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ALGEBRAS OF GELFAND-CONTINUOUS FUNCTIONS INTO ARENS-MICHAEL ALGEBRAS

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Abstract. We characterize Gelfand-continuous functions from a Tychonoff space X into an Arens-Michael algebra A. Then we define several algebras of such functions, and investigate them as topological algebras. Finally, we provide a class of examples of (metrizable) commutative unital complete Arens-Michael algebras A and locally compact spaces X for which all these algebras differ from each other.

1. Introduction

The Gelfand representation plays an important role in the theory of Banach algebras and C^*-algebras. Thanks to it, it is possible to show, for example, the well known Gelfand-Naimark theorem stating that a commutative C^*-algebra with unit is necessarily the algebra of continuous functions on some compact space X (see [3]). This theorem has extensions to involutive Banach algebras and to some general locally convex algebras, see for instance [9]. Recently, the Gelfand representation has been used by J. Rakbud [11] to investigate "the continuity of Banach algebra-valued continuous functions". For a compact topological space K and a unital commutative Banach algebra A with carrier space Δ and Gelfand representation G, he considered A-valued functions f defined on K such that $G \circ f$ is continuous as a $C(\Delta)$-valued function, $C(\Delta)$ being the algebra of all continuous functions on Δ endowed with the topology of uniform convergence. The author dealt with some algebras of such functions and proceeded to a comparison of them. He also provided conditions under which some of these algebras are Banach algebras or coincide with each other.

As it is known, there are algebras which cannot be endowed with any Banach algebra norm, and topological spaces which are not compact. Therefore, one may ask whether similar results as those of Rakbud may hold when the algebra A is no more a Banach one and K is no longer compact. Actually, as we will

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To the memory of my sister Fatima Bassou, Tabassout.

show, it seems that the study of such a situation is richer than the former one. Because there may exist neighborhoods of the origin which are not bounded and bounding subsets of a completely regular space which are not relatively compact, several new phenomena appear and this gives birth to further algebras. In this note, we consider an arbitrary Hausdorff completely regular (i.e., a Tychonoff) space X and an Arens-Michael algebra A and deal with different algebras of A-valued functions f on X such that $G \circ f$ is continuous, G being again the Gelfand representation of A. We first characterize such functions and, depending on the continuity or boundedness condition requested, we exhibit several classes of such functions which turn out to be algebras. We further endow such algebras with appropriate topologies and provide conditions under which some of them are complete Arens-Michael ones. An interesting example, relying on a locally compact space similar to the famous space ψ of [6, page 79, 5I], is given showing that all such algebras (in number of seven + one) may be simultaneously pairwise different.

In Section 2 we give some preliminaries and preliminary results concerning the general theory of Hausdorff completely regular spaces and Arens-Michael algebras. In Section 3 we characterize the Gelfand-continuous functions from a Tychonoff space X into an Arens-Michael algebra A. According to continuity or boundedness conditions requested, we distinguish different algebras of Gelfand-continuous functions and provide a first comparison of them. Section 4 is devoted to the study of the completeness of such algebras with respect to appropriate topologies. In Section 5, we construct a concrete example of a commutative complete Arens-Michael algebra A with unit, we determine its carrier space Δ and give some of the topological properties of Δ. In the last section, we produce an example of a locally compact space X such that all the different algebras considered in Section 3, with domain X and target algebra A, are simultaneously pairwise distinct.

2. Preliminaries and preliminary results

Let X be a Hausdorff completely regular space. We will denote by $C(X)$ the algebra of all $\mathbb{C}$-valued continuous functions on X and by $\mathcal{K}_X$ the set of all compact subsets of X. For every $x \in X$, $\mathcal{V}_x$ will denote the filter of neighborhoods of x in X.

We will say that a subset B of X is bounding if every function $f \in C(X)$ is bounded on B. It will be said to be C-embedded if every continuous function on B extends continuously to the whole of X. The set of all bounding subsets of X will be denoted by $\mathcal{B}_X$, while $\mathcal{C}_X$ will stand for the set of C-embedded ones. It is known that a closed subset of a normal space and a compact subset of a Hausdorff completely regular space are C-embedded [6].

If $\mathcal{P} \subset \mathcal{B}_X$ is a directed upward cover of X with respect to the inclusion, we will consider on $C(X)$ the topology, denoted by $\tau_{\mathcal{P}}$, of uniform convergence on the members of $\mathcal{P}$ (see [12] for details). The obtained topological algebra will be denoted by $C_{\mathcal{P}}(X)$.

We start with the following easy proposition whose proof relies on the fact that the topology of a Tychonoff space X is the one defined by $C(X)$.

Proposition 2.1 ([8]). *Let $\mathcal{F}$ be a family of subsets of X and B an arbitrary (but fixed) separated topological vector space. Then the following are equivalent*:

(1) *Every function from X into an arbitrary Hausdorff completely regular space, whose restrictions to the elements of $\mathcal{F}$ are continuous, must be continuous on X.*

(2) *Every function from X into B, whose restrictions to the elements of $\mathcal{F}$ are continuous must be continuous on X.*

(3) *Every function from X into $\mathbb{R}$, whose restrictions to the elements of $\mathcal{F}$ are continuous, must be continuous on X.*

We will say that X is an $\mathcal{F}_{\mathbb{R}}$-space if it satisfies one of the three assertions of Proposition 2.1. It is clear that, whenever $\mathcal{F} = \mathcal{K}_X$, we retrieve the notion of $k_{\mathbb{R}}$-space and whenever $\mathcal{F} = \mathcal{B}_X$, we retrieve the notion of $b_{\mathbb{R}}$-space of [5].

Recall that a subset $\mathcal{P}_0$ of $\mathcal{P}$ is said to be cofinal, if every element of $\mathcal{P}$ is contained in some element of $\mathcal{P}_0$.

These notions allow us to characterize the complete spaces $C_{\mathcal{P}}(X)$.

Proposition 2.2. *If $\mathcal{P}$ has a cofinal subset $\mathcal{P}_0$ consisting of C-embedded subsets of X, then $C_{\mathcal{P}}(X)$ is complete if and only if X is a $\mathcal{P}_{\mathbb{R}}$-space.*

Proof. Necessity: Assume that $C_{\mathcal{P}}(X)$ is complete and that $f : X \to \mathbb{R}$ has all its restrictions to the elements of $\mathcal{P}$ continuous. For every $B \in \mathcal{P}_0$, denote by f_B a continuous extension of the restriction $f_{|B}$ to B of f. With the inclusion as order on $\mathcal{P}_0$, the net $(f_B)_{B\in\mathcal{P}_0}$ is Cauchy in $C_{\mathcal{P}}(X)$. Since the latter is complete and $\mathcal{P}_0$ is cofinal in $\mathcal{P}$, it converges to some $g \in C(X)$. Now, since $f = g$ pointwise and g is continuous, f is also continuous. Therefore, since f was arbitrary, X is a $\mathcal{P}_{\mathbb{R}}$-space.

Sufficiency: Assume that X is a $\mathcal{P}_{\mathbb{R}}$-space and let $(f_i)_{i\in I}$ be a Cauchy net in $C_{\mathcal{P}}(X)$. Then for every $B \in \mathcal{P}$ and every $r > 0$, there exists some $i_0 \in I$ such that:

$$|f_i(x) - f_j(x)| < r,\ \forall i, j \geq i_0,\ x \in B. \tag{1}$$

Since $\mathcal{P}$ covers X, the evaluations $f \mapsto f(x)$ are continuous. Therefore $(f_i)_{i\in I}$ converges pointwise on X to some function f. Passing to the limit on j in (1), we get the uniform convergence of $(f_i)_{i\in I}$ to f on B. Hence the restriction to B of f is continuous. Since B is arbitrary in $\mathcal{P}$, f is continuous on the whole X. Again, by (1), $(f_i)_{i\in I}$ converges to f in $C_{\mathcal{P}}(X)$. □

Remark 2.3. 1. Notice that the condition that $\mathcal{P}$ contains a cofinal subcollection consisting of C-embedded subsets of X is needed only in the necessity.

2. If $\mathcal{P} = \mathcal{K}_X$, we get the classical result of S. Warner [13], namely: $C_c(X)$ is complete if and only if X is a $k_{\mathbb{R}}$-space. Here $C_c(X)$ is the space $C(X)$ equipped with the compact open topology τ_c.

An Arens-Michael (or a locally multiplicatively convex) algebra is an associative algebra A over the field $\mathbb{C}$ endowed with a locally convex topology given by a family $\mathbb{P}$ of submultiplicative seminorms (see [2] and [9]). This is:

$$P(xy) \leq P(x)P(y),\ x, y \in A \text{ and } P \in \mathbb{P}.$$

A character on A is any nonzero algebra homomorphism from A onto $\mathbb{C}$. The set of all continuous characters on A will be denoted by Δ, while Δ_P will stand for those characters which are continuous with respect to the seminorm $P \in \mathbb{P}$. We will equip Δ with the relative weak topology $\sigma(A', A)$. This makes of Δ a Hausdorff completely regular space which need not be locally compact. It is easily seen that Δ_P is equicontinuous (and then compact since closed) and that the collection $(\Delta_P)_{P\in\mathbb{P}}$ constitutes a basis for the equicontinuous subsets of Δ. Actually we have $|\chi(x)| \leq P(x)$, $\forall \chi \in \Delta_P$, $\forall x \in A$.

Henceforth, unless the contrary is expressly stated, (A, τ) will be a complete Arens-Michael commutative algebra with unit e whose topology is given by a family $\mathbb{P}$ of submultiplicative seminorms such that $P(e) = 1$ for every $P \in \mathbb{P}$. The latter condition is always possible and therefore is not a restriction.

The spectrum of an element $a \in A$ is the set $\mathrm{sp}(a) := \{\alpha \in \mathbb{C} : a - \alpha e$ is not invertible in $A\}$ and its spectral radius is $r(a) := \sup\{|\alpha|,\ \alpha \in \mathrm{sp}(a)\}$. As in the Banach case, one has: $\mathrm{sp}(a) = \{\chi(a), \chi \in \Delta\}$ and therefore $r(a) = \sup\{|\chi(a)|, \chi \in \Delta\}$ (see [9]).

We will endow the algebra $C(\Delta)$ with the topology $\tau_{\mathcal{E}}$, where $\mathcal{E}$ denotes the collection of all equicontinuous subsets of Δ. Of course, this is an Arens-Michael algebra topology. It is generated by the seminorms:

$$r_p(f) := \sup\{|f(\chi)|,\ \chi \in \Delta_P\},\ f \in C(\Delta),\ P \in \mathbb{P}.$$

We will also consider on A the topology $\tau^r_{\mathbb{P}}$ defined by the similar seminorms:

$$r_p(a) := \sup\{|\chi(a)|,\ \chi \in \Delta_P\},\ a \in A,\ P \in \mathbb{P}.$$

The Gelfand representation of A is the mapping $G : A \to C(\Delta)$ defined by $G(x)(\chi) := \chi(x)$, $x \in A$ and $\chi \in \Delta$. Sometimes, we will denote $G(x)$ by $\hat{x}$. The representation G is an algebra homomorphism. It is injective if and only if A is semisimple. Furthermore, since Δ_P is equicontinuous, G is continuous from A into $C_{\mathcal{E}}(\Delta)$. It need be neither surjective nor an isomorphism.

3. Gelfand-continuous functions

In this section we will characterize the A-valued Gelfand-continuous functions and distinguish different classes of them, yielding several algebras of such functions. We will also proceed to a comparison of some of these algebras. First, let us give the following definition whose Banach version is given in [11].

Definition. A function $f : X \to A$ is Gelfand-continuous if the mapping $G \circ f : X \to C_{\mathcal{E}}(\Delta)$ is continuous on X, when $C(\Delta)$ is equipped with the topology $\tau_{\mathcal{E}}$.

It is clear that a continuous function $f : X \to A$ is Gelfand-continuous. The converse need not hold even in the Banach algebra setting as has been shown in [11].

Now, in order to characterize Gelfand-continuous functions, let us fix an upward directed collection $\mathcal{P}$ of bounding subsets of X covering X.

Theorem 3.1. *For a function $f : X \to A$, consider the following assertions*:

(1) *f is Gelfand-continuous,*

(2) *f is continuous when A is endowed with the topology $\tau^r_{\mathbb{P}}$,*

(3) *the set $\{\chi \circ f, \chi \in \Delta_P\}$ is equicontinuous, for every $P \in \mathbb{P}$,*

(4) *the mapping $\psi_f : \chi \mapsto \chi \circ f$ maps continuously Δ into $C_{\mathcal{P}}(X)$.*

Then (1) $\iff$ (2) $\iff$ (3)*.*

If $\mathcal{P} \subset \mathcal{K}_X$ and Δ is an $\mathcal{E}_{\mathbb{R}}$-space, then (3) $\Longrightarrow$ (4)*. Finally if X is a $\mathcal{P}_{\mathbb{R}}$-space, then* (4) $\Longrightarrow$ (1)*.*

Proof. The equivalences (1) $\iff$ (2) $\iff$ (3) are easy.

(3) $\Longrightarrow$ (4): Assume $\mathcal{P} \subset \mathcal{K}_X$ and that Δ is an $\mathcal{E}_{\mathbb{R}}$-space. Since every $\chi \in \Delta$ belongs to some Δ_P and $\{\chi \circ f, \chi \in \Delta_P\}$ is equicontinuous, $\chi \circ f$ is continuous for every $\chi \in \Delta$. Let us show that ψ_f is continuous. Since Δ is an $\mathcal{E}_{\mathbb{R}}$-space, it suffices to show that the restriction of ψ_f to each Δ_P is continuous. Fix then $P \in \mathbb{P}$ and $\chi_0 \in \Delta_P$, and let $B \in \mathcal{P}$ and $\epsilon > 0$ be given. As $\{\chi \circ f, \chi \in \Delta_P\}$ is equicontinuous, for every $x \in B$, there exists $\Omega_x \in \mathcal{V}_x$ such that:

$$|\chi \circ f(y) - \chi \circ f(x)| \leq \epsilon/3, \ y \in \Omega_x, \ \chi \in \Delta_P. \tag{2}$$

But B is compact. Hence there exist $x_1, \dots, x_n \in B$ so that $B \subset \cup\{\Omega_{x_i}, \ i = 1, \dots, n\}$. Let $F := \{f(x_i), \ i = 1, \dots, n\}$ and consider the open neighborhood $V = V(\chi_0, F, \epsilon/3) := \{\chi \in \Delta : |\chi(f(x_i)) - \chi_0(f(x_i))| < \epsilon/3, \ i = 1, \dots, n\}$ of χ_0. Then, for every $x \in B$, there exists some $i \in \{1, \dots, n\}$ satisfying $x \in \Omega_{x_i}$. Therefore, for every $\chi \in V \cap \Delta_P$, we have:

$$\begin{aligned} |\chi(f(x)) - \chi_0(f(x))| &\leq \underbrace{|\chi(f(x)) - \chi(f(x_i))|}_{\leq \epsilon/3 \text{ by } (2)} + \underbrace{|\chi(f(x_i)) - \chi_0(f(x_i))|}_{\leq \epsilon/3 \text{ for } \chi \in V} \\ &\quad + \underbrace{|\chi_0(f(x_i)) - \chi_0(f(x))|}_{\leq \epsilon/3 \text{ by } (2)} \\ &\leq 3\epsilon/3 = \epsilon. \end{aligned}$$

Whence the continuity of ψ_f.

As for (4) $\Longrightarrow$ (1), since X is a $\mathcal{P}_{\mathbb{R}}$-space, it also suffices to show that $G \circ f := \hat{f}$ is continuous on each $B \in \mathcal{P}$. Fix then $B \in \mathcal{P}$, $x \in B$, and given $P \in \mathbb{P}$ and $\epsilon > 0$. Since $\psi_f : \chi \to \chi \circ f$ is continuous on Δ, for every $\chi \in \Delta$, there is a neighborhood V_χ of χ such that:

$$|\phi \circ f(y) - \chi \circ f(y)| \leq \epsilon/3, \ y \in B, \ \phi \in V_\chi. \tag{3}$$

Since Δ_P is compact, there are $\chi_1, \dots, \chi_n \in \Delta_P$ such that $\Delta_P \subset \cup\{V_i := V_{\chi_i},\ i = 1, \dots, n\}$. But $D := \{\chi_i \circ f,\ i = 1, \dots, n\}$ is equicontinuous. Then there is $\Omega \in \mathcal{V}_x$ so that:

$$|\chi_i \circ f(y) - \chi_i \circ f(x)| \leq \epsilon/3,\ y \in \Omega,\ i \in \{1, \dots, n\}. \tag{4}$$

Therefore, for all $\phi \in \Delta_P$, there exists $i \in \{1, \dots, n\}$ with $\phi \in V_i$. Hence, for all $y \in B \cap \Omega$, we get:

$$\begin{aligned} |\phi(f(y)) - \phi(f(x))| &\leq \underbrace{|\phi(f(y)) - \chi_i(f(y))|}_{\leq \epsilon/3, \text{ by } (3)} + \underbrace{|\chi_i(f(y)) - \chi_i(f(x))|}_{\leq \epsilon/3 \text{ by } (4)} \\ &\quad + \underbrace{|\chi_i(f(x)) - \phi(f(x))|}_{\leq \epsilon/3 \text{ by } (3)} \\ &\leq 3\epsilon/3 = \epsilon. \end{aligned}$$

Whereby

$$||G \circ f(y) - G \circ f(x)||_P \leq \epsilon,\ y \in \Omega \cap B,$$

and f is Gelfand-continuous. □

Corollary 3.2. *If $\mathcal{P}$ is a subset of $\mathcal{K}_X$, X is a $\mathcal{P}_{\mathbb{R}}$-space, and Δ is an $\mathcal{E}_{\mathbb{R}}$-space, then the following assertions are equivalent for any mapping $f : X \to A$:*

(1) *f is Gelfand-continuous,*

(2) *f is continuous when A is endowed with the topology $\tau^r_{\mathbb{P}}$.*

(3) *$\forall P \in \mathbb{P}$, the set $\{\chi \circ f, \chi \in \Delta_P\}$ is equicontinuous,*

(4) *the mapping $\psi_f : \chi \mapsto \chi \circ f$ maps continuously Δ into $C_{\mathcal{P}}(X)$.*

In case A is a Banach algebra and X is a compact space, we get:

Corollary 3.3. *If X is compact and A is a Banach algebra, the following assertions are equivalent for every mapping $f : X \to A$:*

(1) *f is Gelfand-continuous,*

(2) *f is continuous when A is endowed with the spectral radius r as a semi-norm.*

(3) *the set $\{\chi \circ f, \chi \in \Delta\}$ is equicontinuous,*

(4) *the mapping $\psi_f : \chi \mapsto \chi \circ f$ maps continuously Δ into $(C(X), \|\ \|_\infty)$.*

Remark 3.4. 1. It follows from Corollary 3.2 that, if X and Δ are $k_{\mathbb{R}}$-spaces and every compact subset of Δ is equicontinuous, then $f : X \to A$ is Gelfand-continuous if and only if ψ_f maps continuously Δ into $C_c(X)$. An instance where every bounded (then also every compact) subset of Δ is equicontinuous is when A is m-barrelled (i.e., every idempotent barrel is a neighborhood of zero).

2. If A happens to be a Q-algebra (i.e., the group of invertible elements is open), then Δ, itself, is both compact and equicontinuous. Therefore, if X is a $k_{\mathbb{R}}$-space and A is a Q-algebra, then f is Gelfand-continuous if and only if ψ_f maps continuously Δ into $C_c(X)$.

One can associate to $\mathcal{P}$ several classes of Gelfand-continuous functions according to continuity or boundedness properties one requests. Before defining such classes, let us say that a subset C of A is uniformly bounded, if there is some $M > 0$ such that $P(x) \leq M$ for every $P \in \mathbb{P}$ and $x \in C$. This makes sense since we assume $P(e) = 1$ for every $P \in \mathbb{P}$. Actually, C is uniformly bounded if and only if it is absorbed by the bounded set $B := \{x \in A : P(x) \leq 1, P \in \mathbb{P}\} = \cap_{P \in \mathbb{P}}\{x \in A : P(x) \leq 1\}$. We then consider:

$$
\begin{aligned}
G(X,A) &:= \{f : X \to A, \text{ Gelfand-continuous}\},\\
G^b(X,A) &:= \{f \in G(X,A) : f(B) \text{ is bounded for every } B \in \mathcal{P}\},\\
G^b_u(X,A) &:= \{f \in G(X,A) : f(B) \text{ is uniformly bounded for every } B \in \mathcal{P}\},\\
G^B(X,A) &:= \{f \in G(X,A) : f(X) \text{ is bounded in } A\},\\
G^B_u(X,A) &:= \{f \in G(X,A) : f(X) \text{ is uniformly bounded in } A\},\\
G^B_r(X,A) &:= \{f \in G(X,A) : \exists M > 0 : r(f(x)) \leq M, \forall x \in X\}.
\end{aligned}
$$

We also consider:

$$G^{\Delta}(X,A) := \{f : X \to A,\ \chi \circ f \text{ is continuous for every } \chi \in \Delta\}.$$

We clearly have the following inclusions (the arrows mean inclusions):

$$
\begin{array}{ccccccc}
G^B_u(X,A) & \longrightarrow & G^b_u(X,A) & \longrightarrow & G^b(X,A) & \longrightarrow & G(X,A)\\
\downarrow & \searrow & & \nearrow & \uparrow & & \downarrow\\
G^B_r(X,A) & & G^B(X,A) & & C(X,A) & \longrightarrow & G^{\Delta}(X,A)
\end{array}
$$

Notice that all the spaces above are algebras for the pointwise operations. Moreover, if A is a Banach algebra and X is compact, the four ones $G^B_u(X,A)$, $G^b_u(X,A)$, $G^B(X,A)$, and $G^b(X,A)$ all coincide. If, in addition, A is a C^*-algebra, then the latter coincide with $G^B_r(X,A)$ too.

One is tempted to consider also the space:

$$G^b_r(X,A) := \{f \in G(X,A) : f(B) \text{ is bounded for the topology } \tau^r_{\mathbb{P}}, B \in \mathcal{P}\}.$$

It is easy to see that this is actually nothing but $G(X,A)$ itself.

In [11], the author showed that, in the Banach algebra setting, if G is an isomorphism (i.e., continuous and open into its range), then the algebras $G^b(Y,A)$ and $G(Y,A)$ coincide for every compact space Y. The following proposition gives instances in which the equality holds although G is not an isomorphism.

Proposition 3.5. (1) *If the topology $\tau^r_{\mathbb{P}}$ defines the same bounded sets as $\tau_{\mathbb{P}}$ in A, then $G^b(Y,A) = G(Y,A)$ for every Hausdorff completely regular space Y and every $\mathcal{P} \subset \mathcal{B}_Y$. The converse is true whenever A is semisimple.*

(2) *If the spectral radius r defines the same bounded sets as the topology $\tau_{\mathbb{P}}$, then $G^B(Y,A) = G^B_r(Y,A)$ for every Hausdorff completely regular space Y. The converse is true whenever A is semisimple.*

Proof. (1): Assume that $\tau_{\mathbb{P}}^r$ and $\tau_{\mathbb{P}}$ define the same bounded sets in A, and let $B \in \mathcal{P}$ and $f \in G(Y, A)$ be given, Y being an arbitrary Hausdorff completely regular space. Then for all ϕ in the topological dual $C_{\mathcal{E}}(\Delta)'$ of $C_{\mathcal{E}}(\Delta)$, we have $\phi \circ G \circ f \in C(X)$. Therefore, since B is bounding, $\phi \circ G \circ f(B)$ is bounded. Hence $G \circ f(B)$ is weakly bounded in $C_{\mathcal{E}}(\Delta)$. Since the weak bounded sets and the bounded ones in an arbitrary locally convex space are the same, $G \circ f(B)$ is bounded in $C_{\mathcal{E}}(\Delta)$. Therefore $f(B)$ is $\tau_{\mathbb{P}}^r$-bounded in A, hence also τ-bounded.

Conversely, suppose, to a contradiction, that $G^b(Y, A) = G(Y, A)$ for every Hausdorff completely regular space Y and every $\mathcal{P} \subset \mathcal{B}_Y$, but that $\tau_{\mathbb{P}}^r$ and $\tau_{\mathbb{P}}$ do not have the same bounded sets. Then there are some sequence $(a_n)_n \subset A$ and some $P \in \mathbb{P}$ so that $(a_n)_n$ is bounded for $(r_P)_{P\in\mathbb{P}}$, but $P(a_n) > n^2$ for every $n \in \mathbb{N}$. Consider then $Y := \{\frac{1}{n}a_n, n \geq 1\} \cup \{0\}$ with the relative topology induced by $\tau_{\mathbb{P}}^r$ and $\mathcal{P} = \{Y\}$. By semisimplicity, Y is Hausdorff and then compact. If $f : Y \to A$, $y \mapsto y$, is the embedding of Y into A, then $f \in G(Y, A) = G^b(Y, A)$. But $\sup\{P(f(y)),\ y \in Y\} = +\infty$ contradicts the assumption $f \in G^b(Y, A)$.

The proof of (2) is similar. □

Remark 3.6. 1. There are commutative complete semisimple Arens-Michael algebras with unit such that the spectral radius r is a norm determining the same bounded sets as $\mathbb{P}$. Take for example the algebra $C_c([0, \Omega[)$, where Ω is the first uncountable ordinal. Since $[0, \Omega[$ is locally compact, it is a $k_{\mathbb{R}}$-space and then $C_c([0, \Omega[)$ is complete. Furthermore, since r is nothing but the uniform norm, it is known (see [4]) that it defines the same bounded sets as $\mathbb{P}$.

2. Actually, if A is uniform (i.e., $P(x^2) = P(x)^2$, $\forall x \in A$, $P \in \mathbb{P}$), then $P = r_P$ for every $P \in P$ and therefore $(r_P)_{P\in\mathbb{P}}$ determines the same bounded sets in A as $\mathbb{P}$.

4. Completeness of the algebras of Gelfand-continuous functions

In this section, we equip different algebras of Gelfand-continuous functions with different topologies and examine when they are complete.

The natural topology $\tau_{\mathbb{P},\mathcal{P}}$ on the algebra $C(X, A)$ is the one given by the family $(P_B)_{P\in\mathbb{P},B\in\mathcal{P}}$ of seminorms, where, for all $f \in C(X, A)$, $P \in \mathbb{P}$, and $B \in \mathcal{P}$,

$$P_B(f) := \sup\{P(f(x)),\ x \in B\}.$$

We will write $C_{\mathbb{P},\mathcal{P}}(X, A)$ to mean the algebra $C(X, A)$ together with the topology $\tau_{\mathbb{P},\mathcal{P}}$. Since every $P \in \mathbb{P}$ is submultiplicative, also P_B is submultiplicative for all $P \in \mathbb{P}$ and all $B \in \mathcal{P}$. Therefore $C_{\mathbb{P},\mathcal{P}}(X, A)$ is an Arens-Michael algebra. Moreover, if $\tau_{\mathbb{P},\mathcal{P}}^r$ is the topology given by the seminorms:

$$r_{P,B}(f) := \sup\{r_P(f(x)), x \in B\},\ P \in \mathbb{P},\ B \in \mathcal{B},\ f \in C(X, A),$$

then $(C(X, A), \tau_{\mathbb{P},\mathcal{P}}^r)$ is also an Arens-Michael algebra and the topology $\tau_{\mathbb{P},\mathcal{P}}^r$ is coarser than $\tau_{\mathbb{P},\mathcal{P}}$.

Now, since every $g \in G^b(X,A)$ is bounded on each $B \in \mathcal{P}$, the seminorms P_B extend to both $G^b(X,A)$ and $G^b_u(X,A)$. We can then also consider the Arens-Michael algebras $(G^b_u(X,A), \tau_{\mathbb{P},\mathcal{P}})$ and $(G^b(X,A), \tau_{\mathbb{P},\mathcal{P}})$. Clearly, $C(X,A)$ and $G^b_u(X,A)$ are Arens-Michael algebras for the relative topology induced by $\tau_{\mathbb{P},\mathcal{P}}$.

Furthermore, on the algebra $G^B(X,A)$ and then also on $G^B_u(X,A)$, one can consider the topology $\tau^\infty_{\mathbb{P}}$ given by the seminorms:

$$P_\infty(f) := \sup\{P(f(x)),\ x \in X\}, P \in \mathbb{P},\ f \in G^B(X,A).$$

We then have:

Proposition 4.1. *Let A be a (non necessarily complete) Arens-Michael commutative algebra with unit. Then*
1. *$G^b_u(X,A)$ is closed in $(G^b(X,A), \tau_{\mathbb{P},\mathcal{P}})$.*
2. *$G^B_u(X,A)$ is closed in $(G^B(X,A), \tau^\infty_{\mathbb{P}})$.*
3. *If X is a $\mathcal{P}_{\mathbb{R}}$-space, then $C(X,A)$ is also closed in $(G^b(X,A), \tau_{\mathbb{P},\mathcal{P}})$.*

Proof. 1. Assume $f \in \overline{G^b_u(X,A)}$, the closure being taken in $G^b_{\mathbb{P},\mathcal{P}}(X,A)$, and given $B \in \mathcal{P}$. Then

$$\forall P \in \mathbb{P}, \exists g \in G^b_u(X,A) : P_B(f-g) < 1.$$

Since $g \in G^b_u(X,A)$, there exists $M > 0$ such that:

$$Q(g(x)) \leq M,\ x \in B,\ Q \in \mathbb{P}.$$

It follows, for every $x \in B$,

$$P(f(x)) \leq P(f(x) - g(x)) + P(g(x)) \leq M + 1.$$

Since P is arbitrary, $f \in G^b_u(X,A)$. Hence the latter is closed.

The proof of 2. is similar to that of 1.

3. Assume $f \in \overline{C(X,A)}$, the closure again taken in $G^b_{\mathbb{P},\mathcal{P}}(X,A)$. We will show that the restriction of f to each $B \in \mathcal{P}$ is continuous. Fix then $B \in \mathcal{P}$ and $x \in B$, and consider arbitrary $P \in \mathbb{P}$ and $\epsilon > 0$. Then

$$\exists g \in C(X,A) : P_B(f-g) < \epsilon/3. \tag{5}$$

By the continuity of g, there exists an open neighborhood Ω of x such that:

$$P(g(y) - g(x)) \leq \epsilon/3,\ y \in \Omega. \tag{6}$$

Then, for $y \in \Omega \cap B$, using (5) and (6), we get:

$$\begin{aligned} P(f(y) - f(x)) &\leq P(f(y) - g(y)) + P(g(y) - g(x)) \\ &\quad + P(g(x) - f(x)) \leq 3\epsilon/3 = \epsilon. \end{aligned}$$

The continuity of f follows, hence $C(X,A)$ is closed in $(G^b(X,A), \tau_{\mathbb{P},\mathcal{P}})$. □

Proposition 4.2. 1. *The algebra $(G^B(X,A), \tau^\infty_{\mathbb{P}})$ is complete, then so is also $(G^B_u(X,A), \tau^\infty_{\mathbb{P}})$.*

2. *If X is a $\mathcal{P}_{\mathbb{R}}$-space, then $G^b_{\mathbb{P},\mathcal{P}}(X,A)$ is complete. Therefore so are also its closed subalgebras $C_{\mathbb{P},\mathcal{P}}(X,A)$ and $G^b_{u,\mathbb{P},\mathcal{P}}(X,A)$.*

Proof. 1. Let $(f_i)_{i\in I}$ be a Cauchy net in $(G^B(X,A),\tau_{\mathbb{P}}^{\infty})$. Since, for every $x\in X$, the evaluation $\delta_x : f\mapsto f(x)$ is continuous from $(G^B(X,A),\tau_{\mathbb{P}}^{\infty})$ into A, the net $(f_i(x))_i$ is Cauchy in A. As A is complete, it converges to some $f(x)\in A$. We claim that the so-defined mapping f belongs to $G^B(X,A)$ and that it is the limit of $(f_i)_i$ in $(G^B(X,A),\tau_{\mathbb{P}}^{\infty})$. Fix $\epsilon>0$, $P\in\mathbb{P}$, and $x_0\in X$. By the Cauchy property, there is some i_0 such that, whenever $i_0\le i,j$

$$P_\infty(f_i-f_j):=\sup\{P(f_i(x)-f_j(x)),x\in X\}<\frac{\epsilon}{3}. \tag{7}$$

Passing in (7) to the limit on j and using the triangular inequality, we get that $f(X)$ is $\tau_{\mathbb{P}}$-bounded in A. But, by the Gelfand-continuity of f_{i_0}, there exists a neighborhood Ω of x_0 such that, for all $x\in\Omega$, $r_P(\widehat{f_{i_0}(x)}-\widehat{f_{i_0}(x_0)})<\frac{\epsilon}{3}$. Hence, for every $x\in\Omega$, we get due to (7):

$$\begin{aligned}
r_P(\widehat{f(x)}-\widehat{f(x_0)}) &\le r_P(\widehat{f(x)}-\widehat{f_{i_0}(x)})+r_P(\widehat{f_{i_0}(x)}-\widehat{f_{i_0}(x_0)})\\
&\quad + r_P(\widehat{f_{i_0}(x)}-\widehat{f(x_0)})\\
&\le P(f(x)-f_{i_0}(x))+r_P(\widehat{f_{i_0}(x)}-\widehat{f_{i_0}(x_0)})\\
&\quad + P(f_{i_0}(x)-f(x_0))\\
&\le 3\frac{\epsilon}{3}=\epsilon.
\end{aligned}$$

Hence $f\in G^B(X,A)$. Now, Passing to the limit in (7), we get that f is the limit in $(G^B(X,A),\tau_{\mathbb{P}}^{\infty})$ of $(f_i)_i$. The completeness follows.

2. A similar proof as for 1. can be carried out. □

From Proposition 2.2 and Remark 2.3 derives the fact that if X is a $\mathcal{P}_{\mathbb{R}}$-space and Δ is an $\mathcal{E}_{\mathbb{R}}$-space, then $C_{\mathcal{E},\mathcal{P}}(X,C_{\mathcal{E}}(\Delta))$ is complete.

5. Example

In the following, we will endow the space of complex sequences $\mathbb{C}^{\mathbb{N}}$, as well as all its subspaces that will be considered, with the pointwise operations. We will let e denote the constant sequence $(1,1,\dots)$, and e_n the sequence whose entries are all 0 but the n^{th} one which equals 1. For any set $N\subset\mathbb{N}$ and any sequence $x=(x_n)_n$, put $e_N:=\sum_{n\in N}e_n$ and $x_N:=xe_N$. By $S_n(x)$ we will denote the finite section $xe_{\{1,2,\dots,n\}}$ of order n of x. Finally, for every $\lambda\in\mathbb{C}$, $x-\lambda$ will denote $x-\lambda e$.

If M is an infinite subset of $\mathbb{N}$, a sequence $x:=(x_n)_n$ of complex numbers is said to converge along M, if the limit $\lim\limits_{\substack{n\to\infty\\ n\in M}}x_n$ exists. Denote such a limit by $\lim_M x$ and set

$$\|x\|_{2,M}:=(\sum_{n\in M}|x_n|^2)^{\frac{1}{2}}.$$

Now, let $\mathcal{F}$ be a family of infinite subsets of $\mathbb{N}$ such that $\mathcal{F}$ covers $\mathbb{N}$ and the intersection of any two different elements of $\mathcal{F}$ is at most finite. We consider the set A of all complex sequences $x := (x_n)_{n\geq 1}$ converging along each $F \in \mathcal{F}$ to some limit $\ell_{x,F} := \lim_F x$ and such that:

$$\forall F \in \mathcal{F}, \quad \|x - \ell_{x,F}\|_{2,F}^2 := \sum_{n\in F} |x_n - \ell_{x,F}|^2 < +\infty.$$

Then A, with the pointwise operations, is obviously a commutative algebra with unit e. Endow A with the topology $\tau_{\mathbb{F}}$ generated by the family $\mathbb{F} := (\| \ \|_F)_{F\in\mathcal{F}}$ of seminorms, where

$$\|x\|_F := \|x - \ell_{x,F}\|_{2,F} + |\ell_{x,F}|, \ x \in A, \quad F \in \mathcal{F}.$$

Notice that the family $\mathbb{F}$ of seminorms is not directed upward. If one wishes to have a directed upward family, one just has to take all the suprema of finitely many elements of $\mathbb{F}$.

Proposition 5.1. *The algebra $(A, \mathbb{F})$ is a complete Arens-Michael algebra whose carrier space is*

$$\Delta := \{\chi_n, n \in \mathbb{N}\} \cup \{\chi_F, F \in \mathcal{F}\},$$

where, for all $x \in A$, $n \in \mathbb{N}$, and all $F \in \mathcal{F}$, $\chi_n(x) = x_n$ and $\chi_F(x) = \lim_F x$.

Proof. It is easily seen that each $\| \ \|_F$ is a seminorm. Now, since $\lim_F xy = \lim_F x \lim_F y$, for every $F \in \mathcal{F}$ and every $x, y \in A$, putting $\lim_F x = \ell$ and $\lim_F y = \ell'$, we get:

$$\begin{aligned}
\|xy\|_F &= \|((x-\ell)+\ell)((y-\ell')+\ell')\|_F \\
&= \|(x-\ell)(y-\ell')+\ell(y-\ell')+\ell'(x-\ell)+\ell\ell'\|_F \\
&:= \|(x-\ell)(y-\ell')+\ell(y-\ell')+\ell'(x-\ell)\|_{2,F}+|\ell\ell'| \\
&\leq \|(x-\ell)\|_{2,F}\|(y-\ell')\|_{2,F}+|\ell|\|y-\ell'\|_{2,F}+|\ell'|\|x-\ell\|_{2,F}+|\ell\ell'| \\
&\leq \|x\|_F\|y\|_F.
\end{aligned}$$

Hence $\| \ \|_F$ is submultiplicative.

Moreover, for every $F \in \mathcal{F}$ and every $n \in F$,

$$|\chi_n(x)| \leq \|x - \ell_{x,F}\|_{2,F} + |\ell_{x,F}| = \|x\|_F.$$

Then $\max(|\chi_n(x)|, |\ell_{x,F}|) \leq \|x\|_F$, whereby both χ_n and χ_F are continuous.

Now, let $\chi \in \Delta$ be given. We claim that, if $\chi \neq \chi_F$ for all $F \in \mathcal{F}$, then there exists exactly one $n \in \mathbb{N}$ such that $\chi = \chi_n$. Indeed, since χ is continuous, there exist $m \in \mathbb{N}$, $F_1, F_2, \ldots, F_m \in \mathcal{F}$, and a constant $c > 0$ such that

$$|\chi(x)| \leq c \max_{i=1,\ldots,m} \|x\|_{F_i}, \ x \in A. \tag{8}$$

Put $x^{(1)} := x_{F_1}$ and, for $i = 1, \ldots, m-1$, $x^{(i+1)} := x_{F_{i+1}\setminus(\cup_{k=1}^{i} F_k)}$. If we set $G := \mathbb{N} \setminus \cup_{i=1}^{m} F_i$, then $x = x_G + \sum_{i=1}^{m} x^{(i)}$. Moreover, if x belongs to A, then so do also x_G and $x^{(i)}$ for every $i = 1, \ldots, m$. Furthermore, thanks to (8), $\chi(x_G) = 0$. Actually, $\chi(x) = 0$ if and only if $\chi(x^{(i)}) = 0$ for every $i = 1, \ldots, m$. Indeed the sufficiency is obvious. Conversely, if $\chi(x) = 0$ and $\chi(x^{(i)}) \neq 0$ for some i. Then $0 = \chi(xx^{(i)}) = \chi(x^{(i)}x^{(i)}) = (\chi(x^{(i)}))^2$. This is a contradiction. Now, since $x^{(i)}x^{(i')} = 0$ whenever $i \neq i'$, if $\chi(x) \neq 0$, there is exactly one $i_x \in \{1, \dots, m\}$ such that $\chi(x^{(i_x)}) \neq 0$. Hence $\chi(x) = \chi(x^{(i_x)})$. We claim that i_x does not depend on x. Indeed, given an arbitrary $y \in A$ such that $\chi(y) \neq 0$ and $i_y \neq i_x$. Then

$$\chi(xy) = \chi(x)\chi(y) = \chi(x^{(i_x)})\chi(y^{(i_y)}) = \chi(x^{(i_x)}y^{(i_y)}) = 0.$$

This is a contradiction. Therefore $i_x = i_y$. Putting $j := i_x$, it follows that $\chi(y) = \chi(y^{(j)})$ for every $y \in A$. But, for $i \neq j$, $\|x^{(j)}\|_{F_i} = 0$. Then, by (8), we get:

$$|\chi(x)| \leq c\|x\|_{F_j}, \ x \in A.$$

In particular, for every $n \in \mathbb{N}$ and every $x \in A$, we also have $|\chi(x)|^n \leq c\|x\|_{F_j}^n$; whereby

$$|\chi(x)| \leq \|x\|_{F_j}, \ x \in A. \tag{9}$$

Suppose $\chi(e_n) = 0$ for every $n \in F_j$. Then, for the kth finite section $S_k(y)$ of Y,

$$\|(x - \ell_{x,F_j}) - S_k(x - \ell_{x,F_j})\|_{F_j}^2 = \sum_{\substack{n>k \\ n\in F_j}} |x_n - \ell_{x,F_j}|^2$$

tends to zero as k tend to infinity. Then also

$$\chi(x) - \ell_{x,F_j} = \chi\left((x - \ell_{x,F_j}) - S_k(x - \ell_{x,F_j})\right)$$

tends to zero by (9). This is $\chi(x) = \ell_{x,F_j} = \chi_{F_j}(x)$ which contradicts our first assumption. Therefore there is some $n \in F_j$ such that $\chi(e_n) \neq 0$. But, if $m \neq n$, then $\chi(e_n)\chi(e_m) = \chi(e_ne_m) = \chi(0) = 0$. Hence $\chi(e_m) = 0$, for all $m \neq n$. Furthermore, $\chi(e_n) = \chi(e_n)^2$. Thus $\chi(e_n) = 1$. But, for $k \geq n$, $\chi\left(S_k(x - \ell_{x,F_j})\right) = x_n - \ell_{x,F_j}$. Since $\chi\left((x - \ell_{x,F_j}) - S_k(x - \ell_{x,F_j})\right)$ tends to 0 when k tends to infinity, $\chi(x) - x_n = 0$, whereby $\chi = \chi_n$.

Concerning the completeness of A, let $(X_i)_{i\in I}$ be a Cauchy net in A, I being an ordered non empty set. Then

$$\forall F \in \mathcal{F}, \forall \epsilon > 0, \exists i_0 \in I : \|X_i - X_j\|_F < \epsilon, \ \forall i, j \geq i_0. \tag{10}$$

The continuity of χ_n and χ_F, gives that $(X_i, n)_i$, as well as $(\ell_{X_i,F})_i$, are Cauchy in $\mathbb{C}$. Hence they converge respectively to some x_n and some ℓ_F. Set $x := (x_n)_n$. By (10), we have:

$$\|X_i - X_j\|_F = \|X_i - X_j\|_{2,F} + |\ell_{X_i,F} - \ell_{X_j,F}| \leq \epsilon, \ \forall i, j \geq i_0. \tag{11}$$

Passing to the limit on j, in (11), we get:

$$\|X_i - x\|_F = \|X_i - x\|_{2,F} + |\ell_{X_i,F} - \ell_F| \leq \epsilon, \ \forall i \geq i_0. \tag{12}$$

As $|x_n - \ell_F| \leq |x_n - X_{i_0,n}| + |X_{i_0,n} - \ell_{X_{i_0},F}| + |\ell_{X_{i_0},F} - \ell_F|$, using (12), we get $|x_n - \ell_F| \leq 3\epsilon$ for n large enough. Therefore $\lim_F x_n = \ell_F$. Furthermore, since $\|x\|_{2,F} \leq \|X_{i_0} - x\|_{2,F} + \|X_{i_0}\|_{2,F} < +\infty$, x belongs to A. Finally, passing to the limit on j in (11), we get that $(X_i)_i$ converges in A to x. Consequently A is complete. $\square$

Notice that, for every $F \in \mathcal{F}$, $\|e\|_F = 1$.

Corollary 5.2. *The algebra A is semisimple. Moreover, for every $x \in A$, the equality sp $x = \{x_n, n \in \mathbb{N}\} \cup \{\lim_F x, F \in \mathcal{F}\}$ holds. Therefore $r(x) = \|x\|_\infty = \sup\{|x_n|, n \in \mathbb{N}\}$.*

The following theorem gives necessary and sufficient conditions under which every element of A is bounded (i.e., where $A \subset \ell^\infty$). To show it, we order by inclusion the class of all infinite families of infinite subsets of $\mathbb{N}$ covering $\mathbb{N}$ and such that the intersection of any two elements of which is at most finite.

Theorem 5.3. *The algebra A is contained in ℓ^∞ if and only if $\mathcal{F}$ is maximal.*

Proof. If $A \subset \ell^\infty$ holds and $\mathcal{F}$ is not maximal, then we can find an infinite subset E of $\mathbb{N}$ such that $E \cap F$ is finite for all $F \in \mathcal{F}$ and $E \notin \mathcal{F}$. Take then the sequence $x = (x_k)_k$ with $x_k = k$ if $k \in E$ and $x_k = 0$ otherwise. Then $x \in A$, since $E \cap F$ is finite, and $\ell_{x,F} = 0$ for all $F \in \mathcal{F}$. However $x \notin \ell^\infty$ which is a contradiction.

Conversely, suppose that $\mathcal{F}$ is maximal but A is not a subset of ℓ^∞. Consider $x \in A \setminus \ell^\infty$. Then there exist a sequence $(n_m)_m$ of positive integers such that $|x_{n_m}| > m$, $m \in \mathbb{N}$. Then $E := \{n_m, m \in \mathbb{N}\}$ is an infinite subset of $\mathbb{N}$ which intersects every $F \in \mathcal{F}$ at most in a finite set. Therefore the family $\mathcal{F} \cup \{E\}$ contains strictly $\mathcal{F}$. This is a contradiction. $\square$

Recall that a topological algebra B with a unit is a Q-algebra if and only if the spectral radius r is (upper semi-) continuous at the origin [9, 10]. The algebra B is said to be strongly sequential [7] or a P-algebra if there is some neighborhood Ω of the origin such that the sequence $(x^n)_n$ converges to zero for all $x \in \Omega$. This is equivalent to the (upper semi-) continuity of the boundedness radius β at the origin [1, 10], where

$$\beta(x) := \inf\{r > 0 : (\frac{x^n}{r^n})_n \text{ converges to zero}\},$$

here $\inf \emptyset = +\infty$.

Remark 5.4. The algebra A is neither a Q-algebra nor a P-algebra. Indeed, since A is a complete Arens-Michael algebra, it is a Q-algebra if and only if it is a P-algebra. Moreover, if A were a Q-algebra, there would exist $F_1, \ldots, F_m \in \mathcal{F}$, for some $m \in \mathbb{N}$, and $c > 0$ such that:

$$r(x) \leq c \max\{\|x\|_{F_i}, i = 1, 2, \ldots, n\},\ x \in A. \tag{13}$$

But this does not hold for $x := ye_{\mathbb{N}\setminus F}$, where $y := (\frac{1}{n})_{n \geq 1}$ and $F := F_1 \cup F_2 \cup \cdots \cup F_n$.

Now, we provide some topological properties of Δ.

Theorem 5.5. *For every $F \in \mathcal{F}$, Δ_F is a clopen set in Δ. Therefore Δ is locally compact. In fact each χ_n is an isolated point in Δ and each χ_F is an isolated point in $\Delta_\infty := \{\chi_E,\ E \in \mathcal{F}\}$.*

Proof. It is clear that, for all $n \in \mathbb{N}$, the open set

$$V(\chi_n, \{e_n\}, \frac{1}{2}) := \{\chi \in \Delta : |\chi(e_n) - \chi_n(e_n)| < \frac{1}{2}\}$$

is reduced to the singleton $\{\chi_n\}$. Therefore χ_n is isolated. Now, if $x_n = 1 + \frac{1}{n}$ for every $n \in F$ and $x_n = 0$ for all $n \notin F$, then $x := (x_n)_n \in A$ and $V(\chi_F, \{x\}, \frac{1}{2}) \cap \Delta_\infty = \{\chi_F\}$. Hence χ_F is isolated in Δ_∞. As Δ_F is compact, it is closed. Moreover, for x as above, we have $V(\chi_F, \{x\}, \frac{1}{2}) \subset \Delta_F$. Hence Δ_F is a neighborhood of each of its elements. Then it is open. It derive from this that Δ is locally compact. $\square$

Remark 5.6. Since A is semisimple, G is injective. However, although A is hermitian for the involution $(x_n)_n^* := (\overline{x_n})_n$, G is not surjective. Indeed, if we put $f_F(\chi_n) = \frac{1}{\sqrt{k}}$, if $n = \alpha_k(F)$ and $f_F(\chi) = 0$ otherwise, then $f_F \in C(\Delta)$, but $f \notin \widehat{A} := G(A)$.

6. *A*-valued Gelfand-continuous functions

In this section, we will provide an example of a Hausdorff locally compact space X for which the eight algebras above are pairwise different. For this purpose, let us consider two infinite subsets E and H of $\mathbb{N}$ such that $E \cap H$ is at most finite, as well as $E \cap F$ and $H \cap F$ for every $F \in \mathcal{F}$. The existence of such E and H not belonging to $\mathcal{F}$ is guarantied whenever the latter is not maximal.

For F and G in $\mathcal{F}' := \mathcal{F} \cup \{E\}$, set $N_F = \max(E \cap F)$, $M_F = \max(H \cap F)$, and $N_{FG} := \max(F \cap G)$, with $\max \emptyset = 0$. If K is an infinite subset of $\mathbb{N}$, denote by $\alpha_k(K)$ the k^{th} element of K starting with the smallest one.

Now, set $X := \mathbb{N} \cup D$, where D consists of pairwise distinct objects w_F not belonging to $\mathbb{N}$, $F \in \mathcal{F}'$, and topologize X in the following way: each $n \in \mathbb{N}$ is an isolated point in X and a neighborhood of w_G is any tail $G^{(m)} := \{n \in G : n > m\}$ of $G \in \mathcal{F}'$ together with w_G, $m \in \mathbb{N}$. Then X is a locally compact space such that $K_G := G \cup \{w_G\}$ is compact for every $G \in \mathcal{F}'$. Moreover, for any mapping f from X into an arbitrary topological space, the three assertions are equivalent:

1. f is continuous,
2. f is continuous at each w_G, $G \in \mathcal{F}$, and
3. the restriction of f to each K_G is continuous.

In case $\mathcal{F}$ is maximal, it necessarily contains E and H, and then X is nothing but the famous space Ψ of [6, page 79, 5I]. Take $\mathcal{P}$ to be the collection of all unions of finitely many K_G's, $G \in \mathcal{F}'$.

In order to avoid the excessive length of the equations, we will omit "(X, A)" from the notations in the statement of the following proposition. We will write, for instance, G, C, and G_u^B instead of $G(X,A)$, $C(X,A)$, and $G_u^B(X,A)$ respectively.

Proposition 6.1. *Assume that $\mathcal{F}$ is not maximal and that E and H do not belong to $\mathcal{F}$. Then the eight algebras above are pairwise distinct. Actually, the following assertions hold*:

$$G^\Delta \notin \{G_u^B, G_r^B, G^B,\ G_u^b,\ G^b,\ C, G\}, \tag{I.1}$$

$$G \notin \{G_u^B, G_r^B, G^B,\ G_u^b,\ G^b, C\}, \tag{I.2}$$

$$G^b \notin \{G_u^B, G_r^B, G^B,\ G_u^b, C\}, \tag{I.3}$$

$$C \notin \{G_u^B, G_r^B, G^B,\ G_u^b\}, \tag{I.4}$$

$$G^B \notin \{G_u^B, G_r^B,\ G_u^b\}, \tag{I.5}$$

$$G_r^B \notin \{G_u^B, G_u^b\}, \tag{I.6}$$

$$G_u^b \notin \{G_u^B\}. \tag{I.7}$$

Proof. In order to show (I.1), it suffices to exhibit a function $f_1 \in G^\Delta(X,A)$ which is not Gelfand-continuous. Let $f_1 : X \to A$ be defined by

$$f_1(x) = \begin{cases} e_n & : \quad x = n \in \mathbb{N}, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Then $\chi_n \circ f_1 = 1_{\{n\}}$ for $n \in \mathbb{N}$ and $\chi \circ f_1 = 0$ identically on X otherwise. Hence $\chi \circ f_1$ is continuous for every $\chi \in \Delta$. Therefore $f_1 \in G^\Delta(X,A)$. However, for every $F \in \mathcal{F}$ and every $n \in F$, we have: $r_F(f_1(n) - f_1(w_F)) := \sup_{m \in F} |f_1(n)(m)| = 1$ showing that f_1 is not Gelfand-continuous at w_F.

To show (I.2), fix some $K \in \mathcal{F}$ and define a function $f_2 : X \to A$ as follows:

$$f_2(x) = \begin{cases} \frac{1}{\sqrt[3]{n}} \sum_{k=1}^{n} e_{\alpha_k(K)} & : \quad x = n \in K, \\ n e_{\alpha_k(K)} & : \quad x = n \in E \setminus K \text{ and } n = \alpha_k(E) \\ 0 & : \quad \text{otherwise.} \end{cases}$$

For every $G \in \mathcal{F}'$, every $F \in \mathcal{F}$, and every $n \in G$, if n is larger than N_G, then $n \notin E$. Therefore $r_F(f_2(n) - f_2(w_G)) \le \frac{1}{\sqrt[3]{n}}$ and f_2 is Gelfand continuous. Now, since $r_K(f_2(n)) = n$ for every $n \in E \setminus K$, we have $f_2 \notin G_r^B(X,A)$. Furthermore, if n is the k^{th} element of E and $n \notin K$, we have $\|f_2(n)\|_K = n$. As $E \setminus K$ is infinite, $f_2(E)$ is not bounded in A. Hence $f_2 \notin G^b(X,A)$ and (I.2) holds.

Now, to show (I.3), let $K \in \mathcal{F}$ be fixed and let f_3 be defined by:

$$f_3(x) = \begin{cases} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{\alpha_k(K)} + \sum_{k=n+1}^{2n} \alpha_k(E) e_{\alpha_k(E)} & : \quad x = n \in K, \\ n e_{\alpha_n(K)} & : \quad x = n \in H \setminus K, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Notice that $f_3 = f + g$, where

$$f(x) = \begin{cases} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{\alpha_k(K)} & : \quad x = n \in K, \\ n e_{\alpha_n(K)} & : \quad x = n \in H \setminus K, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} \sum_{k=n+1}^{2n} \alpha_k(E) e_{\alpha_k(E)} & : \quad x = n \in K, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Since for every $G \in \mathcal{F}'$, every $F \in \mathcal{F}$, and every $n \in G$, one has $\|g(n)\|_F = 0$ whenever $\{\alpha_{n+1}(E), \alpha_{n+2}(E), \ldots, \alpha_{2n}(E)\} \cap F = \emptyset$, a fortiori whenever n is larger than N_F. Hence g is continuous on X and then it is also Gelfand-continuous. Furthermore, if n is larger that M_G, then $n \notin H$ and therefore $f(n)$ is either 0 or $\frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{\alpha_k(K)}$. In both cases, $r_F(f(n)) \leq \frac{1}{\sqrt{n}}$. Therefore f is Gelfand-continuous at w_G. Since G is arbitrary, f is Gelfand-continuous, then so is also f_3. Now, given again F, G, and $n \in G$ as above. If $n \in K$, then $\|f_3(n)\|_F \leq 1$, whenever n is larger than N_F so that $F \cap \{\alpha_{n+1}(E), \alpha_{n+2}(E), \ldots, \alpha_{2n}(E)\} = \emptyset$. If $n \notin K$ and $n > M_G$, then $f_3(n) = 0$. Therefore, since $\{n \in G : n \leq \max(N_F, M_G)\}$ is finite, f_3 is bounded on G, whereby $f_3 \in G^b(X, A)$, as G is arbitrary in $\mathcal{P}$. However $f_3 \notin G_r^B(X, A)$, since $r(f_3(n)) \geq \alpha_{n+1}(E) > n$ for every $n \in K$. Moreover, $f_3 \notin G^B(X, A)$, as for every $n \in H \setminus K$, we have $\|f_3(n)\|_K = n$. But also $f_3 \notin G_u^b(X, A)$ since for every $n \in K$, there exists some F_n such that $F_n \cap \{\alpha_{n+1}(E), \alpha_{n+2}(E), \ldots, \alpha_{2n}(E)\}$ is not empty. Therefore $\|f_3(n)\|_{F_n} \geq \alpha_{n+1}(E) \geq n$ and f_3 is not uniformly bounded on K. Finally, since

$$\|f_3(n) - f_3(w_K)\|_K \geq \|\frac{1}{\sqrt{n}} \sum_{k=1}^{n} e_{\alpha_k(K)}\|_K = 1, \; n \in K,$$

f_3 is not continuous at w_K. This shows (I.3).

For (I.4), let $K \in \mathcal{F}$ be fixed and let us define $f_4 : X \to A$ by:

$$f_4(x) = \begin{cases} n e_n & : \quad x = n \in E, \\ \frac{1}{\sqrt[3]{n}} \sum_{k=1}^{n} e_{\alpha_k(K)} & : \quad x = n \in H \setminus E, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

It is clear that, for every $F \in \mathcal{F}$ and every $n > \max(N_F, M_F)$, $f_4(n) = 0$, then $\|f_4(n)\|_F = 0$. Therefore f_4 is continuous on X. However, $r(f_4(n)) = n$ for every $n \in E$. Hence $f_4 \notin G_r^B(X, A)$. Moreover, since every $n \in E$ belongs to some $F \in \mathcal{F}$, f_4 is not uniformly bounded on E. Hence $f_4 \notin G_u^b(X, A)$. Finally, f_4 is not bounded on X, because, for $F = K \in \mathcal{F}$ and $n \in H \setminus E$, we have $\|f_4(n)\|_F = n^{\frac{1}{6}}$.

For (I.5), let us define f_5 by:

$$f_5(x) = \begin{cases} n e_n & : \quad n \in E \\ 0 & : \quad \text{otherwise.} \end{cases}$$

For every $G \in \mathcal{F}'$ and every $F \in \mathcal{F}$, we have $\|f_4(n) - f_4(w_G)\|_F = 0$ for $n > N_F$. Hence f_4 is continuous, then also Gelfand-continuous. Moreover, for every $F \in \mathcal{F}$ and every $x \in X$, we have $\|f_5(x)\|_F \leq N_F$. Hence $f_5 \in G^B(X, A)$. Now, since every $n \in E$ is in some $F_n \in \mathcal{F}$, one has $\|f_5(n)\|_{F_n} = n$. Therefore f_5 is not uniformly bounded on E. Hence it does not belong to $G_u^b(X, A)$. Finally, $r(f_5(n)) = n$ for every $n \in E$ which gives $f_5 \notin G_r^B(X, A)$.

To show (I.6), we consider some $K \in \mathcal{F}$ and the function f_6 defined by:

$$f_6(x) = \begin{cases} \frac{1}{\sqrt{n}} \sum_{k=1}^{n^2} e_{\alpha_k(K)} & : \quad x = n \in E, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

Since, for every $n \in \mathbb{N}$, $r(f_6(n)) \leq \frac{1}{\sqrt{n}}$, f_6 is Gelfand-continuous on X. Moreover, for every $x \in X$, we have $r(f_6(x)) \leq 1$. Hence $f_6 \in G_r^B(X, A)$. However, $\|f_6(n)\|_K = \sqrt{n}$ for every $n \in E$. Therefore f_6 is not bounded on E, whereby $f_6 \notin G^b(X, A)$ and then also $f_6 \notin G_u^b(X, A) \cup G_u^B(X, A)$.

Concerning (I.7), let us consider again $K \in \mathcal{F}$ and define the function f_7 as follows:

$$f_7(x) = \begin{cases} \frac{1}{\sqrt[3]{k}} \sum_{i=1}^{k} e_{\alpha_i(K)} & : \quad x = \alpha_k(H), \\ 0 & : \quad \text{otherwise.} \end{cases}$$

It is clear that, for every $G \in \mathcal{F}'$, every $F \in \mathcal{F}$ and $n \in G$ with $n > M_G$, $f_7(n) = 0$. This shows that f_7 is continuous, then also Gelfand-continuous. Moreover, for every $G \in \mathcal{F}'$, every $F \in \mathcal{F}$, and every $n \in G$, we have $\|f_7(n)\|_F \leq M_G^{\frac{1}{6}}$ (notice that if $n = \alpha_k(H)$, then $k \leq n \leq N_G$). Hence f_7 belongs to $G_u^b(X, A)$. However, since for every $n \in \mathbb{N}$, $\|f_7(\alpha_n(H))\|_K = n^{\frac{1}{6}}$, f_7 is not uniformly bounded on (H then also on) X. This gives $f_7 \notin G_u^B(X, A)$. □

Remark 6.2. 1. An example of such a collection $\mathcal{F}$ satisfying all the conditions required in Proposition 6.1 is obtained as follows: Denote by $\mathbf{P}$ the set of all prime numbers and, for every $p \in \mathbf{P}$,

$$F_p := \{1, 2, \ldots, p-1\} \cup \{p^k, k \in \mathbb{N}\}.$$

Let $\mathcal{F}$ be the set of all such F_p's, $p \in \mathbf{P}$. Clearly, $\mathcal{F}$ is an (even countable) infinite collection of infinite subsets of $\mathbb{N}$ such that the intersection of any different two of them is finite. Put $E := \mathbf{P}$ and $H := \{p^2, p \in \mathbf{P}\}$. Then $\mathcal{F}$, E, and H satisfy the required conditions.

2. Considering the collection $\mathcal{F}$ above provides actually an example of a Fréchet Arens-Michael commutative algebra A with unit, and a locally compact space so that the eight algebras in consideration are pairwise distinct.

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