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GENERIC LIGHTLIKE SUBMANIFOLDS OF AN INDEFINITE TRANS-SASAKIAN MANIFOLD WITH AN (ℓ, m)-TYPE METRIC CONNECTION

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Abstract. We study generic lightlike submanifolds M of an indefinite trans-Sasakian manifold $\bar{M}$ or an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ endowed with an (ℓ, m)-type metric connection subject such that the structure vector field ζ of $\bar{M}$ is tangent to M.

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *symmetric connection of type* $(\ell,\, m)$ if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}, \tag{1.1}$$

where ℓ and m are smooth functions, J is a tensor field of type $(1, 1)$ and θ is a 1-form associated with a smooth vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if $\bar{\nabla}$ is a metric connection, *i.e.*, it satisfies $\bar{\nabla}\bar{g} = 0$, then $\bar{\nabla}$ is called a *symmetric metric connection of type* $(\ell,\, m)$ or an $(\ell,\, m)$*-type metric connection*.

The notion of (ℓ, m)-type metric connection $\bar{\nabla}$ on indefinite almost contact manifolds $\bar{M}$ was introduced by Jin [9]. In case $(\ell, m) = (1,\, 0)$: $\bar{\nabla}$ becomes a semi-symmetric metric connection, introduced by Hayden [6] and Yano [15]. In case $(\ell, m) = (0,\, 1)$: $\bar{\nabla}$ becomes a quarter-symmetric metric connection, introduced by Yano-Imai [16]. We shall assume that $(\ell, m) \neq (0, 0)$ and the vector field ζ is a unit spacelike one, without loss of generality.

A lightlike submanifold M of an indefinite almost contact manifold $\bar{M}$, with an indefinite almost contact structure J, is called an *generic submanifold* [10] if there exists a screen distribution $S(TM)$ such that

$$J(S(TM)^{\perp}) \subset S(TM), \tag{1.2}$$

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where $S(TM)^{\perp}$ denotes the orthogonal complement of $S(TM)$ in the tangent bundle $T\bar{M}$ of $\bar{M}$ such that $T\bar{M} = S(TM) \oplus_{orth} S(TM)^{\perp}$. The notion of generic lightlike submanifolds was studied by several authors [5, 7, 8, 12].

Remark 1.1. Denote by $\widetilde{\nabla}$ the Levi-Civita connection of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ with respect to $\bar{g}$. It is known [9] that *a linear connection $\bar{\nabla}$ on $\bar{M}$ is an (ℓ, m)-type metric connection if and only if it satisfies*

$$\bar{\nabla}_{\bar{X}}\bar{Y} = \widetilde{\nabla}_{\bar{X}}\bar{Y} + \ell\{\theta(\bar{Y})\bar{X} - \bar{g}(\bar{X}, \bar{Y})\zeta\} - m\theta(\bar{X})J\bar{Y}. \tag{1.3}$$

In this paper, we study generic lightlike submanifolds of an indefinite trans-Sasakian manifold $\bar{M} = (\bar{M}, \zeta, \theta, J, \bar{g})$ endowed with an (ℓ, m)-type metric connection subject to the following two conditions: (1) the tensor field J and the 1-form θ, defined by (1.1), are identical with the indefinite trans-Sasakian structure tensor J and the structure 1-form θ of $\bar{M}$, respectively, and (2) the structure vector field ζ of $\bar{M}$ is tangent to M.

2. (ℓ, m)-type metric connections

An odd-dimensional semi-Riemannian manifold $(\bar{M}, \bar{g})$ is called an *indefinite almost contact metric manifold* if there exists a set $\{J, \zeta, \theta, \bar{g}\}$, where J is a (1, 1)-type tensor field, ζ is a vector field and θ is a 1-form such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \ \bar{g}(J\bar{X}, J\bar{Y}) = \bar{g}(\bar{X}, \bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}), \ \theta(\zeta) = 1, \tag{2.1}$$

where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike respectively. The set $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite almost contact metric structure* of $\bar{M}$.

From (2.1), we show that

$$J\zeta = 0, \quad \theta \circ J = 0, \quad \theta(\bar{X}) = \epsilon\bar{g}(\bar{X}, \zeta), \quad \bar{g}(J\bar{X}, \bar{Y}) = -\bar{g}(\bar{X}, J\bar{Y}).$$

In the entire discussion of this article, we shall assume that the structure vector field ζ is a spacelike one, *i.e.*, $\epsilon = 1$, without loss of generality.

Definition. An indefinite almost contact metric manifold $\bar{M}$ is said to be an *indefinite trans-Sasakian manifold* [14] if, for the Levi-Civita connection $\widetilde{\nabla}$, there exist two smooth functions α and β such that

$$(\widetilde{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.$$

We say that $\{J, \zeta, \theta, \bar{g}\}$ is an *indefinite trans-Sasakian structure of type* (α, β).

Note that the notion of a (Riemannian) trans-Sasakian manifold of type (α, β) was introduced by Oubina [14]. Sasakian, Kenmotsu and cosymplectic manifolds are important kinds of the trans-Sasakian manifold such that

$$\alpha = 1, \ \beta = 0; \quad \alpha = 0, \ \beta = 1; \quad \alpha = \beta = 0, \quad \text{respectively.}$$

Replacing the Levi-Civita connection $\widetilde{\nabla}$ by the (ℓ, m)-type metric connection $\bar{\nabla}$, the equation in the above Definition is reduce to

$$\begin{aligned}(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \ & \alpha\{\bar{g}(\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})\bar{X}\} \\ & + \ (\beta + \ell)\{\bar{g}(J\bar{X}, \bar{Y})\zeta - \theta(\bar{Y})J\bar{X}\}.\end{aligned} \tag{2.2}$$

Replacing $\bar{Y}$ by ζ to (2.2) and using $J\zeta = 0$ and $\theta(\bar{\nabla}_{\bar{X}}\zeta) = 0$, we obtain

$$\bar{\nabla}_{\bar{X}}\zeta = -\alpha J\bar{X} + (\beta + \ell)\{\bar{X} - \theta(\bar{X})\zeta\}. \tag{2.3}$$

Let (M, g) be an m-dimensional lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$, of dimension $(m + n)$. Then the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ on M is a subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank $r\,(1 \leq r \leq \min\{m,\, n\})$. In general, there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$, respectively, which are called the *screen distribution* and *co-screen distribution* of M such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \;\; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}),$$

where $\oplus_{orth}$ denotes the orthogonal direct sum. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. Also denote by $(2.1)_i$ the i-th equation of (2.1). We use the same notations for any others. Let X, Y and Z be the vector fields on M, unless otherwise specified. We use the following range of indices:

$$i,\, j,\, k,\, \ldots \in \{1,\, \ldots\, ,\, r\}, \qquad a,\, b,\, c,\, \ldots \in \{r\, +\, 1,\, \ldots\, ,\, n\}.$$

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}_{|M}$ and $TM^{\perp}$ in $S(TM)^{\perp}$ respectively and let $\{N_1,\, \cdots\, ,\, N_r\}$ be a lightlike basis of $ltr(TM)_{|\mathcal{U}}$, where $\mathcal{U}$ is a coordinate neighborhood of M, such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0,$$

where $\{\xi_1,\, \ldots,\, \xi_r\}$ is a lightlike basis of $Rad(TM)_{|\mathcal{U}}$. In this case,

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned}$$

A lightlike submanifold $M = (M, g, S(TM), S(TM^{\perp}))$ of $\bar{M}$ is called an *r-lightlike submanifold* [4] if $1 \leq r < \min\{m,\, n\}$. For an r-lightlike M, we see that $S(TM) \neq \{0\}$ and $S(TM^{\perp}) \neq \{0\}$. In the sequel, by saying that M is a lightlike submanifold we shall mean that it is an r-lightlike submanifold with following local quasi-orthonormal field of frames of $\bar{M}$:

$$\{\xi_1,\, \ldots,\, \xi_r\,,\; N_1,\, \ldots,\, N_r\,,\; F_{r+1},\, \ldots,\, F_m\,,\; E_{r+1},\, \ldots,\, E_n\},$$

where $\{F_{r+1}, \ldots, F_m\}$ and $\{E_{r+1},\, \ldots,\, E_n\}$ are orthonormal bases of $S(TM)$ and $S(TM^{\perp})$, respectively. Denote $\epsilon_a = \bar{g}(E_a, E_a)$. Then $\epsilon_a\delta_{ab} = \bar{g}(E_a, E_b)$.

Let P be the projection morphism of TM on $S(TM)$. Then the local Gauss-Weingarten formulae of M and $S(TM)$ are given respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + \sum_{i=1}^{r} h_i^{\ell}(X, Y)N_i + \sum_{a=r+1}^{n} h_a^{s}(X, Y)E_a, \tag{2.4}$$

$$\bar{\nabla}_X N_i = -A_{N_i}X + \sum_{j=1}^{r} \tau_{ij}(X)N_j + \sum_{a=r+1}^{n} \rho_{ia}(X)E_a, \tag{2.5}$$

$$(2.6)\qquad \bar{\nabla}_X E_a = -A_{E_a}X + \sum_{i=1}^{r}\lambda_{ai}(X)N_i + \sum_{b=r+1}^{n}\mu_{ab}(X)E_b;$$

$$(2.7)\qquad \nabla_X PY = \nabla^*_X PY + \sum_{i=1}^{r} h^*_i(X,PY)\xi_i,$$

$$(2.8)\qquad \nabla_X \xi_i = -A^*_{\xi_i}X - \sum_{j=1}^{r}\tau_{ji}(X)\xi_j,$$

where ∇ and ∇^* are induced linear connections on M and $S(TM)$ respectively, h^ℓ_i and h^s_a are called the *local second fundamental forms* on M, h^*_i are called the *local second fundamental forms* on $S(TM)$. A_{N_i}, A_{E_a} and $A^*_{\xi_i}$ are called the *shape operators*, and τ_{ij}, ρ_{ia}, λ_{ai} and μ_{ab} are 1-forms.

Let M be a generic lightlike submanifold of $\bar{M}$. From (1.2) we show that $J(Rad(TM))$, $J(ltr(TM))$ and $J(S(TM^\perp))$ are vector subbundles of $S(TM)$. Thus there exist two non-degenerate almost complex distributions H_o and H with respect to J, *i.e.*, $J(H_o) = H_o$ and $J(H) = H$, such that

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^\perp)) \oplus_{orth} H_o,$$
$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

In this case, the tangent bundle TM on M is decomposed as follows:

$$(2.9)\qquad TM = H \oplus J(ltr(TM)) \oplus_{orth} J(S(TM^\perp)).$$

Consider local null vector fields U_i and V_i for each i, local non-null unit vector fields W_a for each a, and their 1-forms u_i, v_i and w_a defined by

$$(2.10)\qquad U_i = -JN_i, \qquad V_i = -J\xi_i, \qquad W_a = -JE_a,$$

$$(2.11)\qquad u_i(X) = g(X, V_i), \quad v_i(X) = g(X, U_i), \quad w_a(X) = \epsilon_a g(X, W_a).$$

Denote by S the projection morphism of TM on H and by F the tensor field of type $(1,1)$ globally defined on M by $F = J \circ S$. Then JX is expressed as

$$(2.12)\qquad JX = FX + \sum_{i=1}^{r} u_i(X)N_i + \sum_{a=r+1}^{n} w_a(X)E_a.$$

3. Structure equations

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with an (ℓ, m)-type metric connection $\bar{\nabla}$. We shall assume that ζ is tangent to M. Călin [2] proved that if ζ is tangent to M, then it belongs to $S(TM)$ which we assumed in this paper. Using (1.2), (1.3), (2.3) and (2.11), we see that

$$(3.1)\qquad (\nabla_X g)(Y,Z) = \sum_{i=1}^{r}\{h^\ell_i(X,Y)\eta_i(Z) + h^\ell_i(X,Z)\eta_i(Y)\},$$

$$(3.2)\qquad T(X,Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\},$$

$$h_i^\ell(X,Y) - h_i^\ell(Y,X) = m\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}, \tag{3.3}$$

$$h_a^s(X,Y) - h_a^s(Y,X) = m\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}, \tag{3.4}$$

where η_i's are 1-forms such that $\eta_i(X) = \bar{g}(X, N_i)$. From the facts that $h_i^\ell(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi_i)$ and $\epsilon_a h_a^s(X,Y) = \bar{g}(\bar{\nabla}_X Y, E_a)$, we know that h_i^ℓ and h_a^s are independent of the choice of $S(TM)$. The above local second fundamental forms are related to their shape operators by

$$h_i^\ell(X,Y) = g(A^*_{\xi_i}X, Y) - \sum_{k=1}^{r} h_k^\ell(X,\xi_i)\eta_k(Y), \tag{3.5}$$

$$\epsilon_a h_a^s(X,Y) = g(A_{E_a}X, Y) - \sum_{k=1}^{r} \lambda_{ak}(X)\eta_k(Y), \tag{3.6}$$

$$h_i^*(X, PY) = g(A_{N_i}X, PY). \tag{3.7}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(E_a, E_b) = \epsilon\delta_{ab}$, $g(\xi_i, \xi_j) = 0$, $\bar{g}(\xi_i, E_a) = 0$, $\bar{g}(N_i, N_j) = 0$ and $\bar{g}(N_i, E_a) = 0$ by turns, we obtain $\epsilon_b\mu_{ab} + \epsilon_a\mu_{ba} = 0$ and

$$\begin{aligned} &h_i^\ell(X,\xi_j) + h_j^\ell(X,\xi_i) = 0, \qquad && h_a^s(X,\xi_i) = -\epsilon_a\lambda_{ai}(X), \\ &\eta_j(A_{N_i}X) + \eta_i(A_{N_j}X) = 0, \qquad && \bar{g}(A_{E_a}X, N_i) = \epsilon_a\rho_{ia}(X). \end{aligned} \tag{3.8}$$

Furthermore, using (3.3) and $(3.8)_1$ we see that

$$h_i^\ell(X,\xi_i) = 0, \qquad h_i^\ell(\xi_j,\xi_k) = 0, \qquad A^*_{\xi_i}\xi_i = 0. \tag{3.9}$$

Replacing Y by ζ to (2.4) and using (2.3) and (2.12), we have

$$\nabla_X\zeta = -\alpha FX + (\beta + \ell)\{X - \theta(X)\zeta\}, \tag{3.10}$$

$$h_i^\ell(X,\zeta) = -\alpha u_i(X), \qquad h_a^s(X,\zeta) = -\alpha w_a(X). \tag{3.11}$$

Applying $\bar{\nabla}_X$ to $\bar{g}(\zeta, N_i) = 0$ and using (2.3), (2.5) and (3.7), we have

$$h_i^*(X,\zeta) = -\alpha v_i(X) + (\beta + \ell)\eta_i(X). \tag{3.12}$$

Applying $\bar{\nabla}_X$ to $(2.10)_{1,2,3}$ and (2.12) by turns and using (2.2), (2.4) $\sim$ (2.8), (2.10) $\sim$ (2.12) and (3.5) $\sim$ (3.7), we have

$$\begin{aligned} &h_j^\ell(X,U_i) = h_i^*(X,V_j), \qquad && \epsilon_a h_i^*(X,W_a) = h_a^s(X,U_i), \\ &h_j^\ell(X,V_i) = h_i^\ell(X,V_j), \qquad && \epsilon_a h_i^\ell(X,W_a) = h_a^s(X,V_i), \\ &\epsilon_b h_b^s(X,W_a) = \epsilon_a h_a^s(X,W_b), && \end{aligned} \tag{3.13}$$

$$\begin{aligned} \nabla_X U_i \;=\; & F(A_{N_i}X) + \sum_{j=1}^{r}\tau_{ij}(X)U_j + \sum_{a=r+1}^{n}\rho_{ia}(X)W_a \\ & - \{\alpha\eta_i(X) + (\beta+\ell)v_i(X)\}\zeta, \end{aligned} \tag{3.14}$$

$$\nabla_X V_i \;=\; F(A^*_{\xi_i}X) - \sum_{j=1}^{r}\tau_{ji}(X)V_j + \sum_{j=1}^{r} h_j^\ell(X,\xi_i)U_j \tag{3.15}$$

$$- \sum_{a=r+1}^{n} \epsilon_a \lambda_{ai}(X) W_a - (\beta + \ell) u_i(X) \zeta,$$

$$\nabla_X W_a = F(A_{E_a} X) + \sum_{i=1}^{r} \lambda_{ai}(X) U_i + \sum_{b=r+1}^{n} \mu_{ab}(X) W_b - \epsilon_a(\beta + \ell) w_a(X) \zeta, \tag{3.16}$$

$$\begin{aligned}(\nabla_X F)(Y) = & \sum_{i=1}^{r} u_i(Y) A_{N_i} X + \sum_{a=r+1}^{n} w_a(Y) A_{E_a} X \\ & - \sum_{i=1}^{r} h_i^{\ell}(X, Y) U_i - \sum_{a=r+1}^{n} h_a^s(X, Y) W_a \\ & + \alpha \{ g(X, Y) \zeta - \theta(Y) X \} \\ & + (\beta + \ell) \{ \bar{g}(JX, Y) \zeta - \theta(Y) FX \},\end{aligned} \tag{3.17}$$

$$\begin{aligned}(\nabla_X u_i)(Y) = & - \sum_{j=1}^{r} u_j(Y) \tau_{ji}(X) - \sum_{a=r+1}^{n} w_a(Y) \lambda_{ai}(X) \\ & - h_i^{\ell}(X, FY) - (\beta + \ell) \theta(Y) u_i(X),\end{aligned} \tag{3.18}$$

$$\begin{aligned}(\nabla_X v_i)(Y) = & \sum_{j=1}^{r} v_j(Y) \tau_{ij}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y) \rho_{ia}(X) \\ & + \sum_{j=r+1}^{r} u_j(Y) \eta_i(A_{N_j} X) - g(A_{N_i} X, FY) \\ & - \theta(Y) \{ \alpha \eta_i(X) + (\beta + \ell) v_i(X) \}.\end{aligned} \tag{3.19}$$

Definition. We say that a lightlike submanifold M is

(1) *irrotational* [13] if $\bar{\nabla}_X \xi_i \in \Gamma(TM)$ for all $i \in \{1, \ldots, r\}$,
(2) *solenoidal* [11] if A_{E_a} and A_{N_i} are $S(TM)$-valued,
(3) *statical* [11] if M is both irrotational and solenoidal.

From (2.3) and $(3.8)_2$, the item (1) is equivalent to

$$h_j^{\ell}(X, \xi_i) = 0, \qquad h_a^s(X, \xi_i) = \lambda_{ai}(X) = 0. \tag{3.20}$$

By using $(3.8)_4$, the item (2) is equivalent to

$$\eta_j(A_{N_i} X) = 0, \qquad \rho_{ia}(X) = \eta_i(A_{E_a} X) = 0. \tag{3.21}$$

Theorem 3.1. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type metric connection subject such that ζ is tangent to M. If F is parallel with respect to the connection ∇, then*

(1) *$\bar{M}$ is an indefinite β-Kenmotsu manifold with $\alpha = 0$ and $\beta = -\ell$,*
(2) *M is statical,*
(3) *H, $J(ltr(TM))$ and $J(S(TM^{\perp}))$ are parallel distributions on M, and*

(4) *M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H, respectively.*

Proof. (1) Taking $X = \xi_k$ and $Y = \zeta$ to (3.17) and using (3.11), we get

$$\alpha\xi_k = (\beta + \ell)V_k.$$

Taking the scalar product with N_k and U_k to this equation by turns, we have $\alpha = 0$ and $\beta = -\ell$. Thus $\bar{M}$ is an indefinite β-Kenmotsu manifold.

(2) Taking $Y = \xi_j$ to (3.17) with $\nabla_X F = 0$, we obtain

$$\sum_{i=1}^{r} h_i^\ell(X, \xi_j)U_i + \sum_{a=r+1}^{n} h_a^s(X, \xi_j)W_a = 0.$$

Taking the scalar product with V_i and W_a to this by turns, we obtain (3.20). Thus M is irrotational. Taking the scalar product with N_j to (3.17), we get

$$\sum_{i=1}^{r} u_i(Y)\eta_j(A_{N_i}X) + \sum_{a=r+1}^{n} w_a(Y)\eta_j(A_{E_a}X) = 0.$$

Taking $Y = U_i$ and $Y = W_a$ to this, we have (3.21). Thus M is solenoidal. As M is both irrotational and solenoidal, M is statical.

(3) Taking the scalar product with V_i and W_a to (3.17) by turns, we have

$$h_i^\ell(X, Y) = \sum_{k=1}^{r} u_k(Y)u_i(A_{N_k}X) + \sum_{a=r+1}^{n} w_a(Y)u_i(A_{E_a}X),$$
$$h_a^s(X, Y) = \sum_{i=1}^{r} u_i(Y)w_a(A_{N_i}X) + \sum_{b=r+1}^{n} w_b(Y)w_a(A_{E_b}X).$$

Taking $Y = V_j$ and $Y = FZ$, $Z \in \Gamma(TM)$ to the last two equations by turns and using the facts that $u_i(FZ) = w_a(FZ) = 0$, we obtain

$$h_i^\ell(X, V_j) = 0, \qquad h_i^\ell(X, FZ) = 0,$$
$$h_a^s(X, V_j) = 0, \qquad h_a^s(X, FZ) = 0.$$

Using these, (2.1), (2.8), (2.12), (3.1), (3.13), (3.15) and (3.20), we derive

$$g(\nabla_X\xi_i, V_j) = -h_i^\ell(X, V_j) = 0, \quad g(\nabla_X\xi_i, W_a) = -\epsilon_a h_a^s(X, V_i) = 0,$$
$$g(\nabla_X V_i, V_j) = h_j^\ell(X, \xi_i) = 0, \quad g(\nabla_X V_i, W_a) = h_a^s(X, \xi_i) = 0,$$
$$g(\nabla_X Z_o, V_j) = h_i^\ell(X, FZ_o) = 0, \quad g(\nabla_X Z_o, W_a) = h_a^s(X, FZ_o) = 0,$$

where $Z_o \in \Gamma(H_o)$, that is,

$$\nabla_X Y \in \Gamma(H), \quad \forall\, X \in \Gamma(TM), \quad \forall\, Y \in \Gamma(H).$$

It follows that H is a parallel distribution on M.

On the other hand, taking $Y = U_i$ and $Y = W_a$ to (3.17) by turns, we have

$$A_{N_i}X = \sum_{k=1}^{r} h_k^\ell(X, U_i)U_k + \sum_{a=r+1}^{n} h_a^s(X, U_i)W_a, \tag{3.22}$$

$$A_{E_a}X = \sum_{i=1}^{r} h_i^\ell(X, W_a)U_i + \sum_{b=r+1}^{n} h_b^s(X, W_a)W_b.$$

Applying F to these equations and using $FU_i = FW_a = 0$, we have

$$F(A_{N_i}X) = 0, \qquad F(A_{E_a}X) = 0.$$

Using these results, $(3.20)_2$ and $(3.21)_2$, Eqs. (3.14) and (3.16) reduce

$$\nabla_X U_i = \sum_{j=1}^{r} \tau_{ij}(X)U_j, \qquad \nabla_X W_a = \sum_{b=r+1}^{n} \mu_{ab}(X)W_b. \tag{3.23}$$

Thus $J(ltr(TM))$ and $J(S(TN^\perp))$ are also parallel distributions on M, *i.e.*,

$$\nabla_X U_i \in \Gamma(J(ltr(TM))), \quad \nabla_X W_a \in \Gamma(J(S(TM^\perp)), \quad \forall\, X \in \Gamma(TM).$$

(4) As $J(ltr(TM))$, $J(S(TM^\perp))$ and H are parallel distributions and satisfy (2.9), by the decomposition theorem of de Rham [3], M is locally a product manifold $M_r \times M_{n-r} \times M^\sharp$, where M_r, M_{n-r} and $M^\sharp$ are leaves of $J(ltr(TM))$, $J(S(TM^\perp))$ and H, respectively. □

Theorem 3.2. *Let M be a generic lightlike submanifold of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type metric connection such that ζ is tangent to M. If U_is are parallel with respect to ∇, then $\tau = 0$, M is solenoidal and $\bar{M}$ is an indefinite β-Kenmotsu manifold, i.e., $\alpha = 0$ and $\beta = -\ell$.*

Proof. Taking the scalar product with ζ, V_j, U_j, W_a and N_j to (3.14) with $\nabla_X U_i = 0$ by turns and using the fact that $g(FX, \zeta) = 0$, we obtain

$$\alpha = 0,\ \beta = -\ell\,;\ \tau_{ij} = 0,\ \eta_j(A_{N_i}X) = 0,\ \rho_{ia} = 0,\ h_i^*(X, U_j) = 0, \tag{3.24}$$

respectively. As $\alpha = 0$ and $\beta = -\ell$, $\bar{M}$ is an indefinite β-Kenmotsu manifold. As $\eta_j(A_{N_i}X) = 0$ and $\rho_{ia}(X) = \eta_i(A_{E_a}X) = 0$, M is solenoidal. □

Theorem 3.3. *Let M be a generic lightlike hypersurface of an indefinite trans-Sasakian manifold $\bar{M}$ with an (ℓ, m)-type metric connection such that ζ is tangent to M. If V_is are parallel with respect to the connection ∇, then $\tau_{ij} = 0$, $\alpha = -m$, $\beta = -\ell$ and M is irrotational.*

Proof. Taking the scalar product with ζ, U_j, V_j, W_a and N_j to (3.15) with $\nabla_X V_i = 0$ by turns and using the fact that $g(FX, \zeta) = 0$, we obtain

$$\beta = -\ell, \quad \tau_{ij} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0. \tag{3.25}$$

As $h_j^\ell(X, \xi_i) = 0$ and $\lambda_{ai}(X) = h_a^s(X, \xi_i) = 0$, M is irrotational. On the other hand, replacing Y by U_i to (3.3) and using $(3.25)_5$, we have

$$h_i^\ell(U_i, X) = m\theta(X).$$

Replacing X by ζ to this equation and using $(3.11)_1$, we have $\alpha = -m$. □

4. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $\bar{M}$ is said to be a *indefinite generalized Sasakian space form* [1] and denote it by $\bar{M}(f_1, f_2, f_3)$ if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$
\begin{aligned}
(4.1)\qquad \widetilde{R}(\bar{X},\bar{Y})\bar{Z} = {} & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\
& + f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\
& + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\
& \qquad + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\},
\end{aligned}
$$

where $\widetilde{R}$ is the curvature tensor of the Levi-Civita connection $\bar{\nabla}$ of $\bar{M}$.

Denote by $\bar{R}$ the curvature tensors of the (ℓ, m)-type metric connection $\bar{\nabla}$ on $\bar{M}$, By directed calculations from (1.1), (1.3) and (2.2), we see that

$$
\begin{aligned}
(4.2)\qquad \bar{R}(\bar{X},\bar{Y})\bar{Z} = {} & \widetilde{R}(\bar{X},\bar{Y})\bar{Z} \\
& + (X\ell)\{\theta(Z)Y - g(Y,Z)\zeta\} - (Xm)\theta(Y)JZ \\
& - (Y\ell)\{\theta(Z)X - g(X,Z)\zeta\} + (Ym)\theta(X)JZ \\
& + \ell\{(\bar{\nabla}_X\theta)(Z)Y - (\bar{\nabla}_Y\theta)(Z)X \\
& \qquad + \alpha[g(Y,Z)JX - g(X,Z)JY] \\
& \qquad - \beta[g(Y,Z)X - g(X,Z)Y] \\
& \qquad + (\beta+\ell)[g(Y,Z)\theta(X) - g(X,Z)\theta(Y)]\zeta \\
& \qquad + m[\theta(Y)JX - \theta(X)JY]\theta(Z)\} \\
& - m\{[(\bar{\nabla}_X\theta)(Y) - (\bar{\nabla}_Y\theta)(X)]JZ \\
& \qquad + \alpha[\theta(Y)g(X,Z) - \theta(X)g(Y,Z)]\zeta \\
& \qquad - \alpha[\theta(Y)X - \theta(X)Y]\theta(Z) \\
& \qquad + (\beta+\ell)[\theta(Y)g(JX,Z) - \theta(X)g(JY,Z)]\zeta \\
& \qquad - \beta[\theta(Y)JX - \theta(X)JY]\theta(Z)\}.
\end{aligned}
$$

Denote by R and R^* the curvature tensors of ∇ and ∇^* respectively. Then we obtain Gauss equations for M and $S(TM)$, respectively:

$$
\begin{aligned}
(4.3)\qquad \bar{R}(X,Y)Z = {} & R(X,Y)Z \\
& + \sum_{i=1}^{r}\{h_i^\ell(X,Z)A_{N_i}Y - h_i^\ell(Y,Z)A_{N_i}X\} \\
& + \sum_{a=r+1}^{n}\{h_a^s(X,Z)A_{E_a}Y - h_a^s(Y,Z)A_{E_a}X\} \\
& + \sum_{i=1}^{r}\{(\nabla_X h_i^\ell)(Y,Z) - (\nabla_Y h_i^\ell)(X,Z)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{r}[\tau_{ji}(X)h_j^\ell(Y,Z)-\tau_{ji}(Y)h_j^\ell(X,Z)]\\
&+\sum_{a=r+1}^{n}[\lambda_{ai}(X)h_a^s(Y,Z)-\lambda_{ai}(Y)h_a^s(X,Z)]\\
&-\ell[\theta(X)h_i^\ell(Y,Z)-\theta(Y)h_i^\ell(X,Z)]\\
&-m[\theta(X)h_i^\ell(FY,Z)-\theta(Y)h_i^\ell(FX,Z)]\}N_i\\
&+\sum_{a=r+1}^{n}\{(\nabla_X h_a^s)(Y,Z)-(\nabla_Y h_a^s)(X,Z)\\
&+\sum_{i=1}^{r}[\rho_{ia}(X)h_i^\ell(Y,Z)-\rho_{ia}(Y)h_i^\ell(X,Z)]\\
&+\sum_{b=r+1}^{n}[\mu_{ba}(X)h_b^s(Y,Z)-\mu_{ba}(Y)h_b^s(X,Z)]\\
&-\ell[\theta(X)h_a^s(Y,Z)-\theta(Y)h_a^s(X,Z)]\\
&-m[\theta(X)h_a^s(FY,Z)-\theta(Y)h_a^s(FX,Z)]\}E_a,
\end{aligned}
$$

$$
\begin{aligned}
R(X,Y)PZ = R^*(X,Y)PZ
&+\sum_{i=1}^{r}\{h_i^*(X,PZ)A^*_{\xi_i}Y-h_i^*(Y,PZ)A_{\xi_i}X\}\\
&+\sum_{i=1}^{r}\{(\nabla_X h_i^*)(Y,PZ)-(\nabla_Y h_i^*)(X,PZ)\\
&+\sum_{k=1}^{r}[\tau_{ik}(Y)h_k^*(X,PZ)-\tau_{ik}(X)h_k^*(Y,PZ)]\\
&-\ell[\theta(X)h_i^*(Y,PZ)-\theta(Y)h_i^*(FX,PZ)]\\
&-m[\theta(X)h_i^*(FY,PZ)-\theta(Y)h_i^*(FX,PZ)]\}\xi_i.
\end{aligned}
\tag{4.4}
$$

Applying $\bar{\nabla}_X$ to $\theta(\xi_i)=0$, $\theta(V_i)=0$, $\theta(U_i)=0$, $\theta(W_a)=0$ and $\theta(\zeta)=1$ by turns and using (2.4), (2.8), (3.5), $(3.11)_1$, (3.14), (3.15), (3.16) and the facts that $g(FX,\zeta)=0$, $\bar{g}(\zeta,\zeta)=1$ and $\bar{\nabla}$ is metric, we obtain

$$
\begin{aligned}
&(\bar{\nabla}_X\theta)(\xi_i)=-\alpha u_i(X), \qquad (\bar{\nabla}_X\theta)(V_i)=(\beta+\ell)u_i(X),\\
&(\bar{\nabla}_X\theta)(U_i)=\alpha\eta_i(X)+(\beta+\ell)v_i(X),\\
&(\bar{\nabla}_X\theta)(W_a)=\epsilon_a(\beta+\ell)w_a(X), \qquad (\bar{\nabla}_X\theta)(\zeta)=0.
\end{aligned}
\tag{4.5}
$$

Taking the scalar product with ξ_i, E_a and N_i to (4.2) by turns and using (4.1), (4.3), (4.4) and the facts that $\zeta\in\Gamma(S(TM))$ and $\bar{\nabla}$ is metric, we get

$$
(\nabla_X h_i^\ell)(Y,Z)-(\nabla_Y h_i^\ell)(X,Z)
\tag{4.6}
$$

$$
\begin{aligned}
&+ \sum_{j=1}^{r}\{\tau_{ji}(X)h_j^\ell(Y,Z) - \tau_{ji}(Y)h_j^\ell(X,Z)\} \\
&+ \sum_{a=r+1}^{n}\{\lambda_{ai}(X)h_a^s(Y,Z) - \lambda_{ai}(Y)h_a^s(X,Z)\} \\
&- \ell\{\theta(X)h_i^\ell(Y,Z) - \theta(Y)h_i^\ell(X,Z)\} \\
&- m\{\theta(X)h_i^\ell(FY,Z) - \theta(Y)h_i^\ell(FX,Z)\} \\
&+ \{(Xm)\theta(Y) + m(\bar\nabla_X\theta)(Y) \\
&\qquad - (Ym)\theta(X) - m(\bar\nabla_Y\theta)(X)\}u_i(Z) \\
&-\ell\alpha\{g(Y,Z)u_i(X) - g(X,Z)u_i(Y)\} \\
&- m(\beta+\ell)\{\theta(Y)u_i(X) - \theta(X)u_i(Y)\}\theta(Z) \\
= {} & f_2\{u_i(Y)\bar g(X,JZ) - u_i(X)\bar g(Y,JZ) + 2u_i(Z)\bar g(X,JY)\},
\end{aligned}
$$

$$
\begin{aligned}
&(\nabla_X h_a^s)(Y,Z) - (\nabla_Y h_a^s)(X,Z) \\
&+ \sum_{i=1}^{r}\{\rho_{ia}(X)h_i^\ell(Y,Z) - \rho_{ia}(Y)h_j^\ell(X,Z)\} \\
&+ \sum_{b=r+1}^{n}\{\mu_{ba}(X)h_a^s(Y,Z) - \mu_{ba}(Y)h_a^s(X,Z)\} \\
&- \ell\{\theta(X)h_a^s(Y,Z) - \theta(Y)h_a^s(X,Z)\} \\
&- m\{\theta(X)h_a^s(FY,Z) - \theta(Y)h_a^s(FX,Z)\} \\
&+ \{(Xm)\theta(Y) + m(\bar\nabla_X\theta)(Y) \\
&\qquad - (Ym)\theta(X) - m(\bar\nabla_Y\theta)(X)\}w_a(Z) \\
&-\ell\alpha\{g(Y,Z)w_a(X) - g(X,Z)w_a(Y)\} \\
&- m(\beta+\ell)\{\theta(Y)w_a(X) - \theta(X)w_a(Y)\}\theta(Z) \\
= {} & f_2\{w_a(Y)\bar g(X,JZ) - w_a(X)\bar g(Y,JZ) + 2w_a(Z)\bar g(X,JY)\},
\end{aligned} \tag{4.7}
$$

$$
\begin{aligned}
&(\nabla_X h_i^*)(Y,\, PZ) - (\nabla_Y h_i^*)(X,\, PZ) \\
&- \sum_{j=1}^{r}\{\tau_{ij}(X)h_j^*(Y,\, PZ) - \tau_{ij}(Y)h_j^*(X,\, PZ)\} \\
&+ \sum_{j=1}^{r}\{h_j^\ell(X,\, PZ)\eta_i(A_{N_j}Y) - h_j^\ell(Y,\, PZ)\eta_i(A_{N_j}X)\} \\
&- \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y,\, PZ) - \rho_{ia}(Y)h_a^s(X,\, PZ)\} \\
&- \ell\{\theta(X)h_i^*(Y,PZ) - \theta(Y)h_i^*(X,PZ)\} \\
&- m\{\theta(X)h_i^*(FY,PZ) - \theta(Y)h_i^*(FX,PZ)\}
\end{aligned} \tag{4.8}
$$

$$\begin{aligned}
&- \{(X\ell)\theta(PZ) + \ell(\bar{\nabla}_X\theta)(PZ)\}\eta_i(Y)\\
&\qquad + \{(Y\ell)\theta(PZ) + \ell(\bar{\nabla}_Y\theta)(PZ)\}\eta_i(X)\\
&+ \{(Xm)\theta(Y) + m(\bar{\nabla}_X\theta)(Y)\\
&\qquad - (Ym)\theta(X) - \ell(\bar{\nabla}_Y\theta)(X)\}v_i(PZ)\\
&- \ell\alpha\{g(Y, PZ)v_i(X) - g(X, PZ)v_i(Y)\}\\
&+ \ell\beta\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\}\\
&-m\alpha\{\theta(Y)\eta_i(X) - \theta(X)\eta_i(Y)\}\theta(PZ)\\
&-m(\beta+\ell)\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\}\theta(PZ)\\
= \;& f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\}\\
&+ f_2\{v_i(Y)\bar{g}(X, JPZ) - v_i(X)\bar{g}(Y, JPZ) + 2v_i(PZ)\bar{g}(X, JY)\}\\
&+ f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}$$

Theorem 4.1. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type metric connection such that ζ is tangent to M. Then the functions α, β, f_1, f_2 and f_3 satisfy*

(1) *α is a constant on M,*
(2) *$\alpha\beta = 0$, and*
(3) *$f_1 - f_2 = \alpha^2 - \beta^2$ and $f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta$.*

Proof. Applying ∇_X to $(3.13)_1$: $h^\ell_j(Y, U_i) = h^*_i(Y, V_j)$ and using (2.1), (2.12), (3.5), (3.7), $(3.11)_1$, (3.12), $(3.13)_{1,2,3}$, (3.14) and (3.15), we obtain

$$\begin{aligned}
(\nabla_X h^\ell_j)(Y, U_i) = \;& (\nabla_X h^*_i)(Y, V_j)\\
&- \sum_{k=1}^{r}\{\tau_{kj}(X)h^\ell_k(Y, U_i) + \tau_{ik}(X)h^*_k(Y, V_j)\}\\
&- \sum_{a=r+1}^{n}\{\lambda_{aj}(X)h^s_a(Y, U_i) + \epsilon_a\rho_{ia}(X)h^s_a(Y, V_j)\}\\
&+ \sum_{k=1}^{r}\{h^*_i(Y, U_k)h^\ell_k(X, \xi_j) + h^*_i(X, U_k)h^\ell_k(Y, \xi_j)\}\\
&- g(A^*_{\xi_j}X, F(A_{N_i}Y)) - g(A^*_{\xi_j}Y, F(A_{N_i}X))\\
&- \sum_{k=1}^{r} h^\ell_j(X, V_k)\eta_k(A_{N_i}Y)\\
&- \alpha(\beta+\ell)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}\\
&- \alpha^2 u_j(Y)\eta_i(X) - (\beta+\ell)^2 u_j(X)\eta_i(Y).
\end{aligned}$$

Substituting this equation and $(3.13)_1$ into (4.6) [which is changed i by j] such that $Z = U_i$ and using $(3.8)_3$, $(3.13)_3$ and $(4.5)_3$, we have

$$(\nabla_X h^*_i)(Y, V_j) - (\nabla_Y h^*_i)(X, V_j)$$

$$
\begin{aligned}
&- \sum_{k=1}^{r}\{\tau_{ik}(X)h_k^*(Y,V_j) - \tau_{ik}(Y)h_k^*(X,V_j)\} \\
&+ \sum_{k=1}^{r}\{h_k^\ell(X,V_j)\eta_i(A_{N_k}Y) - h_k^\ell(Y,V_j)\eta_i(A_{N_k}X)\} \\
&- \sum_{a=r+1}^{n}\epsilon_a\{\rho_{ia}(X)h_a^s(Y,V_j) - \rho_{ia}(Y)h_a^s(X,V_j)\} \\
&- \ell\{\theta(X)h_i^*(Y,V_j) - \theta(Y)h_i^*(X,V_j)\} \\
&- m\{\theta(X)h_i^*(FY,V_j) - \theta(Y)h_i^*(FX,V_j)\} \\
&+ \{(Xm)\theta(Y) + m(\bar\nabla_X\theta)(Y) \\
&\qquad - (Ym)\theta(X) - m(\bar\nabla_Y\theta)(X)\}\delta_{ij} \\
&- \alpha(2\beta+\ell)\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\} \\
&- \{\alpha^2 - (\beta+\ell)^2\}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} \\
= \;& f_2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y) + 2\delta_{ij}\bar g(X,JY)\}.
\end{aligned}
$$

Comparing this with (4.8) such that $PZ = V_j$ and using $(4.5)_2$, we obtain

$$
\begin{aligned}
&\{f_1 - f_2 - \alpha^2 + \beta^2\}\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} \\
= \;& 2\alpha\beta\{u_j(Y)v_i(X) - u_j(X)v_i(Y)\}.
\end{aligned}
$$

Taking $Y = U_j$, $X = \xi_i$ and $Y = U_j$, $X = V_i$ to this by turns, we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0.$$

Applying $\bar\nabla_X$ to $\eta_i(Y) = \bar g(Y, N_i)$ and using (2.5), we obtain

$$(\nabla_X\eta_i)(Y) = -g(A_{N_i}X, Y) + \sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y). \tag{4.9}$$

Applying ∇_Y to (3.12) and using (3.7), (3.10), (3.12), (3.19) and (4.9), we have

$$
\begin{aligned}
(\nabla_X h_i^*)(Y,\zeta) = \;& -(X\alpha)v_i(Y) + X(\beta+\ell)\eta_i(Y) \\
&+ \alpha\{g(A_{N_i}X, FY) + g(A_{N_i}Y, FX) - \sum_{j=1}^{r}v_j(Y)\tau_{ij}(X) \\
&- \sum_{a=r+1}^{n}\epsilon_a w_a(Y)\rho_{ia}(X) - \sum_{j=1}^{r}u_j(Y)\eta_i(A_{N_j}X)\} \\
&- (\beta+\ell)\{g(A_{N_i}X, Y) + g(A_{N_i}Y, X) - \sum_{j=1}^{r}\tau_{ij}(X)\eta_j(Y)\} \\
&+ \alpha^2\theta(Y)\eta_i(X) + (\beta+\ell)^2\theta(X)\eta_i(Y) \\
&+ \alpha\ell\{\theta(Y)v_i(X) - \theta(X)v_i(Y)\}.
\end{aligned}
$$

Substituting this and $(3.12)_2$ into (4.8) with $PZ = \zeta$ and using $(4.5)_5$, we get

$$\begin{aligned}&\{X\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(X)\}\eta_i(Y)\\ &- \{Y\beta + (f_1 - f_3 - \alpha^2 + \beta^2)\theta(Y)\}\eta_i(X)\\ =\ &(X\alpha)v_i(Y) - (Y\alpha)v_i(X).\end{aligned}$$

Taking $X = \zeta$, $Y = \xi_i$ and $X = U_j$, $Y = V_i$ to this by turns, we have

$$f_1 - f_3 = \alpha^2 - \beta^2 - \zeta\beta, \qquad U_j\alpha = 0.$$

Applying ∇_Y to $(3.11)_1$ and using (3.10), $(3.11)_1$ and (3.18), we obtain

$$\begin{aligned}(\nabla_X h_i^\ell)(Y,\zeta) = &-(X\alpha)u_i(Y) - (\beta+\ell)h_i^\ell(Y,X)\\ &+ \alpha\{\sum_{j=1}^{r} u_j(Y)\tau_{ji}(X) + \sum_{a=r+1}^{n} \epsilon_a w_a(Y)\lambda_{ai}(X)\\ &+ h_i^\ell(X,FY) + h_i^\ell(Y,FX)\\ &+ \ell[\theta(Y)u_i(X) - \theta(X)u_i(Y)]\}.\end{aligned}$$

Substituting this into (4.6) with $Z = \zeta$ and using (3.3) and (3.11), we have

$$(X\alpha)u_i(Y) = (Y\alpha)u_i(X).$$

Taking $Y = U_i$ to this result and using the fact that $U_i\alpha = 0$, we have $X\alpha = 0$. Therefore α is a constant. This completes the proof of the theorem. □

Definition. (1) A screen distribution $S(TM)$ is said to be *totally umbilical* [5] in M if there exist smooth functions γ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X,PY) = \gamma_i g(X,PY).$$

In case $\gamma_i = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) A lightlike submanifold M is said to be *screen conformal* [7] if there exist non-vanishing smooth functions φ_i on a neighborhood $\mathcal{U}$ such that

$$h_i^*(X,PY) = \varphi_i h_i^\ell(X,PY). \tag{4.10}$$

Theorem 4.2. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1,f_2,f_3)$ with an (ℓ,m)-type metric connection such that ζ is tangent to M. If one of the following three conditions satisfies*;

(1) *F is parallel with respect to the connection ∇,*
(2) *U_is are parallel with respect to the connection ∇,*
(3) *$S(TM)$ is totally umbilical, or*
(4) *M is screen conformal,*

then $\bar{M}(f_1,f_2,f_3)$ is an indefinite β-Kenmotsu manifold such that

$$\alpha = 0, \quad \beta = -\ell, \qquad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta\beta.$$

Proof. (1) Assume that F is parallel with respect to ∇. As $\alpha = 0$ and $\beta = -\ell$, $\bar{M}(f_1, f_2, f_3)$ is an indefinite β-Kenmotsu manifold and $f_1 - f_2 = -\beta^2$. Taking the scalar product with U_j to $(3.22)_1$ and using $(3.23)_1$, we get

$$h_i^*(Y, U_j) = 0.$$

Applying ∇_X to this equation and using (3.20), we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting these equations into (4.6) with $PZ = U$ and using (3.21), we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0,$$

due to $f_1 = -\beta^2$. Taking $X = V_j$ and $Y = \xi_i$ to this equation, we obtain $f_2 = 0$. Therefore, $f_1 = -\beta^2$, $f_2 = 0$ and $f_3 = \zeta\beta$ by Theorem 4.1.

(2) If U_is are parallel with respect to ∇, then we have (3.24):

$$\alpha = 0,\ \beta = -\ell\,;\ \tau_{ij} = 0,\ \eta_j(A_{N_i}X) = 0,\ \rho_{ia} = 0,\ h_i^*(X, U_j) = 0.$$

As $\alpha = 0$ and $\beta = -\ell$, we get $f_1 + \beta^2 = f_2$ and $f_1 - f_3 = -\beta^2 - \zeta\beta$ by Theorem 4.1. Applying ∇_Y to $(3.24)_6$ and using the fact that $\nabla_Y U_j = 0$, we obtain

$$(\nabla_X h_i^*)(Y, U_j) = 0.$$

Substituting this into (4.8) with $PZ = U_j$ and using (3.24), we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0,$$

due to the facts: $f_1 + \beta^2 = f_2$ and $(\bar{\nabla}_X\theta)(U_i) = 0$ by $(4.5)_3$. Taking $X = \xi_i$ and $Y = V_j$ to the last equation, we get $f_2 = 0$. Thus $f_1 = -\beta^2$ and $f_3 = \zeta\beta$.

(3) If $S(TM)$ is totally umbilical, then (3.12) is reduced to

$$\gamma_i\theta(X) = -\alpha v_i(X) + (\beta + \ell)\eta_i(X).$$

Taking $X = \zeta$, $X = V_i$ and $X = \xi_i$ to this equation by turns, we have

$$\gamma_i = 0, \qquad \alpha = 0, \qquad \beta = -\ell. \tag{4.11}$$

As $\alpha = 0$ and $\beta = -\ell \neq 0$, $\bar{M}$ is an indefinite β-Kenmotsu manifold and $f_1 + \beta^2 = f_2$ by Theorem 4.1. As $\gamma_i = 0$, $S(TM)$ is totally geodesic.

As $h_i^* = 0$, from $(3.13)_{1,\,2}$, we get

$$h_j^\ell(X, U_i) = 0, \qquad h_a^s(X, U_i) = 0. \tag{4.12}$$

Taking $PZ = U_j$ to (4.8) and using $(4.5)_3$, (4.11) and (4.12), we have

$$f_2\{[v_j(Y)\eta_i(X) - v_j(X)\eta_i(Y)] + [v_i(Y)\eta_j(X) - v_i(X)\eta_j(Y)]\} = 0,$$

due to $f_1 + \beta^2 = f_2$. Taking $X = \xi_i$ and $Y = U_j$, we obtain $f_2 = 0$. Therefore,

$$\alpha = 0, \quad \beta = -\ell \neq 0, \qquad f_1 = -\beta^2, \quad f_2 = 0, \quad f_3 = \zeta\beta.$$

(4) If M is screen conformal, then, from $(3.11)_1$, (3.12) and (4.10), we have

$$\alpha v_i(X) - (\beta + \ell)\eta_i(X) = \alpha\varphi_i u_i(X)\}.$$

Taking $X = V_i$ and $X = \xi_i$ to this by turns, we see that

$$\alpha = 0, \qquad \beta = -\ell. \tag{4.13}$$

Denote by $\mathcal{U}_i^*$ the r-th vector fields on $S(TM)$ such that $\mathcal{U}_i^* = U_i - \varphi_i V_i$. Using $(3.13)_{1,3}$, $(3.13)_{2,4}$ and (4.10), we see that

$$h_j^\ell(X, \mathcal{U}_i^*) = 0, \qquad h_a^s(X, \mathcal{U}_i^*) = 0, \qquad J\mathcal{U}_i^* = N_i - \varphi_i \xi_i. \tag{4.14}$$

Applying ∇_X to $\mathcal{U}_i^* = U_i - \varphi_i V_i$ and using (3.14) and (3.15), and then, taking the scalar product with ζ to the resulting equation, we obtain $g(\nabla_X \mathcal{U}_i^*, \zeta) = 0$. Applying $\bar{\nabla}_X$ to $\theta(\mathcal{U}_i^*) = 0$ and using (2.4) and the last equation, we get

$$(\bar{\nabla}_X \theta)(\mathcal{U}_i^*) = 0. \tag{4.15}$$

Applying ∇_Y to (4.10), we have

$$(\nabla_X h_i^*)(Y, PZ) = (X\varphi_i) h_i^\ell(Y, PZ) + \varphi_i (\nabla_X h_i^\ell)(Y, PZ).$$

Substituting this equation and (4.10) into (4.8) and using (4.6), we have

$$\begin{aligned}
&\sum_{j=1}^{r} \{(X\varphi_i)\delta_{ij} - \varphi_i \tau_{ji}(X) - \varphi_j \tau_{ij}(X) - \eta_i(A_{N_j} X)\} h_j^\ell(Y, PZ) \\
&- \sum_{j=1}^{r} \{(Y\varphi_i)\delta_{ij} - \varphi_i \tau_{ji}(Y) - \varphi_j \tau_{ij}(Y) - \eta_i(A_{N_j} Y)\} h_j^\ell(X, PZ) \\
&- \sum_{a=r+1}^{n} \{\epsilon_a \rho_{ia}(X) + \varphi_i \lambda_{ai}(X)\} h_a^s(Y, PZ) \\
&+ \sum_{a=r+1}^{n} \{\epsilon_a \rho_{ia}(Y) + \varphi_i \lambda_{ai}(Y)\} h_a^s(X, PZ) \\
&- \{(X\ell)\theta(PZ) + \ell(\bar{\nabla}_X \theta)(PZ) + \ell\beta g(X, PZ) - m\alpha\theta(X)\theta(PZ)\}\eta_i(Y) \\
&+ \{(Y\ell)\theta(PZ) + \ell(\bar{\nabla}_Y \theta)(PZ) + \ell\beta g(Y, PZ) - m\alpha\theta(Y)\theta(PZ)\}\eta_i(X) \\
&+ \{(Xm)\theta(Y) + m(\bar{\nabla}_X \theta)(Y) \\
&\qquad - (Ym)\theta(X) - \ell(\bar{\nabla}_Y \theta)(X)\} g(PZ, \mathcal{U}_i^*)) \\
&- \ell\alpha\{g(Y, PZ) g(X, \mathcal{U}_i^*) - g(X, PZ) g(Y, \mathcal{U}_i^*)\} \\
&- m(\beta + \ell)\{\theta(Y) g(X, \mathcal{U}_i^*) - \theta(X) g(Y, \mathcal{U}_i^*)\}\theta(PZ) \\
&= f_1\{g(Y, PZ)\eta_i(X) - g(X, PZ)\eta_i(Y)\} \\
&\quad + f_2\{g(\mathcal{U}_i^*, Y)\bar{g}(X, JPZ) - g(\mathcal{U}_i^*, X)\bar{g}(Y, JPZ) + 2g(\mathcal{U}_i^*, PZ)\bar{g}(X, JY)\} \\
&\quad + f_3\{\theta(X)\eta_i(Y) - \theta(Y)\eta_i(X)\}\theta(PZ).
\end{aligned}$$

Taking $X = \xi_i$, $Y = V_j$ and $PZ = \mathcal{U}_j^*$ to this equation and using $(4.5)_{1,2}$ and $(4.13) \sim (4.15)$, we have $f_1 + f_2 = -\beta^2$. As $f_1 - f_2 = -\beta^2$ by Theorem 4.1, we have $f_2 = 0$ and $f_1 = -\beta^2$. Consequently, we obtain $f_3 = \zeta\beta$. □

Theorem 4.3. *Let M be a generic lightlike submanifold of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with an (ℓ, m)-type metric connection such that ζ is tangent to M. If V_is are parallel with respect to ∇, then $\bar{M}(f_1, f_2, f_3)$ is an indefinite space form such that*

$$\alpha = -m, \quad \beta = -\ell, \qquad f_1 = -\beta^2, \quad f_2 = -\alpha^2, \quad f_3 = -\alpha^2 + \zeta\beta.$$

Proof. If V_is are parallel with respect to ∇, then we have (3.25):

$$\beta = -\ell, \quad \tau_{ij} = 0, \quad h_j^\ell(X, \xi_i) = 0, \quad \lambda_{ai} = 0, \quad h_i^\ell(X, U_j) = 0.$$

Taking $Y = \xi_j$ and $Y = U_j$ to (3.3) by turns and using $(3.25)_{3,5}$, we have

$$h_i^\ell(\xi_j, X) = 0, \qquad h_i^\ell(U_j, X) = m\theta(X)\delta_{ij}.$$

Using these two equations and $(3.13)_4$, we see that

$$\begin{aligned} &h_k^\ell(\xi_i, V_j) = 0, \qquad h_a^s(\xi_i, V_j) = \epsilon_a h_j^\ell(\xi_i, W_a) = 0,\\ &h_k^\ell(U_j, V_j) = 0, \qquad h_a^s(U_j, V_j) = \epsilon_a h_j^\ell(U_j, W_a) = 0. \end{aligned} \tag{4.16}$$

From $(3.13)_1$ and $(3.25)_5$, we have

$$h_i^*(Y, V_j) = 0.$$

Applying ∇_X to this equation and using the fact that $\nabla_X V_j = 0$, we have

$$(\nabla_X h_i^*)(Y, V_j) = 0.$$

Substituting the last two equations into (4.8) such that $PZ = V_j$ and using (3.25), $(4.5)_2$: $(\bar{\nabla}_X\theta(V_j) = 0$ and the fact that $\alpha\ell = -\alpha\beta = 0$, we obtain

$$\begin{aligned} &\sum_{k=1}^{r} \{h_k^\ell(X, V_j)\eta_i(A_{N_k}Y) - h_k^\ell(Y, V_j)\eta_i(A_{N_k}X)\}\\ &\quad + \sum_{a=r+1}^{n} \epsilon_a\{\rho_{ia}(Y)h_a^s(X, V_j) - \rho_{ia}(X)h_a^s(Y, V_j)\}\\ &\quad + \{(Xm)\theta(Y) + m(\bar{\nabla}_X\theta)(Y) - (Ym)\theta(X) - m(\bar{\nabla}_Y\theta)(X)\}\}\delta_{ij}\\ &\quad - \beta^2\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\}\\ &= f_1\{u_j(Y)\eta_i(X) - u_j(X)\eta_i(Y)\} + 2f_2\delta_{ij}\bar{g}(X, JY). \end{aligned}$$

Taking $X = \xi_i$ and $Y = U_j$ to this equation and using $(4.5)_{1,3}$, (4.16) and the fact that $\alpha = -m$, we obtain $f_1 + 2f_2 = -2\alpha^2 - \beta^2$. As $f_1 - f_2 = \alpha^2 - \beta^2$, we get $f_2 = -\alpha^2$. Thus $f_1 = -\beta^2$ and $f_3 = -\alpha^2 + \zeta\beta$. □

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