique common fixed point.

Proof. From Theorem 2.2, $C(A, S) \neq \emptyset$ and $C(B, T) \neq \emptyset$. Since the pair A, S is occasionally weakly compatible so there exists $u \in C(A, S)$ such that $Au = Su = z$ (say) and $ASu = SAu = z'$ (say). Hence we have $Az = Sz = z'$ (say). Since B, T

is occasionally weakly compatible, there exists $v \in C(B,T)$ such that $Bv = Tv = w$ (say) and $BTv = TBv$. Hence we have $Bw = Tw = w'$ (say).

Now, we claim that $z' = w'$. Putting $x = z$ and $y = w$ in $(C2)$, we get

$$\begin{aligned}&[1 + pd(Az, Bw)]d^2(Sz, Tw)\\&\le p\max\{1/2[d^2(Az, Sz)d(Bw, Tw) + d(Az, Sz)d^2(Bw, Tw)],\\&\qquad d(Az, Sz)d(Az, Tw)d(Bw, Sz), d(Az, Tw)d(Bw, Sz)d(Bw, Tw)\}\\&\quad + m(Az, Bw) - \phi(m(Az, Bw)),\end{aligned}$$

where

$$\begin{aligned}&m(Az, Bw)\\&= \max\{d^2(Az, Bw), d(Az, Sz)d(Bw, Tw), d(Az, Tw)d(Bw, Sz),\\&\qquad 1/2[d(Az, Sz)d(Az, Tw) + d(Bw, Sz)d(Bw, Tw)]\},\end{aligned}$$

which implies that

$$\begin{aligned}&[1 + pd(z', w')]d^2(z', w')\\&\le p\max\{1/2[d^2(z', z')d(w', w') + d(z', z')d^2(w', w')],\\&\qquad d(z', z')d(z', w')d(w', z'), d(z', w')d(w', z')d(w', w')\}\\&\quad + m(z', w') - \phi(m(z', w')),\end{aligned}$$

where

$$\begin{aligned}m(z', w') &= \max\{d^2(z', w'), d(z', z')d(w', w'), d(z', w')d(w', z'),\\&\qquad 1/2[d(z', z')d(z', w') + d(w', z')d(w', w')]\}\\&= d^2(z', w'),\end{aligned}$$

which implies that

$$[1 + pd(z', w')]d^2(z', w') \le p\max\{1/2[0 + 0], 0, 0\} + d^2(z', w') - \phi(d^2(z', w')),$$

we get $z' = w'$ and hence $Az' = Sz' = w'$.

Next we claim that $z = w'$. Putting $x = u$ and $y = w$ in $(C2)$, we get

$$\begin{aligned}&[1 + pd(Au, Bw)]d^2(Su, Tw)\\&\le p\max\{1/2[d^2(Au, Su)d(Bw, Tw) + d(Au, Su)d^2(Bw, Tw)],\\&\qquad d(Au, Su)d(Au, Tw)d(Bw, Su), d(Au, Tw)d(Bw, Su)d(Bw, Tw)\}\\&\quad + m(Au, Bw) - \phi(m(Au, Bw)),\end{aligned}$$

where

$$\begin{aligned}&m(Au, Bw)\\&= \max\{d^2(Au, Bw), d(Au, Su)d(Bw, Tw), d(Au, Tw)d(Bw, Su),\\&\quad 1/2[d(Au, Su)d(Au, Tw) + d(Bw, Su)d(Bw, Tw)]\},\end{aligned}$$

which implies that

$$\begin{aligned}&1 + pd(z, w')]d^2(z, w')\\&\leq p\max\{1/2[d^2(z, z)d(w', w') + d(z, z)d^2(w', w')],\\&\quad d(z, z)d(z, w')d(w', z), d(z, w')d(w', z)d(w', w')\}\\&\quad + m(z, w') - \phi(m(z, w')),\end{aligned}$$

where

$$\begin{aligned}m(z, w') &= \max\{d^2(z, w'), d(z, z)d(w', w'), d(z, w')d(w', z),\\&\qquad 1/2[d(z, z)d(z, w') + d(w', z)d(w', w')]\}\\&= d^2(z, w').\end{aligned}$$

Thus

$$[1 + pd(z, w')]d^2(z, w') \leq p\max\{1/2[0 + 0], 0, 0\} + d^2(z, w') - \phi(d^2(z, w')),$$

which gives that $z = w'$ and hence $Az = Sz = z$ and $Bw = Tw = z$.

Next we show that $z = w$. Putting $x = z$ and $y = v$ in $(C2)$, we get

$$\begin{aligned}&[1 + pd(Az, Bv)]d^2(Sz, Tv)\\&\leq p\max\{1/2[d^2(Az, Sz)d(Bv, Tv) + d(Az, Sz)d^2(Bv, Tv)],\\&\quad d(Az, Sz)d(Az, Tv)d(Bv, Sz), d(Az, Tv)d(Bv, Sz)d(Bv, Tv)\}\\&\quad + m(Az, Bv) - \phi(m(Az, Bv)),\end{aligned}$$

where

$$\begin{aligned}m(Az, Bv) &= \max\{d^2(Az, Bv), d(Az, Sz)d(Bv, Tv), d(Az, Tv)d(Bv, Sz),\\&\qquad 1/2[d(Az, Sz)d(Az, Tv) + d(Bv, Sz)d(Bv, Tv)]\},\end{aligned}$$

which implies that

$$\begin{aligned}&[1 + pd(z, w)]d^2(z, w)\\&\leq p\max\{1/2[d^2(z, z)d(w, w) + d(z, z)d^2(w, w)],\\&\quad d(z, z)d(z, w)d(w, z), d(z, w)d(w, z)d(w, w)\}\\&\quad + m(z, w) - \phi(m(z, w)),\end{aligned}$$

where

$$\begin{aligned} m(z,w) &= \max\{d^2(z,w), d(z,z)d(w,w), d(z,w)d(w,z), \\ &\qquad 1/2[d(z,z)d(z,w) + d(w,z)d(w,w)]\} \\ &= d^2(z,w). \end{aligned}$$

This implies that $d^2(z,w) = 0$, that is, $z = w$. Hence we have $Az = Bz = Sz = Tz = z$. Therefore z is a common fixed point of A, B, S and T.

Uniqueness follows easily. This completes the proof. □

Example 2.4. Let $X = [2, 20]$ with the metric d defined by $d(x,y) = |x-y|$. Define the mappings A, B, S and $T : X \to X$ by

$$Ax = \begin{cases} 2 & \text{if } x = 2, \\ 12 & \text{if } 2 < x \le 5, \\ x-3 & \text{if } x > 5, \end{cases} \quad Bx = \begin{cases} 2 & \text{if } x = 2, \\ 6 & \text{if } x > 2, \end{cases}$$

$$Sx = \begin{cases} x & \text{if } x = 2, \\ 6 & \text{if } 2 < x \le 5, \\ 2 & \text{if } x > 5, \end{cases} \quad Tx = \begin{cases} x & \text{if } x = 2, \\ 3 & \text{if } x > 2, \end{cases}$$

Taking $\{x_n\} = \{2\}$, it is clear that the pairs A, S and B, T are subcompatible, reciprocally continuous and occasionally weakly compatible mappings. Therefore, all the condition of Theorem 2.1 and Theorem 2.3 are satisfied, then we can obtain $S2 = T2 = A2 = B2 = 2$, so 2 is a common fixed point of A, B, S and T. In fact, 2 is the unique common fixed point of A, B, S and T.

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