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OPTIMIZATIONS ON TOTALLY REAL SUBMANIFOLDS OF LCS-MANIFOLDS USING CASORATI CURVATURES

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Abstract. In the present paper, we derive two optimal inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to Levi-Civita connection and quarter symmetric metric connection by using T. Oprea's optimization method.

1. Introduction

The concept of Lorentzian concircular structure manifold (LCS-manifolds) is studied by A. A. Shaikh as a generalization of LP-Sasakian manifolds in [18]. These manifolds are of great interest in the general theory of relativity and cosmology [19, 20]. Many researchers have studied LCS-manifolds (for example [1, 8–11, 21]).

The notion of semi-symmetric linear connection on smooth manifolds is initiated by Friedmann and Schouten in [4]. Later on, Golab has introduced the idea of quarter symmetric linear connection on such smooth manifolds as a generalization of semi-symmetric connection in [6].

In 1890, F. Casorati [2] has defined Casorati curvature and used it at the place of traditional Gauss curvature. The geometrical importance of the Casorati curvatures has been discussed by many researchers [3, 7, 12, 25]. Due to its vast geometric significance it drew attention of researchers to construct optimal inequalities for Casorati curvatures for different set ups [5, 13, 14, 22–24, 26, 27].

The outline of the present paper is as follows: Section 2 is preliminary in nature. Section 3 deals with the study of Casorati curvatures. Section 4 derives the optimal inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to Levi-Civita connection. Section 5 gives the proof of the geometric inequalities for totally real submanifolds and C-totally real submanifolds of LCS-manifolds with respect to quarter symmetric metric connection.

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2. LCS-manifolds and their submanifolds

Definition ([16, 18])**.** A Lorentzian manifold $\overline{M}$ together with the unit time-like concircular vector field ξ, its associated 1-form η and an $(1,1)$ tensor field φ is said to be a Lorentzian concircular structure manifold (or LCS-manifold).

In an n-dimensional $(\mathrm{LCS})_n$-manifold $\overline{M}$, $n > 2$, the following relations hold [18]:

$$\eta(\xi) = -1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \tag{1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \varphi^2 X = X + \eta(X)\xi, \tag{2}$$

$$\overline{R}(X, Y)Z = \varphi\overline{R}(X, Y)Z + (\alpha^2 - \rho)\Big[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\Big]\xi \tag{3}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

We consider $\overline{\nabla}$ is the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfying the following:

$$\overline{\nabla}_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X) \tag{4}$$

for any $X \in \Gamma(T\overline{M})$, where ρ is a certain scalar function given by

$$\rho = -(\xi\alpha). \tag{5}$$

Also,

$$\begin{aligned} \overline{R}(X, Y, Z, W) = \overline{R}(X, Y, Z, \varphi W) + (\alpha^2 - \rho)\Big[&g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\Big]\eta(W) \end{aligned} \tag{6}$$

for any $X, Y, Z, W \in \Gamma(T\overline{M})$.

Remark 2.1. If we assume that $\alpha = 1$, then Lorentzian concircular structure becomes LP-Sasakian structure [15].

Let M be an m-dimensional submanifold of an n-dimensional manifold $\overline{M}$ with induced metric g. The Gauss equation is given by [29]

$$\begin{aligned} \overline{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ g(\zeta(X, Z), \zeta(Y, W)) \\ &- g(\zeta(X, W), \zeta(Y, Z)) \end{aligned} \tag{7}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Here ζ is the second fundamental form of M in $\overline{M}$.

Definition ([6])**.** A linear connection $\hat{\overline{\nabla}}$ in an n-dimensional smooth manifold $\overline{M}$ is said to be a quarter symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \hat{\overline{\nabla}}_X Y - \hat{\overline{\nabla}}_Y X - [X, Y] = \eta(Y)\varphi X - \eta(X)\varphi Y,$$

where η is an 1-form and φ is a tensor of type $(1,1)$.

Remark 2.2. If we assume that $\varphi X = X$, then the quarter symmetric connection reduces to semi-symmetric connection.

Definition ([6]). The quarter symmetric connection $\hat{\overline{\nabla}}$ is said to be a quarter symmetric metric connection if $\hat{\overline{\nabla}}$ satisfies the following condition:

$$(\hat{\overline{\nabla}}_X g)(Y, Z) = 0$$

for any $X, Y, Z, W \in \Gamma(T\overline{M})$.

The relation between quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$ on a $(\mathrm{LCS})_n$-manifold $\overline{M}$ is given by [11]

$$\hat{\overline{\nabla}}_X Y = \overline{\nabla}_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi.$$

If $\hat{\overline{R}}$ and $\overline{R}$ are the curvature tensors of a $(\mathrm{LCS})_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and Riemannian connection $\overline{\nabla}$, then [10]

$$\begin{aligned} \hat{\overline{R}}(X, Y, Z, W) = \overline{R}(X, Y, Z, W) &+ (2\alpha - 1)\Big[g(\varphi X, Z)g(\varphi Y, W) \\ &- g(\varphi Y, Z)g(\varphi X, W)\Big] + \alpha\Big[\eta(Y)g(X, W) \\ &- \eta(X)g(Y, W)\Big]\eta(Z) + \alpha\Big[g(Y, Z)\eta(X) \\ &- g(X, Z)\eta(Y)\Big]\eta(W) \end{aligned} \tag{8}$$

for any $X, Y, Z, W \in \Gamma(TM)$.

Let M be an m-dimensional submanifold of an n-dimensional $(\mathrm{LCS})_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection $\hat{\overline{\nabla}}$ and $\hat{\nabla}$ be the induced connection of M associated to the quarter symmetric metric connection. Also let $\hat{\zeta}$ be the second fundamental form of M with respect to $\hat{\nabla}$. Then the relation (7) becomes

$$\begin{aligned} \hat{\overline{R}}(X, Y, Z, W) = \hat{R}(X, Y, Z, W) &+ g(\hat{\zeta}(X, Z), \hat{\zeta}(Y, W)) \\ &- g(\hat{\zeta}(X, W), \hat{\zeta}(Y, Z)) \end{aligned} \tag{9}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Here $\hat{R}$ is the curvature tensor of M with respect to the induced connection associated to the quarter symmetric metric connection.

Definition ([28, 29]). (i) A submanifold M of a contact metric manifold $\overline{M}$ is said to be anti-invariant if for any X tangent to M, φX is normal to M, i.e., $\varphi(TM) \subset T^{\perp}M$ at every point of M, where $T^{\perp}M$ denotes the normal bundle of M.

(ii) A submanifold M in a contact metric manifold $\overline{M}$ is called a C-totally real submanifold in $\overline{M}$ if every tangent vector of M belongs to the contact distribution.

Remark 2.3. We note that if a submanifold M of a contact metric manifold $\overline{M}$ is normal to the structure vector field ξ, then it is anti-invariant. Also, a submanifold M in a contact metric manifold $\overline{M}$ is a C-totally real submanifold if the structure vector field ξ is normal to M. Therefore it is clear that C-totally real submanifolds in a contact metric manifold are anti-invariant, as they are normal to ξ.

For a totally real submanifold and a C-totally real submanifold of a $(\mathrm{LCS})_n$-manifold $\overline{M}$, $\hat{\zeta}$ is given by [10]

$$\hat{\zeta}(X,Y) = \zeta(X,Y) + \eta(Y)\varphi X \tag{10}$$

and

$$\hat{\zeta}(X,Y) = \zeta(X,Y), \tag{11}$$

respectively, for any $X, Y \in \Gamma(TM)$.

3. Casorati curvatures

Let $\overline{M}$ be an n-dimensional $(\mathrm{LCS})_n$-manifold and M be an m-dimensional submanifold in $\overline{M}$. Let $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ be an orthonormal basis of $T_\wp M$ and $\{\mathcal{E}_{m+1}, \ldots, \mathcal{E}_n\}$ be an orthonormal basis of $T^\perp_\wp M$ at any $\wp \in M$. Then the scalar curvature $\sigma(\wp)$ at $\wp$ is given by

$$\sigma(\wp) = \sum_{1\le \imath<\jmath\le m} K(\mathcal{E}_\imath \wedge \mathcal{E}_\jmath) \tag{12}$$

and the normalized scalar curvature ϱ is given by

$$\varrho = \frac{2\sigma}{m(m-1)}, \tag{13}$$

where $K(\Lambda)$ denotes the sectional curvature of the plane section $\Lambda \subset T_\wp M$.

The mean curvature vector $\mathcal{H}$ is defined as

$$\mathcal{H} = \frac{1}{m}\sum_{\imath,\jmath=1}^{m} \zeta(\mathcal{E}_\imath, \mathcal{E}_\imath) \tag{14}$$

and the squared norm of mean curvature is given by

$$\|\mathcal{H}\|^2 = \frac{1}{m^2}\sum_{a=m+1}^{n}\Big(\sum_{\imath=1}^{m}\zeta^a_{\imath\imath}\Big)^2. \tag{15}$$

We also put

$$\zeta^a_{\imath\jmath} = g(\zeta(\mathcal{E}_\imath, \mathcal{E}_\jmath), \mathcal{E}_a),\ \imath,\jmath \in \{1,2,\ldots,m\},\ a \in \{m+1, m+2, \ldots, n\}.$$

The Casorati curvature $\mathcal{C}$ of M is defined by

$$\mathcal{C} = \frac{1}{m} \sum_{a=m+1}^{n} \sum_{\imath,\jmath=1}^{m} \left(\zeta_{\imath\jmath}^{a}\right)^2. \tag{16}$$

Let us assume an r-dimensional subspace Ψ of TM, $r \geq 2$, whose orthonormal basis is $\{\mathcal{E}_1, \mathcal{E}_2, \ldots, \mathcal{E}_r\}$. Then we have

$$\sigma(\Psi) = \sum_{1\leq\alpha<\beta\leq r} K(\mathcal{E}_\alpha \wedge \mathcal{E}_\beta) \tag{17}$$

and

$$\mathcal{C}(\Psi) = \frac{1}{r} \sum_{a=m+1}^{n} \sum_{\imath,\jmath=1}^{m} \left(\zeta_{\imath\jmath}^{a}\right)^2, \tag{18}$$

where $\sigma(\Psi)$ and $\mathcal{C}(\Psi)$ are the scalar curvature and Casorati curvature of Ψ, respectively.

The following δ-Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\widehat{\delta}_{\mathcal{C}}(m-1)$

$$[\delta_{\mathcal{C}}(m-1)]_{\wp} = \frac{1}{2}\mathcal{C}_{\wp} + \frac{m+1}{2m} \inf\{\mathcal{C}(\Psi)|\Psi : \text{a hyperplane of } T_{\wp}M\} \tag{19}$$

and

$$[\widehat{\delta}_{\mathcal{C}}(m-1)]_{\wp} = 2\mathcal{C}_{\wp} + \frac{2m-1}{2m} \sup\{\mathcal{C}(\Psi)|\Psi : \text{a hyperplane of } T_{\wp}M\} \tag{20}$$

are known as the normalized δ-Casorati curvatures.

Definition ([14])**.** A point $p \in M$ is said to be an invariantly quasi-umbilical point if there exist $n-m$ orthogonal unit normal vectors $\{\mathcal{E}_{m+1}, \ldots, \mathcal{E}_n\}$ such that the shape operator with respect to all directions $\mathcal{E}_r$ have an eigenvalue of multiplicity $m-1$ and that for each $\mathcal{E}_r$ the distinguished eigendirection is the same. The submanifold M is said to be an invariantly quasi-umbilical submanifold if each of its point is an invariantly quasi-umbilical point.

For the main results, we need following lemma:

Lemma 3.1 ([17])**.** *Let M be a Riemannian submanifold of Riemannian manifold $(\overline{M}, \overline{g})$, where g is the induced metric on M from $\overline{g}$ and $\iota : M \to \mathbb{R}$ is a differentiable function. If $y \in M$ is the solution of the constrained extremum problem* $\min_{x\in M} \iota(x)$*, then*

(i) $(grad\ \iota)(y) \in T_y^{\perp}M$;

(ii) *the bilinear form* $Ł : T_yM \times T_yM \to \mathbb{R}$;

$$Ł(X,Y) = \overline{g}(\varsigma(X,Y), (grad\ \iota)(y)) + \mathcal{H}ess_{\iota}(X,Y)$$

is positive semi-definite, where ς is the second fundamental form of M in $\overline{M}$.

4. Main result 1

Theorem 4.1. *Let M be an m-dimensional totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$. Then*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \delta_{\mathcal{C}}(m-1) - \frac{\alpha^2 - \rho}{m}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \dots, n\}$, are of the following form

$$\mathcal{S}_{m+1} = \begin{pmatrix} \beta & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \beta & 0 \\ 0 & \dots & 0 & 2\beta \end{pmatrix}, \ \mathcal{S}_{m+2} = \dots = \mathcal{S}_n = 0. \tag{21}$$

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \widehat{\delta}_{\mathcal{C}}(m-1) - \frac{\alpha^2 - \rho}{m}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1, \dots, n\}$, are of the following form

$$\mathcal{S}_{m+1} = \begin{pmatrix} 2\beta & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 2\beta & 0 \\ 0 & \dots & 0 & \beta \end{pmatrix}, \ \mathcal{S}_{m+2} = \dots = \mathcal{S}_n = 0. \tag{22}$$

Proof. Let $\{\mathcal{E}_1, \dots, \mathcal{E}_m\}$ be an orthonormal frame of $T_\wp M$ and $\{\mathcal{E}_{m+1}, \dots, \mathcal{E}_n\}$ be an orthonormal frame of $T_\wp^\perp M$, $\wp \in M$. From [10], we get

$$2\sigma = -(m-1)(\alpha^2 - \rho) + m^2||\mathcal{H}||^2 - m\mathcal{C}.$$

Let us take a quadratic polynomial $\mathbb{K}$ in the components of the second fundamental form

$$\mathbb{K} = \frac{m(m-1)}{2}\mathcal{C} + \frac{m^2-1}{2}\mathcal{C}(\Psi) + 2\sigma - (m-1)(\alpha^2 - \rho). \tag{23}$$

Without loss of generality, we assume that Ψ is spanned by $\mathcal{E}_1, \dots, \mathcal{E}_{m-1}$ and together (23), we find that

$$\mathbb{K} = \frac{m+1}{2} \sum_{a=m+1}^{n} \left[\sum_{\imath,\jmath=1}^{m} (\zeta_{\imath\jmath}^a)^2 + \sum_{\imath,\jmath=1}^{m-1} (\zeta_{\imath\jmath}^a)^2 \right] - \sum_{a=m+1}^{n} \left[\sum_{\imath=1}^{m} \zeta_{\imath\imath}^a \right]^2. \tag{24}$$

Also,

$$\begin{aligned} \mathbb{K} = & \sum_{a=m+1}^{n} \sum_{\imath=1}^{m-1} \left[m(\zeta_{\imath\imath}^a)^2 + (m+1)(\zeta_{\imath m}^a)^2 \right] \\ & + \sum_{a=m+1}^{n} \left[2(m+1) \sum_{1\le\imath<\jmath\le m-1} (\zeta_{\imath\jmath}^a)^2 - 2 \sum_{1\le\imath<\jmath\le m} \zeta_{\imath\imath}^a \zeta_{\jmath\jmath}^a + \frac{m-1}{2} (\zeta_{mm}^a)^2 \right] \\ \ge & \sum_{a=m+1}^{n} \sum_{\imath=1}^{m-1} m(\zeta_{\imath\imath}^a)^2 + \sum_{a=m+1}^{n} \left[-2 \sum_{1\le\imath<\jmath\le m} \zeta_{\imath\imath}^a \zeta_{\jmath\jmath}^a + \frac{m-1}{2} (\zeta_{mm}^a)^2 \right]. \end{aligned} \tag{25}$$

For $a = m+1, \dots, n$, we suppose the following quadratic form $f_a : \mathbb{R}^m \to \mathbb{R}$,

$$f_a\big(\zeta_{11}^a, \dots, \zeta_{mm}^a\big) = \sum_{\imath=1}^{m-1} m(\zeta_{\imath\imath}^a)^2 - 2 \sum_{1\le\imath<\jmath\le m} \zeta_{\imath\imath}^a \zeta_{\jmath\jmath}^a + \frac{m-1}{2} (\zeta_{mm}^a)^2 \tag{26}$$

and the constrained extremum problem $\min f_a$ subject to the component of trace $\mathcal{H}$,

$$\varphi : \zeta_{11}^a + \dots + \zeta_{mm}^a = \gamma^a,$$

where γ^a is a real constant.

The function f_a has the following partial derivatives:

$$\begin{aligned} \tfrac{\partial f_a}{\partial \zeta_{11}^a} &= 2m\zeta_{11}^a - 2\textstyle\sum_{\imath=2}^{m} \zeta_{\imath\imath}^a, \\ \tfrac{\partial f_a}{\partial \zeta_{22}^a} &= 2m\zeta_{22}^a - 2\zeta_{11}^a - 2\textstyle\sum_{\imath=3}^{m} \zeta_{\imath\imath}^a, \\ &\vdots \\ \tfrac{\partial f_a}{\partial \zeta_{m-1\ m-1}^a} &= 2m\zeta_{m-1\ m-1}^a - 2\textstyle\sum_{\imath=1}^{m-2} \zeta_{\imath\imath}^a - 2\zeta_{mm}^a, \\ \tfrac{\partial f_a}{\partial \zeta_{mm}^a} &= -2\textstyle\sum_{\imath=1}^{m-1} \zeta_{\imath\imath}^a + (m-1)\zeta_{mm}^a. \end{aligned} \tag{27}$$

For an optimal solution $\big(\zeta_{11}^a, \dots, \zeta_{mm}^a\big)$ of the problem in question, the vector *grad* f_a is normal at φ. From (27), we have a following critical point of the considered problem:

$$\zeta_{11}^a = \zeta_{22}^a = \dots = \zeta_{m-1\ m-1}^a = \frac{\gamma^a}{m+1},\ \zeta_{mm}^a = \frac{2\gamma^a}{m+1}. \tag{28}$$

Now, we use Lemma 3.1 and for this, we fix an arbitrary point $y \in \varphi$. The bilinear form

$$Ł : T_y\varphi \times T_y\varphi \to \mathbb{R}$$

is defined by

$$Ł(X,Y) = \langle \hbar(X,Y), (grad\ f_a)(y)\rangle + \mathcal{H}ess_{f_a}(X,Y),$$

where $\hbar$ denotes the second fundamental form of φ in $\mathbb{R}^m$ and $\langle , \rangle$ denotes the standard inner product on $\mathbb{R}^m$. So, we have the following:

$$\begin{aligned}
Ł(Z,Z) = & -2(Z_1,\ldots,Z_{m-1},Z_m) \\
& \begin{pmatrix} -m & 1 & 1 & \ldots & 1 & 1 \\ 1 & -m & 1 & \ldots & 1 & 1 \\ 1 & 1 & -m & \ldots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & \ldots & -m & 1 \\ 1 & 1 & 1 & \ldots & 1 & \frac{1-m}{2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ \vdots \\ Z_{m-1} \\ Z_m \end{pmatrix} \\
&= 2(m+1)\sum_{\imath=1}^{m-1} Z_\imath^2 + (m+1)Z_m^2 - 2(Z_1+\cdots+Z_m)^2 \\
&= 2(m+1)\sum_{\imath=1}^{m-1} Z_\imath^2 + (m+1)Z_m^2 \\
&\geq 0,
\end{aligned}$$

where we have used the relation $\sum_{\imath=1}^{m} Z_\imath^2 = 0$ (because a vector Z is tangent to φ at $y \in \varphi$ and φ is totally geodesic in $\mathbb{R}^m$). Thus, the point $(\zeta_{11}^a,\ldots,\zeta_{mm}^a)$ (see (28)) is a global minimum point. From relations (25) and (28), we get $\mathbb{K} \geq 0$ and hence we have

$$2\sigma \leq m(m-1)\mathcal{C} + \frac{m^2-1}{2}\mathcal{C}(\Psi) - (m-1)(\alpha^2-\rho).$$

Further, we find that

$$\varrho \leq \mathcal{C} + \frac{(m+1)}{2m}\mathcal{C}(\Psi) - \frac{\alpha^2-\rho}{m}.$$

This is the required inequality in (i). The equality in (i) holds if and only if

$$\zeta_{\imath\jmath}^a = 0,\ \forall\ \imath,\jmath \in \{1,\ldots,m\},\ \imath \neq \jmath,\ a \in \{m+1,\ldots,n\} \tag{29}$$

and

$$\zeta_{mm}^a = 2\zeta_{11}^a = \cdots = 2\zeta_{m-1\ m-1}^a\ \forall a \in \{m+1,\ldots,n\}. \tag{30}$$

With the help of (29) and (30), we find that the submanifold is invariantly quasi-umbilical and the shape operators are given by (21).

Similarly, one can easily prove the geometric inequality (ii). □

Corollary 4.2. *Let M be an m-dimensional C-totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$. Then:*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \delta_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1,\ldots,n\}$, is given by (21).

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq \widehat{\delta}_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1,\ldots,n\}$, is given by (22).

5. Main result 2

Let $\{\mathcal{E}_1,\ldots,\mathcal{E}_n\}$ be an orthonormal basis of the tangent space $\overline{M}$ and $\mathcal{N}$ be a unit tangent vector at $\wp \in \overline{M}^n$ such that $\mathcal{E}_1 = \mathcal{N}$ refracting to M^m, $\{\mathcal{E}_1,\ldots,\mathcal{E}_m\}$ is the orthonormal basis to the tangent space $T_\wp M$ with respect to induced quarter symmetric metric connection. Let us denote the scalar curvature and normalized scalar curvature of M with respect to induced connection associated to the quarter symmetric metric connection by $\hat{\sigma}(\wp)$ at $\wp$ and $\hat{\varrho}$, respectively. Then we prove the following:

Theorem 5.1. *Let M be an m-dimensional totally real submanifold in an n-dimensional $(LCS)_n$-manifold $\overline{M}$ with respect to quarter symmetric metric connection. Then*

(i) *The normalized δ-Casorati curvature $\delta_{\mathcal{C}}(m-1)$ satisfies*

$$\hat{\varrho} \leq \delta_{\mathcal{C}}(m-1) - \frac{(2m-1)\alpha}{m(m-1)} - \frac{\alpha\eta^2(\mathcal{N})}{m-1}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, such that with respect to orthonormal frames of $T_\wp M$ and $T_\wp^\perp M$, respectively, the shape operators $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$, $a \in \{m+1,\ldots,n\}$, is given by (21).

(ii) *The normalized δ-Casorati curvature $\widehat{\delta}_{\mathcal{C}}(m-1)$ satisfies*

$$\hat{\varrho} \leq \widehat{\delta}_{\mathcal{C}}(m-1) - \frac{(2m-1)\alpha}{m(m-1)} - \frac{\alpha\eta^2(\mathcal{N})}{m-1}.$$

Furthermore, the equality sign holds if and only if M is an invariantly quasi-umbilical submanifold with trivial normal connection in $\overline{M}(c)$, *such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (22)*.*

Proof. Following [10], we have

$$2\hat{\sigma} = -(2m-1)\alpha - m\alpha\eta^2(\mathcal{N}) + m^2||\mathcal{H}||^2 - ||\zeta||^2. \tag{31}$$

Again by using T. Oprea's optimization technique, one can prove the theorem. □

Note that the scalar curvature and hence the normalized scalar curvature of C-totally real submanifold of a $(\mathrm{LCS})_n$-manifold with respect to induced Levi-Civita connection and induced quarter symmetric metric connection are identical (see [10]). Thus, we have the following:

Corollary 5.2. *Let* M *be an* m*-dimensional* C*-totally real submanifold in an* n*-dimensional* $(LCS)_n$*-manifold* $\overline{M}$ *with respect to quarter symmetric metric connection. Then*

(i) *The normalized* δ*-Casorati curvature* $\delta_{\mathcal{C}}(m-1)$ *satisfies*

$$\hat{\varrho} \leq \delta_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\overline{M}(c)$*, such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (21)*.*

(ii) *The normalized* δ*-Casorati curvature* $\widehat{\delta}_{\mathcal{C}}(m-1)$ *satisfies*

$$\hat{\varrho} \leq \widehat{\delta}_{\mathcal{C}}(m-1).$$

Furthermore, the equality sign holds if and only if M *is an invariantly quasi-umbilical submanifold with trivial normal connection in* $\overline{M}(c)$*, such that with respect to orthonormal frames of* $T_\wp M$ *and* $T_\wp^\perp M$*, respectively, the shape operators* $\mathcal{S}_a \equiv \mathcal{S}_{\mathcal{E}_a}$*,* $a \in \{m+1, \ldots, n\}$*, is given by* (22)*.*

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