Commun. Korean Math. Soc. **34** (2019), No. 2, pp. 573–583
https://doi.org/10.4134/CKMS.c180205
pISSN: 1225-1763 / eISSN: 2234-3024

BOUNDS AND INEQUALITIES OF THE MODIFIED LOMMEL FUNCTIONS

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Abstract. This article studies the monotonicity, log-convexity of the modified Lommel functions by using its power series and infinite product representation. Some properties for the ratio of the modified Lommel functions with the Lommel function, sinh and cosh are also discussed. As a consequence, Turán type and reverse Turán type inequalities are given. A Rayleigh type function for the Lommel functions are derived and as an application, we obtain the Redheffer-type inequality.

1. Introduction

The Lommel functions [10, 11] are the particular solution of the inhomogeneous Bessel differential equations

$$x^2\mathtt{f}''_{\mu,\nu}(x)+x\mathtt{f}'_{\mu,\nu}(x)-(\nu^2-x^2)\mathtt{f}_{\mu,\nu}(x)=x^{\mu+1}, \tag{1.1}$$

which are usually denoted as $s_{\mu,\nu}$ and $S_{\mu,\nu}$ [2, 18] and $\mathbf{S}_{\mu,\nu}$ [17] given by

$$\mathtt{f}_{\mu,\nu}(x):=\begin{cases} \mathtt{S}_{\mu,\nu}(x)=\frac{x^{\mu+1}}{(\mu+1)^2-\nu^2}\,{}_1F_2\left(1;\frac{\mu-\nu+3}{2},\frac{\mu+\nu+3}{2};-\frac{x^2}{4}\right), \\ \mathbb{S}_{\mu,\nu}(x)=\mathtt{S}_{\mu,\nu}(x)+\frac{(2i)^{\mu-1}}{i^\nu}\Gamma(\frac{\mu-\nu+1}{2})\Gamma(\frac{\mu+\nu+1}{2})J_\nu(x), \\ S_{\mu,\nu}(x)=\mathbb{S}_{\mu,\nu}(x)+i2^{\mu-1}\cos\left(\frac{\pi(\mu-\nu)}{2}\right)\Gamma(\frac{\mu-\nu+1}{2})\Gamma(\frac{\mu+\nu+1}{2})H_\nu^{(1)}(x), \end{cases} \tag{1.2}$$

where J_ν and $H_\nu^{(1)}$ are respectively the Bessel and the Hankel function of the first kind. The above functions satisfy the recurrence relation

$$\mathtt{f}_{\mu+2,\nu}(x)=x^{\mu+1}-\left((\mu+1)^2-\nu^2\right)\mathtt{f}_{\mu,\nu}(x). \tag{1.3}$$

The application of the Lommel functions can be seen in various branches of mathematics and mathematical physics. The mathematical properties of the Lommel functions are available in the literature [5–7, 9, 13, 14, 16, 17]. Like the modified Bessel functions, the analogous of the Lommel functions is the modified Lommel functions. This functions first appear in the theory of screw

Received May 11, 2018; Accepted July 24, 2018.
2010 *Mathematics Subject Classification.* 33C10, 26D7, 26D15.
Key words and phrases. Lommel functions, modified Lommel functions, Turán-type inequality, monotonicity properties, log-convexity.

propeller [8] and later analysed in [12, 15, 17]. The modified Lommel function $\mathtt{g}_{\mu,\nu}$ is a particular solution of the differential equation

$$x^2y''(x) + xy'(x) - (x^2 + \nu^2)y(x) = x^{\mu+1}, \tag{1.4}$$

and satisfies the relation $\mathtt{g}_{\mu,\nu}(x) = i^{-(\mu+1)}\mathtt{f}_{\mu,\nu}(ix)$. Clearly, $\mathtt{g}_{\mu,\nu}$ satisfies the recurrence relation

$$\mathtt{g}_{\mu+2,\nu}(x) = \left((\mu+1)^2 - \nu^2\right)\mathtt{g}_{\mu,\nu}(x) - x^{\mu+1}. \tag{1.5}$$

For $k \in \{0, 1, 2, \ldots\}$, consider the function

$$\varphi_k(x) := {}_1F_2\left(1; \tfrac{\mu-k+2}{2}, \tfrac{\mu-k+3}{2}; -\tfrac{x^2}{4}\right), \tag{1.6}$$

where $x \in \mathbb{R}$ and $\mu \in \mathbb{R}$ such that $\mu - k$ is not in $\{0, -1, -2, \ldots\}$. In [3], it is shown that φ_k is an even real entire function of order one and poses the Hadamard factorization

$$\varphi_k(x) = \prod_{j=1}^{\infty}\left(1 - \frac{x^2}{\eta^2_{\mu,k,n}}\right), \tag{1.7}$$

where $\pm\eta_{\mu,k,n}$ are all zeroes of φ_k. The infinite product in (1.7) is absolutely convergent. The function φ_k have close association with the Lommel function $\mathtt{S}_{\mu,\nu}$ by the relation

$$\mathtt{S}_{\mu-k-1/2,1/2}(x) = \frac{x^{\mu-k+1/2}}{(\mu-k)(\mu-k+1)}\varphi_k(x). \tag{1.8}$$

For $\mu \in (0, 1)$, it is shown in [9] that $\mathtt{S}_{\mu-1/2,1/2}$ has only one zero in each of the interval

$$I_{2n-1}(\mu) = \left(\left(2n-1+\frac{\mu}{2}\right)\pi, (2n-1+\mu)\,\pi\right) \text{ and } I_{2n}(\mu) = \left(2n\pi, \left(2n+\frac{\mu}{2}\right)\pi\right).$$

In this article we consider the function $\mathtt{L}_{\mu,\nu}$ as

$$\begin{aligned}\mathtt{L}_{\mu,\nu}(x) &:= i^{-(\mu+1)}\mathtt{S}_{\mu,\nu}(ix)\\ &= \frac{x^{\mu+1}}{(\mu-\nu+1)(\mu+\nu+1)}\,{}_1F_2\left(1; \frac{\mu-\nu+3}{2}; \frac{\mu+\nu+3}{2}; \frac{x^2}{4}\right).\end{aligned} \tag{1.9}$$

The function $\mathtt{L}_{\mu,\nu}$ is known as the modified Lommel function. We also consider the normalized modified Lommel functions as

$$\begin{aligned}\lambda_{\mu,\nu}(x) &= (\mu-\nu+1)(\mu+\nu+1)x^{-\mu-1}\mathtt{L}_{\mu,\nu}(x)\\ &= \sum_{n=0}^{\infty}\frac{x^{2n}}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n 4^n}.\end{aligned} \tag{1.10}$$

More details about the modified Lommel functions can be seen in [15, 18, 19].

The Section 2 in this article is devoted for the investigation of the monotonicity properties of $\lambda_{\mu,\nu}$. Log-concavity and log-convexity properties in terms of the parameters μ and variable x are also investigated. As a consequence, direct and reverse Turán-type inequalities are obtained. The ratio of the derivatives of $\lambda_{\mu,\nu}$ with sinh and cosh also considered in this section.

In Section 3, the special case for the Lommel and the modified Lommel functions related to φ_k are considered. This section investigate the monotonicity and log-convexity for the product and the ratio of the Lommel and the modified Lommel functions. At the end a Redheffer-type inequality for both the Lommel and the modified Lommel functions is derived.

Following lemma is required in sequel.

Lemma 1.1 ([4])**.** *Suppose $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$, where $a_k \in \mathbb{R}$ and $b_k > 0$ for all k. Further suppose that both series converge on $|x| < r$. If the sequence $\{a_k/b_k\}_{k\geq 0}$ is increasing (or decreasing), then the function $x \mapsto f(x)/g(x)$ is also increasing (or decreasing) on $(0, r)$.*

Notably, the above lemma also holds true when both f and g are even, or both are odd functions.

2. Monotonicity pattern

Theorem 2.1. *Suppose that $\mu, \mu_1 > -1$ and $\nu, \nu_1 \in \mathbb{R}$ such that $\mu \pm \nu$ and $\mu_1 \pm \nu_1$ are not negative odd integer. Then the following assertion are true.*

(i) *Suppose that $\mu_1 \geq \mu > -1$ and $(\mu_1 - \mu)(\mu_1 + \mu + 6) \geq \nu_1^2 - \nu^2$. Then, the function $x \mapsto \lambda_{\mu,\nu}(x)/\lambda_{\mu_1,\nu_1}(x)$ is increase on $(0, \infty)$.*

(ii) *If $\mu \pm \nu + 3 > 0$, then the function $\mu \mapsto \lambda_{\mu,\nu}(x)$ is decreasing and log-convex on $(-1, \infty)$ for each fixed $\nu \in \mathbb{R}$ and $x > 0$.*

(iii) *If $\mu \pm \nu + 3 > 0$, then the function $\nu \mapsto \lambda_{\mu,\nu}(x)$ is log-convex on $\mathbb{R}$ for each fixed $\mu > -1$ and $x > 0$.*

(iv) *The function $x \mapsto \lambda^{2k}_{\mu,\nu}(x)/\cosh(x)$ is strictly decreasing if $(\mu - \nu + 3)(\mu + \nu + 3) > 2$.*

(v) *The function $x \mapsto \lambda^{2k+1}_{\mu,\nu}(x)/\sinh(x)$ is strictly decreasing provided $(\mu - \nu + 5)(\mu + \nu + 5) > 12$.*

Proof. First consider a sequence $\{w_n\}$ defined by

$$w_n := \frac{(a-b)_n (a+b)_n}{(c-d)_n (c+d)_n},$$

where a, b, c, d are real numbers such that $a \pm b$ and $c \pm d$ are not negative integers or zero.

Then a calculation yield

$$\frac{w_{n+1}}{w_n} = \frac{(a-b+n)(a+b+n)}{(c-d+n)(c+d+n)} \geq 1,$$

provided $a^2 - b^2 + 2an + n^2 \geq c^2 - d^2 + 2cn + n^2$, which is equivalent to

$$2(a-c)n + a^2 - b^2 - c^2 + d^2 \geq 0.$$

The last inequality holds for all $n \geq 0$ if $a \geq c$ and $a^2 - b^2 - c^2 + d^2 \geq 0$.

Choose $a = (\mu_1+3)/2$, $b = \nu_1/2$, $c = (\mu+3)/2$ and $d = \nu/2$. Then, $a \geq c$ is equivalent to $\mu_1 \geq \mu$ and $a^2 - b^2 - c^2 + d^2 \geq 0$ reduces to $(\mu_1 - \mu)(\mu_1 + \mu + 6) > \nu_1^2 - \nu^2$. This establish the fact that under the hypothesis in (i) the sequence $\{w_n\}$ is increasing. Since in this case $\{w_n\}$ represent the ratio of the coefficients of $\lambda_{\mu,\nu}(x)$ and $\lambda_{\mu_1,\nu_1}(x)$, the result in (i) follows, in view of Lemma 1.1.

Two prove (ii) and (iii), consider the function

$$g_n(\mu,\nu) := \frac{\Gamma\left(\frac{\mu-\nu+3}{2}\right)\Gamma\left(\frac{\mu+\nu+3}{2}\right)}{\Gamma\left(\frac{\mu-\nu+3}{2}+n\right)\Gamma\left(\frac{\mu+\nu+3}{2}+n\right)}.$$

The first and second partial differentiation of $\log(g_n(\mu,\nu))$ with respect to μ,

$$\begin{aligned}
&\frac{\partial}{\partial\mu}\log(g_n(\mu,\nu))\\
&= \frac{1}{2}\left(\Psi\left(\tfrac{\mu-\nu+3}{2}\right) + \Psi\left(\tfrac{\mu+\nu+3}{2}\right) - \Psi\left(\tfrac{\mu-\nu+3}{2}+n\right) - \Psi\left(\tfrac{\mu+\nu+3}{2}+n\right)\right),\\
&\frac{\partial^2}{\partial\mu^2}\log(g_n(\mu,\nu))\\
&= \frac{1}{4}\left(\Psi'\left(\tfrac{\mu-\nu+3}{2}\right) + \Psi'\left(\tfrac{\mu+\nu+3}{2}\right) - \Psi'\left(\tfrac{\mu-\nu+3}{2}+n\right) - \Psi'\left(\tfrac{\mu+\nu+3}{2}+n\right)\right)\\
&= \frac{\partial^2}{\partial\nu^2}\log(g_n(\mu,\nu)).
\end{aligned}$$

Here, $\Psi(x) = \Gamma'(x)/\Gamma(x)$ is the digamma function which is increasing and concave on $(0,\infty)$. Thus

$$\begin{aligned}
&\frac{\frac{\partial}{\partial\mu}g_n(\mu,\nu)}{g_n(\mu,\nu)} = \frac{\partial}{\partial\mu}\log(g_n(\mu,\nu)) < 0 \quad \text{and}\\
&\frac{\partial^2}{\partial\mu^2}\log(g_n(\mu,\nu)) = \frac{\partial^2}{\partial\nu^2}\log(g_n(\mu,\nu)) \geq 0.
\end{aligned}$$

This conclude that $\mu \mapsto \lambda_{\mu,\nu}(x)$ is decreasing and log-convex on $(-1,\infty)$. Also, $\nu \mapsto \lambda_{\mu,\nu}(x)$ is log-convex on $\mathbb{R}$ for each fixed $\mu > -1$ and $x \in \mathbb{R}$. This prove (ii) and (iii) in view of the fact that the sum of log-convex functions is also log-convex.

A computation yield

$$\lambda_{\mu,\nu}^{(2k)}(x) = \sum_{n=0}^{\infty} \frac{(2n+2k)!}{\left(\frac{\mu-\nu+3}{2}\right)_{n+k}\left(\frac{\mu+\nu+3}{2}\right)_{n+k} 4^{n+k}(2n)!}x^{2n}. \tag{2.1}$$

It is well-known that

$$\cosh(x) = \sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}.$$

In view of Lemma 1.1, it is enough to know the monotonicity of the sequence $\{\alpha_n\}_{n\geq 0}$ where

$$\alpha_n = \frac{(2n+2k)!}{\left(\frac{\mu-\nu+3}{2}\right)_n \left(\frac{\mu+\nu+3}{2}\right)_n 4^{n+k}}.$$

Now, for all $n \geq 0$ and $k \geq 0$, the ratio

$$\frac{\alpha_{n+1}}{\alpha_n} = \frac{(2n+2k+2)(2n+2k+1)}{4\left(\frac{\mu-\nu+3}{2}+n+k\right)\left(\frac{\mu+\nu+3}{2}+n+k\right)} < 1,$$

provided $(\mu-\nu+3)(\mu+\nu+3) > 2$.

Similarly,

$$\begin{aligned} \lambda_{\mu,\nu}^{(2k+1)}(x) &= \sum_{n=0}^{\infty} \frac{(2n+2k+2)!x^{2n+1}}{\left(\frac{\mu-\nu+3}{2}\right)_{n+k+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+k+1} 4^{n+k+1}(2n+1)!} \quad \text{and} \\ \sin(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}, \end{aligned} \tag{2.2}$$

together with Lemma 1.1 yields that $\lambda_{\mu,\nu}^{(2k+1)}(x)/\sin(x)$ is decreasing if the sequence $\{\beta_n\}_{n\geq 0}$ where

$$\beta_n = \frac{(2n+2k+2)!}{\left(\frac{\mu-\nu+3}{2}\right)_{n+k+1}\left(\frac{\mu+\nu+3}{2}\right)_{n+k+1} 4^{n+k+1}(2n+1)!},$$

is also decreasing. Again for all $n \geq 0$ and $k \geq 0$, the ratio

$$\frac{\beta_{n+1}}{\beta_n} = \frac{(2n+2k+4)(2n+2k+3)}{4\left(\frac{\mu-\nu+3}{2}+n+k+1\right)\left(\frac{\mu+\nu+3}{2}+n+k+1\right)} < 1,$$

provided $(\mu-\nu+5)(\mu+\nu+5) > 12$. Hence the conclusion. □

From Theorem 2.1, we have few interesting consequence. For example, the log-convexity of $\mu \mapsto \lambda_{\mu,\nu}(x)$ means, for any $\alpha \in [0,1]$ and for $\mu_1, \mu_2 > -1$,

$$\lambda_{\alpha\mu_1+(1-\alpha)\mu_2,\nu}(x) \leq \lambda_{\mu_1,\nu}^{\alpha}(x)\lambda_{\mu_1,\nu}^{1-\alpha}(x). \tag{2.3}$$

In particular, if $\mu_1 = \mu+a > -1$ and $\mu_2 = \mu-a > -1$ for $\mu, a \in \mathbb{R}$, and $\alpha = 1/2$, then the above inequality gives the reverse of the Turàn's type inequality for the modified Lommel functions as

$$\lambda_{\mu,\nu}^2(x) \leq \lambda_{\mu+a,\nu}(x)\lambda_{\mu-a,\nu}(x).$$

Similarly, the log-convexity of $\nu \mapsto \lambda_{\mu,\nu}(x)$ gives

$$\lambda_{\mu,\nu}^2(x) \leq \lambda_{\mu,\nu+a}(x)\lambda_{\mu,\nu-a}(x).$$

3. Redheffer type bound

In this section we prove the Redheffer-type inequality for some special kind Lommel and modified Lommel functions. From (1.7) and (1.8) it follows that

$$\mathtt{S}_{\mu-1/2,1/2}(x)=\frac{z^{\mu+1/2}}{(\mu)(\mu+1)}\prod_{j=1}^{\infty}\left(1-\frac{x^2}{\eta^2_{\mu,0,n}}\right), \tag{3.1}$$

and this implies

$$\mathtt{L}_{\mu-1/2,1/2}(x)=i^{-\mu-1/2}\mathtt{S}_{\mu-1/2,1/2}(iz)=\frac{z^{\mu+1/2}}{(\mu)(\mu+1)}\prod_{j=1}^{\infty}\left(1+\frac{x^2}{\eta^2_{\mu,0,n}}\right). \tag{3.2}$$

From (1.10) we have

$$\lambda_{\mu-1/2,1/2}(x)=\mu(\mu+1)z^{-\mu-1/2}\mathtt{L}_{\mu-1/2,1/2}(x)=\prod_{j=1}^{\infty}\left(1+\frac{x^2}{\eta^2_{\mu,0,n}}\right). \tag{3.3}$$

Also consider the normalized Lommel function as

$$\Lambda_{\mu-1/2,1/2}(x)=\mu(\mu+1)z^{-\mu-1/2}\mathtt{S}_{\mu-1/2,1/2}(x)=\prod_{j=1}^{\infty}\left(1-\frac{x^2}{\eta^2_{\mu,0,n}}\right). \tag{3.4}$$

Applying logarithmic differentiation on (3.3) gives

$$\frac{\lambda'_{\mu-1/2,1/2}(x)}{z\lambda_{\mu-1/2,1/2}(x)}=\sum_{n=1}^{\infty}\frac{2}{\eta^2_{\mu,0,n}+x^2}. \tag{3.5}$$

A calculation gives

$$\lim_{z\to 0}\frac{\lambda'_{\mu,\nu}(x)}{z\lambda_{\mu,\nu}(x)}=\lim_{z\to 0}\frac{\sum_{n=1}^{\infty}\frac{2nz^{2n-2}}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n4^n}}{\sum_{n=0}^{\infty}\frac{z^{2n}}{\left(\frac{\mu-\nu+3}{2}\right)_n\left(\frac{\mu+\nu+3}{2}\right)_n4^n}}=\frac{2}{(\mu+3)^2-\nu^2}. \tag{3.6}$$

Now (3.5) and (3.6) together give the useful identity

$$\sum_{n=1}^{\infty}\frac{1}{\eta^2_{\mu,0,n}}=\frac{4}{(2\mu+5)^2-1}=\frac{1}{(\mu+2)(\mu+3)}. \tag{3.7}$$

For simplicity in sequel, we will use the notation $\eta_{\mu,n}$ for $\eta_{\mu,0,n}$, the n^{th} positive zero of $\varphi_0(x)$.

Next we state and proof some results involving the function $\lambda_{\mu-1/2,1/2}$ and $\Lambda_{\mu-1/2,1/2}$. The monotonic properties of l'Hospital' rule as state in the following result are useful in sequel.

Lemma 3.1 ([1, Lemma 2.2]). *Suppose that $-\infty<a<b<\infty$ and $p,q:[a,b)\mapsto\infty$ are differentiable functions such that $q'(x)\neq 0$ for $x\in(a,b)$. If p'/q' is increasing (decreasing) on (a,b), then so is $(p(x)-p(a))/(q(x)-q(a))$.*

Next we will state and proof our main result in this section.

Theorem 3.1. *Suppose that* $\mu > -1$ *and* $I_\mu := (-\eta_{\mu,1}, \eta_{\mu,1})$.

(1) *The function* $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ *is increasing on* $(0,\infty)$.

(2) *The function* $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ *is strictly log-convex on* I_μ *and strictly geometrically convex on* $(0,\infty)$.

(3) *The modified Lommel functions* $\lambda_{\mu-1/2,1/2}(x)$ *satisfies the sharp exponential Redheffer-type inequality*

$$\left(\frac{\eta_{\mu,1}^2 + x^2}{\eta_{\mu,1}^2 - x^2}\right)^{a_\mu} \le \lambda_{\mu-1/2,1/2}(x) \le \left(\frac{\eta_{\mu,1}^2 + x^2}{\eta_{\mu,1}^2 - x^2}\right)^{b_\mu} \tag{3.8}$$

on I_μ. *Here,* $a_\mu = 0$ *and* $b_\mu = \frac{2\eta_{\mu,1}^2}{(\mu+2)(\mu+3)}$ *are the best possible constant.*

(4) *The function* $x \mapsto \lambda_{\mu-1/2,1/2}(x)\Lambda_{\mu-1/2,1/2}(x)$ *is increasing on* $(-\eta_{\mu,1}, 0]$ *and decreasing on* $[0, \eta_{\mu,1})$.

(5) *The function*

$$x \mapsto \frac{\lambda_{\mu-1/2,1/2}(x)}{\Lambda_{\mu-1/2,1/2}(x)} = \frac{L_{\mu-1/2,1/2}(x)}{S_{\mu-1/2,1/2}(x)}$$

is strictly log-convex on I_μ.

(6) *The Lommel functions* $\Lambda_{\mu-1/2,1/2}(x)$ *satisfies the sharp exponential Redheffer-type inequality*

$$\left(\frac{\eta_{\mu,1}^2 - x^2}{\eta_{\mu,1}^2}\right)^{a_\mu} \le \Lambda_{\mu-1/2,1/2}(x) \le \left(\frac{\eta_{\mu,1}^2 - x^2}{\eta_{\mu,1}^2}\right)^{b_\mu} \tag{3.9}$$

on I_μ. *Here,* $a_\mu = 0$ *and* $b_\mu = \frac{2\eta_{\mu,1}^2}{(\mu+2)(\mu+3)}$ *are the best possible constant.*

Proof. Consider $\mu > -1$ and $x \in (-\eta_{\mu,1}, \eta_{\mu,1})$.

(1) From (3.5) it is evident that

$$(\log(\lambda_{\mu-1/2,1/2}(x)))' = \frac{\lambda'_{\mu-1/2,1/2}(x)}{\lambda_{\mu-1/2,1/2}(x)} = \sum_{n=1}^{\infty} \frac{2x}{\eta_{\mu,n}^2 + x^2} > 0$$

on $(0,\infty)$. Thus, for $\mu > -1$, the function $x \mapsto \log(\lambda_{\mu-1/2,1/2}(x))$ is strictly increasing on $(0,\infty)$ and consequently $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ is also increasing on $(0,\infty)$.

(2) Again from (3.5) it follows that

$$\left(\frac{\lambda'_{\mu-1/2,1/2}(x)}{\lambda_{\mu-1/2,1/2}(x)}\right)' = \sum_{n=1}^{\infty} \frac{2(\eta_{\mu,n}^2 - x^2)}{(\eta_{\mu,n}^2 + x^2)^2}.$$

Clearly, the function $x \mapsto \lambda'_{\mu-1/2,1/2}(x)/\lambda_{\mu-1/2,1/2}(x)$ is increasing for $x \in (-\eta_{\mu,1}, \eta_{\mu,1})$. This is equivalent to say that the function $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ is log-convex on $(-\eta_{\mu,1}, \eta_{\mu,1})$.

Another calculation from (3.5) yields

$$\left(\frac{z\lambda'_{\mu-1/2,1/2}(x)}{\lambda_{\mu-1/2,1/2}(x)}\right)' = \sum_{n=1}^{\infty} \frac{4z\eta_{\mu,n}^2}{(\eta_{\mu,n}^2 + x^2)^2}.$$

This implies that the function $x \mapsto x\lambda'_{\mu-1/2,1/2}(x)/\lambda_{\mu-1/2,1/2}(x)$ is strictly increasing for $x \in (0,\infty)$ and hence $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ is geometrically convex on $(0,\infty)$.

(3) Consider the function

$$g_\mu(x) := \frac{\log(\lambda_{\mu-1/2,1/2}(x))}{\log(\eta_{\mu,1}^2 + x^2) - \log(\eta_{\mu,1}^2 - x^2)}.$$

Denote $p(x) := \log(\lambda_{\mu-1/2,1/2}(x))$ and $q(x) := \log(\eta_{\mu,1}^2 + x^2) - \log(\eta_{\mu,1}^2 - x^2)$ on $x \in [0,\infty)$. In view of (3.5), it follows that

$$\frac{p'(x)}{q'(x)} = \frac{\eta_{\mu,n}^4 - x^4}{4x\eta_{\mu,1}^2} \frac{\lambda'_{\mu-1/2,1/2}(x)}{\lambda_{\mu-1/2,1/2}(x)} = \frac{1}{2\eta_{\mu,1}^2} \sum_{n=1}^{\infty} \frac{\eta_{\mu,1}^4 - x^4}{\eta_{\mu,n}^2 + x^2}$$

and then

$$\frac{d}{dx}\left(\frac{p'(x)}{q'(x)}\right) = -\frac{x}{\eta_{\mu,1}^2} \sum_{n=1}^{\infty} \frac{z^4 + 2x^2\eta_{\mu,1}^2 + \eta_{\mu,n}^4}{(\eta_{\mu,n}^2 + x^2)^2} \leq 0.$$

Thus, $p'(x)/q'(x)$ is decreasing.

Therefore,

$$g_\mu(x) = \frac{p(x) - p(0)}{q(x) - q(0)} = \frac{p(x)}{q(x)}$$

is decreasing too on $[0, \eta_{\mu,1})$ and hence

$$a_\mu = \lim_{x\to\eta_{\mu,1}} g_\mu(x) < g_\mu(x) < \lim_{x\to 0} g_\mu(x) = b_\mu.$$

Finally,

$$\lim_{x\to\eta_{\mu,1}} \frac{p'(x)}{q'(x)} = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{p'(x)}{q'(x)} = \frac{\eta_{\mu,1}^2}{2} \sum_{n=1}^{\infty} \frac{1}{\eta_{\mu,n}^2} = \frac{\eta_{\mu,1}^2}{2(\mu+2)(\mu+3)}$$

implies $a_\mu = 0$ and $b_\mu = \eta_{\mu,1}^2/(2(\mu+2)(\mu+3))$.

(4) From (3.6) and (3.4), it is evident that

$$\lambda_{\mu-1/2,1/2}(x)\Lambda_{\mu-1/2,1/2}(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^4}{\eta_{\mu,n}^4}\right).$$

Thus, by the logarithmic differentiation it follows that

$$\frac{\Big(\lambda_{\mu-1/2,1/2}(x)\Lambda_{\mu-1/2,1/2}(x)\Big)'}{\lambda_{\mu-1/2,1/2}(x)\Lambda_{\mu-1/2,1/2}(x)} = -\sum_{n=1}^{\infty} \frac{4x^3}{\eta_{\mu,n}^4 - x^4}.$$

Since $x \in I_\mu$, the conclusion follows.

(5) From (3.4), we have the logarithmic differentiation of $(\Lambda_{\mu-1/2,1/2}(x))^{-1}$ as

$$\left(\log\left((\Lambda_{\mu-1/2,1/2}(x))^{-1}\right)\right)' = \sum_{n=1}^{\infty} \frac{2x}{\eta_{\mu,n}^2 - x^2}$$

and

$$\left(\log\left((\Lambda_{\mu-1/2,1/2}(x))^{-1}\right)\right)'' = 2\sum_{n=1}^{\infty} \frac{\eta_{\mu,n}^2 + x^2}{(\eta_{\mu,n}^2 - x^2)^2} > 0.$$

This conclude that the function $x \mapsto (\Lambda_{\mu-1/2,1/2}(x))^{-1}$ is strictly log-convex on I_μ. Finally, being the product of two strictly log-convex functions, the function

$$x \mapsto \frac{\lambda_{\mu-1/2,1/2}(x)}{\Lambda_{\mu-1/2,1/2}(x)} = \frac{L_{\mu-1/2,1/2}(x)}{S_{\mu-1/2,1/2}(x)}$$

is also strictly log-convex. Note that the log-convexity of $x \mapsto \lambda_{\mu-1/2,1/2}(x)$ follows from part (2) of this theorem.

(6) To prove this result first we need to set up a Rayleigh type functions for the Lommel function. Define the function

$$\alpha_{n,\mu}^{(2m)} := \sum_{n=1}^{\infty} \eta_{\mu,n}^{-2m}, \quad m = 1, 2, \ldots. \tag{3.10}$$

Logarithmic differentiation of (3.4) yield

$$\begin{aligned}\frac{x\Lambda_{\mu-1/2,1/2}'(x)}{\Lambda_{\mu-1/2,1/2}(x)} &= -2\sum_{n=1}^{\infty} \frac{x^2}{\eta_{\mu,n}^2 - x^2} \\ &= \sum_{n=1}^{\infty} \frac{x^2}{\eta_{\mu,n}^2}\left(1 - \frac{x^2}{\eta_{\mu,n}^2}\right)^{-1} \\ &= \sum_{n=1}^{\infty} \frac{x^2}{\eta_{\mu,n}^2} \sum_{m=0}^{\infty} \frac{x^{2m}}{\eta_{\mu,n}^{2m}}.\end{aligned}$$

Interchanging the order of the summation it follows that

$$\frac{x\Lambda_{\mu-1/2,1/2}'(x)}{\Lambda_{\mu-1/2,1/2}(x)} = -2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{x^{2m+2}}{\eta_{\mu,n}^{2m+2}} = -2\sum_{m=1}^{\infty} \alpha_{n,\mu}^{(2m)} x^{2m}. \tag{3.11}$$

Consider the function

$$\varphi_\mu(x) := \frac{\log(\Lambda_{\mu-1/2,1/2}(x))}{\log\left(1 - \frac{x^2}{\eta_{\mu,1}^2}\right)} = \frac{\mathtt{p}_\mu(x)}{\mathtt{q}_\mu(x)}. \tag{3.12}$$

The binomial series together with (3.11) gives the ratio of $\mathtt{p}_\mu'$ and $\mathtt{q}_\mu'$ as

$$\frac{\mathtt{p}_\mu'(x)}{\mathtt{q}_\mu'(x)} = \frac{\frac{x\Lambda_{\mu-1/2,1/2}'(x)}{\Lambda_{\mu-1/2,1/2}(x)}}{\frac{-2x^2}{\eta_{\mu,1}^2}\left(1 - \frac{x^2}{\eta_{\mu,1}^2}\right)^{-1}} = \frac{\sum_{m=1}^{\infty} \alpha_{n,\mu}^{(2m)} x^{2m}}{\sum_{m=1}^{\infty} \eta_{\mu,1}^{-2m} x^{2m}}. \tag{3.13}$$

Denote $d_m = \eta_{\mu,1}^{2m}\alpha_{n,\mu}^{(2m)}$. Then

$$d_{m+1} - d_m = \eta_{\mu,1}^{2m+2}\alpha_{n,\mu}^{(2m+2)} - \eta_{\mu,1}^{2m}\alpha_{n,\mu}^{(2m)} = \sum_{n=1}^{\infty} \frac{\eta_{\mu,1}^{2m}}{\eta_{\mu,n}^{2m}} \left(\frac{\eta_{\mu,1}^2}{\eta_{\mu,n}^2} - 1 \right) < 0.$$

This is equivalent to say that the sequence $\{d_m\}$ is decreasing, and hence by Lemma 1.1 it follows that the ratio p'_μ/q'_μ is decreasing. In view of Lemma 3.1, we have $\varphi_\mu = p_\mu/q_\mu$ is decreasing.

From (3.12) and (3.13), it can be shown that

$$\lim_{x\to 0} \varphi_\mu(x) = \lim_{x\to 0} \frac{p'_\mu(x)}{q'_\mu(x)} = \lim_{x\to 0} \frac{p''_\mu(x)}{q''_\mu(x)} = \lim_{x\to 0} \frac{p''_\mu(x)}{q''_\mu(x)} = \eta_{\mu,1}^2\alpha_{\mu,n}^{(2)}$$

and

$$\lim_{x\to \eta_{\mu,1}} \varphi_\mu(x) = \lim_{x\to \eta_{\mu,1}} \frac{p'_\mu(x)}{q'_\mu(x)} = \lim_{x\to \eta_{\mu,1}} \sum_{n=1}^{\infty} \frac{\eta_{\mu,1}^2 - x^2}{\eta_{\mu,n}^2 - x^2} = 1.$$

It is easy to seen that $\eta_{\mu,1}^2\alpha_{\mu,n}^{(2)} = b_\mu$. □

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