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# A NOTE ON CONCIRCULAR STRUCTURE SPACE-TIMES

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Abstract. In this note we show that Lorentzian Concircular Structure manifolds $(LCS)_n$ coincide with Generalized Robertson-Walker space-times.

*Generalized Robertson-Walker* (GRW) space-times were introduced in 1995 by Alías, Romero and Sánchez [1] as the warped product $-1 \times_{q^2} M^*$, where $(M^*, g^*)$ is a Riemannian submanifold. In other terms, they are Lorentzian manifolds characterised by a metric

$$g_{ij}dx^i dx^j = -(dt)^2 + q(t)^2 g^*_{\mu\nu}(x^1, \ldots, x^{n-1})dx^\mu dx^\nu. \tag{1}$$

They are interesting not only for geometry [2, 5, 9–11], but also for physics: they include relevant space-times such as Robertson-Walker, Einstein-de Sitter, static Einstein, de-Sitter, the Friedmann cosmological models. They are a wide generalization of space-times for cosmological models.

In 2003 A. A. Shaikh [12] introduced the notion of *Lorentzian Concircular Structure* $(LCS)_n$. It is a Lorentzian manifold endowed with a unit time-like concircular vector field, i.e., $u^i u_i = -1$ and

$$\nabla_k u_j = \varphi(u_k u_j + g_{kj}), \tag{2}$$

where $\varphi \neq 0$ is a scalar function obeying

$$\nabla_j \varphi = \mu u_j \tag{3}$$

being $\mu$ a scalar function. Various authors studied the properties of $(LCS)_n$ manifolds [6, 13–15].

We show that GRW and $(LCS)_n$ are the same space-times.

We recall few definitions that will be used in this note. The first ones are the definitions of "torse-forming" and "concircular" vector fields, by Yano:

**Definition 1** (Yano, [16, 17])**.** A vector field $X_j$ is named torse-forming if $\nabla_k X_j = \omega_k X_j + \varphi g_{kj}$, being $\varphi$ a scalar function and $\omega_k$ a non vanishing one-form. It is named concircular if $\omega_k$ is a gradient or locally a gradient of a scalar function.

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Fialkow gave a definition different from Yano's:

**Definition 2** (Fialkow [4])**.** A vector field $X_j$ is named concircular if it satisfies $\nabla_k X_j = \rho g_{kj}$, being $\rho$ a scalar function.

The following simple but deep result was recently proven:

**Theorem 3** (Bang-Yen Chen, [3])**.** *A $n > 3$ dimensional Lorentzian manifold is a GRW space-time if and only if it admits a time-like concircular vector field (in the sense of Fialkow).*

It is worth noticing that for a unit time-like vector field, the torse-forming property by Yano becomes precisely Eq. (2), with generic scalar field $\varphi$. Based on Chen's theorem we proved:

**Proposition 4** (Mantica and Molinari, [7,8])**.** *A $n > 3$ dimensional Lorentzian manifold is a GRW space-time if and only if it admits a unit time-like torse-forming vector,* (2)*, that is also an eigenvector of the Ricci tensor.*

Now comes the equivalence: from (2) (holding either for GRW and $(LCS)_n$ space-times) we evaluate

$$R_{jkl}{}^m u_m = [\nabla_j, \nabla_k] u_l = (h_{kl}\nabla_j - h_{jl}\nabla_k)\varphi - \varphi^2(u_j g_{kl} - u_k g_{jl}),$$

where $h_{kl} = u_k u_l + g_{kl}$. Contraction with $g^{jl}$ gives

$$R_k{}^m u_m = u_k[u^m \nabla_m \varphi + (n-1)\varphi^2] - (n-2)\nabla_k \varphi. \tag{4}$$

If (3) holds, then $u_k$ is an eigenvector of the Ricci tensor, and we conclude that *a $(LCS)_n$ manifold is a GRW space-time.*

If $R_{km}u^m = \xi u_k$ it is $(n-2)\nabla_k \varphi = \alpha u_k$ for some scalar field $\alpha$, i.e., (3) holds. Then we conclude that *a GRW space-time is a $(LCS)_n$ manifold.*

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