Commun. Korean Math. Soc. **34** (2019), No. 2, pp. 565–572
https://doi.org/10.4134/CKMS.c180194
pISSN: 1225-1763 / eISSN: 2234-3024

GLOBAL EXISTENCE AND STABILITY OF A KORTEWEG-DE VRIES EQUATION IN NONCYLINDRICAL DOMAIN

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Abstract. In this paper, we consider a Korteweg-de Vries equation in noncylindrical domain. This work is devoted to prove existence and uniqueness of global solutions employing Faedo-Galerkin's approximation and transformation of the noncylindrical domain with moving boundary into cylindrical one. Moreover, we estimate the exponential decay of solutions in the asymptotically cylindrical domain.

1. Introduction

In this paper, we are concerned with global existence and stability of a Korteweg-de Vries equation given by

$$\begin{cases} u_t + u_x + u_{xxx} + uu_x + a(x)u = 0 \ \text{ in } \hat{Q}, \\ u(\alpha(t), t) = u(\beta(t), t) = u_x(\beta(t), t) = 0 \ \text{ for } t \geq 0, \\ u(x, 0) = u_0(x) \ \text{ for } x \in [\alpha_0, \beta_0], \end{cases} \tag{1.1}$$

where α, $\beta \in C^2([0, \infty))$, $\alpha(0) = \alpha_0 < \beta_0 = \beta(0)$ and $\hat{Q} = \{(x, t) \in \mathbb{R}^2 : \alpha(t) < x < \beta(t),\ 0 < t < T\}$.

The Korteweg-de Vries equation was initially derived by Korteweg and de Vries [5] as a model for one-directional water waves of small amplitude in shallow water. At present it is known that the Korteweg-de Vries equation is not only a good model for water waves but also a very useful approximation model in nonlinear studies whenever one wishes to include and balance weak nonlinear and dispersive effect.

In the case of Linares and Pazoto [7], they studied the problem (1.1) in the cylindrical domain. On the other hand, the case of noncylindrical domain problems has been studied less than cylindrical domain problems (cf. [1–4]). For

Received May 4, 2018; Revised October 11, 2018; Accepted October 18, 2018.
2010 *Mathematics Subject Classification.* 35Q53, 35B40.
Key words and phrases. existence of solution, energy decay, noncylindrical domain.
This paper was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (NRF-2016 R1D1A1B03932096).

example, Benabidallah and Ferreira [1] studied the existence of solutions for the hyperbolic-parabolic equation in noncylindrical domain. Also Clark et al. [2] proved the global solvability, uniqueness of solutions and the exponential decay of solution for the Boussinesq equation in the noncylindrical domain. Recently, Ha [3] proved the existence of solutions and decay of the energy of solutions for the Kirchhoff type wave equation in the noncylindrical domain. However, there is few research for the Korteweg-de Vries equation in noncylindrical domain.

In this paper, we study existence and uniqueness of global solutions for the Korteweg-de Vries equation in noncylindrical domain as well as the exponential decay of small solutions in asymptotically close to cylindrical domain.

This paper is organized as follows: In Section 2, we recall notations and hypotheses and introduce our main results. In Section 3, we prove the existence and uniqueness of solution employing Faedo-Galerkin's method. In Section 4, we prove the exponential decay rate for the solution.

2. Hypotheses and main results

We begin this section introducing some hypotheses and our main results. Throughout this paper we use standard functional spaces. And $'$ denotes the derivative with respect to time t.

The idea that we use to prove the result of existence and uniqueness is based on the transformation of our problem into another initial boundary value problem defined over a cylindrical domain whose sections are not time dependent. This is done using a suitable change of variable. Our existence result on noncylindrical domain will follow using the inverse transformation. That is, using the diffeomorphism $\tau : \hat{Q} \to Q = (0,1) \times [0,\infty)$ defined by

$$\tau(x,t) = (y,t) = \left(\frac{x-\alpha(t)}{\gamma(t)}, t\right) \tag{2.1}$$

and $\tau^{-1} : Q \to \hat{Q}$ defined by

$$\tau^{-1}(y,t) = (x,t) = (\alpha(t) + \gamma(t)y, t). \tag{2.2}$$

Denoting by v function

$$v(y,t) = u \circ \tau^{-1}(y,t) = u(\alpha(t) + \gamma(t)y, t) \tag{2.3}$$

the initial boundary value problem (1.1) becomes

$$\begin{cases} v_t - \frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y - 1)v_y + \frac{1}{\gamma^3(t)}v_{yyy} + \frac{1}{\gamma(t)}vv_y \\ +a(\alpha(t) + \gamma(t)y)v = 0 \quad \text{in } Q, \\ v(0,t) = v(1,t) = v_y(1,t) = 0 \quad \text{for } t \geq 0, \\ v(y,0) = v_0(y) \quad \text{for } y \in [0,1]. \end{cases} \tag{2.4}$$

We now give hypotheses for the main result.

($\mathbf{H_1}$) Hypotheses on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

(2.5) $\alpha, \beta \in C^2([0,\infty))$, (2.6) There exist δ_0 and δ_1 such that $0 < \delta_0 \leq \gamma := \gamma(t) = \beta(t) - \alpha(t) \leq \delta_1$ for all $t \geq 0$,

(2.7) $\alpha'(t) > 0,\ \beta'(t) < 0$ for all $t \geq 0$.

($\mathbf{H_2}$) Hypotheses on $\boldsymbol{a := a(x)}$.

$$a \in L^\infty(\alpha(t), \beta(t)) \text{ for all } t \geq 0, \tag{2.8}$$

$$a \geq a_0 > 0 \text{ for all } x. \tag{2.9}$$

Note that the assumption (2.7) means that $\hat{Q}$ is decreasing in the sense that if $t_1 > t_2$, then the projection of $[\alpha(t_2), \beta(t_2)]$ on the subspace $t = 0$ contains the projection of $[\alpha(t_1), \beta(t_1)]$ on the same subspace.

Now we are in a position to state our main results.

Theorem 2.1. *Let $v_0 \in L^2(0,1)$ and (H_1)-(H_2) hold. Then for all finite $T > 0$ there exists a unique weak solution v of the problem (2.4) satisfying*

$$u \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)).$$

As a consequence of the above theorem and using the change variable given in (2.1), we obtain the next result.

Theorem 2.2. *Let I_t and I_0 the intervals $(\alpha(t), \beta(t))$ $(0 < t < T)$ and $(\alpha(0), \beta(0))$ respectively. Let $u_0 \in L^2(I_0)$ and (H_1)-(H_2) hold. Then for all finite $T > 0$ there exists a unique weak solution u of the problem (1.1) satisfying*

$$u \in L^\infty(0,T; L^2(I_t)) \cap L^2(0,T; H^1(I_t)).$$

Theorem 2.3. *Let u be the solution of problem (1.1) given by Theorem 2.2. Suppose $\alpha', \beta' \in L^1(0,\infty) \cup L^\infty(0,\infty)$ and $||u_0||_{L^2(I_0)} \leq \frac{3}{8K\gamma^2}$, where $K = \exp\Big\{\frac{1}{2}\int_0^T \big(\frac{1}{\gamma(t)} - \frac{\gamma'(t)}{\gamma(t)}\big)dt\Big\}$ for all $t \geq 0$. Then there exist positive constants c and ω such that*

$$||u||^2_{L^2(I_t)} \leq c||u_0||^2_{L^2(I_0)} e^{-\omega t}$$

holds for all $t > 0$.

3. Existence of solutions

3.1. Proof of Theorem 2.1

In this subsection we prove the existence and uniqueness of weak solutions to problem (2.4). First of all we note that by [6], for every fixed $m \in \mathbb{N}$ there exists a unique strong solution to problem (2.4) in the class

$$v^m \in L^\infty(0,\infty; H^3(0,1)), \quad v_t^m \in L^\infty(0,\infty; L^2(0,1) \cap L^2(0,\infty; H^1(0,1)). \tag{3.1}$$

We now suppose $v_0(x) \in L^2(0,1)$ and let $(w_j)_{j\in\mathbb{N}}$ be a basis in $L^2(0,1)$, and let V_m be the finite dimensional subspace of $L^2(0,1)$ spanned by the first m

vectors $\{w_1, w_2, \dots, w_m\}$. We construct approximations

$$v^m = \sum_{j=1}^{m} g_j^m(t) w_j(x),$$

where $g_j^m(t)$ are solutions for the nonlinear system of ordinary differential equations

$$(3.2)\quad (v_t^m, w_j) - (\frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y - 1)v_y^m, w_j) + \frac{1}{\gamma^3(t)}(v_{yyy}^m, w_j) + \frac{1}{\gamma(t)}(v^m v_y^m, w_j) + (a(\alpha(t) + \gamma(t)y)v^m, w_j) = 0, \qquad (j = 1, \dots, m),$$

$$(3.3)\qquad v^m(y, 0) = v_0^m(y) \to v_0(y) \quad \text{in} \quad L^2(0,1).$$

Here (u, v) is the inner product in $L^2(0,1)$. From the ODE theory we have the local solution $v^m = v^m(y,t)$ well defined on the interval $[0, t_m)$. The estimates that follow permit us to extend the solution v^m to the whole interval $[0, T)$ and take the limit in v^m as $m \to \infty$. Henceforth the symbol C_i, $i \in \mathbb{N}$ indicates positive constants, which may be different.

3.1.1. *The first estimate.* Setting $w_j = v^m$ in (3.2) and integrating over $(0, t)$ with $t \in (0, t_m)$, we get

$$\frac{1}{2}||v^m||^2_{L^2(0,1)} + \frac{1}{2\delta_1^3}\int_0^t |v_y^m(0,s)|^2 ds \le C_1 \int_0^t ||v^m(y,s)||^2_{L^2(0,1)} ds + \frac{1}{2}||v_0^m(y)||^2_{L^2(0,1)}.$$

Therefore, by Gronwall's lemma we have

$$(3.4)\qquad ||v^m||^2_{L^2(0,1)} + \int_0^t |v_y^m(0,s)|^2 ds \le C_2,$$

where C_2 is a positive constant which is independent m and t.

3.1.2. *The second estimate.* Setting $w_j = yv^m$ in (3.2), we get

$$(3.5)\qquad \frac{1}{2}\frac{d}{dt}\int_0^1 y|v^m|^2 dy + \frac{\alpha'}{2\gamma}\int_0^1 |v^m|^2 dy + \frac{\gamma'}{\gamma}\int_0^1 y|v^m|^2 dy - \frac{1}{2\gamma}\int_0^1 |v^m|^2 dy + \frac{3}{2\gamma^3}\int_0^1 |v_y^m|^2 dy - \frac{1}{3}\int_0^1 (v^m)^3 dy + \int_0^1 a(\alpha(t) + \gamma(t)y)(v^m)^2 y dy = 0.$$

Integrating (3.5) over $(0, t)$ with $t \in (0, t_m)$ and using (H_1), (H_2) and (3.4), we have

$$(3.6)\qquad \int_0^t \int_0^1 |v_y^m(y,s)|^2 dy ds \le \frac{2\delta_1^3}{9}\int_0^t \int_0^1 |v^m(y,s)|^3 dy ds + C_3.$$

On the other hand, Hölder's inequality, Sobloev imbedding theorem and Young's inequality imply

$$\int_0^t \int_0^1 |v^m(y,s)|^3 dyds \leq \int_0^t ||v^m||_{L^\infty(0,1)} ||v^m||^2_{L^2(0,1)} ds$$
$$\leq \epsilon \int_0^t ||v^m||^2_{H_0^1(0,1)} + C(\epsilon) \int_0^t ||v^m||^4_{L^2(0,1)} ds.$$

Replacing above inequality in (3.6) and choosing $\epsilon > 0$ sufficiently small, we have

$$\int_0^t \int_0^1 |v_y^m(y,s)|^2 dyds \leq C_4. \tag{3.7}$$

From (3.4) and (3.7), we have

$$v^m \to v \quad \text{weakly star in} \quad L^\infty(0,T;L^2(0,1)),$$

and

$$v^m \to v \quad \text{weakly in} \quad L^2(0,T;H^1(0,1)).$$

Moreover, convergences (3.4) and (3.7), and the regularity (3.1) permit us to pass to the limit as $m \to \infty$, consequently we obtain, for all $w \in H^2(0,1) \cap H^1(0,1)$

$$(v_t, w) - (A(y,t)v_y, w) + \frac{1}{\gamma^3(t)}(v_y, w_{yy}) + \frac{1}{\gamma(t)}(vv_y, w) + (B(y,t)v, w) = 0, \tag{3.8}$$

where $A(y,t) = \frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y - 1)$ and $B(y,t) = a(\alpha(t) + \gamma(t)y)$. This implies that the existence part of Theorem 2.1 is proved.

3.1.3. *Uniqueness.* Let v and $\bar{v}$ be two solutions of (2.4). Define $z = v - \bar{v}$, we have

$$z_t - \frac{1}{\gamma(t)}(\alpha'(t) + \gamma'(t)y - 1)z_y + \frac{1}{\gamma^3(t)} z_{yyy} + \frac{1}{\gamma(t)}(vz_y + \bar{v}_y z) \tag{3.9}$$
$$+ a(\alpha(t) + \gamma(t)y)z = 0.$$

Multiplying (3.9) by z, we obtain

$$(z_t, z) - \frac{1}{\gamma(t)}((\alpha'(t) + \gamma'(t)y - 1)z_y, z) + \frac{1}{\gamma^3(t)}(z_{yyy}, z) \tag{3.10}$$
$$+ \frac{1}{\gamma(t)}(vz_y + \bar{v}_y z, z) + (a(\alpha(t) + \gamma(t)y)z, z) = 0.$$

Due to regularity of solutions v and $\bar{v}$,

$$\sup_{\bar{Q}}\{|v(y,t)|, |\bar{v}(y,t)|, |v_y(y,t)|, |\bar{v}_y(y,t)|\} \leq \nu < \infty.$$

Then (3.10) becomes

$$\frac{1}{2}\frac{d}{dt}||z||^2_{L^2(0,1)} + \frac{1}{2} z_y^2(0,t) \tag{3.11}$$

$$\leq \frac{\nu}{\delta_0}\int_0^1 |z_y|\ |z|dy + \frac{\nu}{\delta_0}||z||^2_{L^2(0,1)} - \frac{\gamma'}{2\gamma}||z||^2_{L^2(0,1)}.$$

Integrating (3.11) over $(0,t)$ and then using the fact $\gamma > 0$, $\gamma' < 0$ and Gronwall's lemma, we have $||z||_{L^2(0,1)} = 0$. This completes the proof of uniqueness.

3.2. Proof of Theorem 2.2

To show the existence in noncylindrical domain, we return to our original problem in the noncylindrical domain by using the change variable given in (2.1) by $(y,t) = \tau(x,t)$, $(x,t) \in \hat{Q}$. Let v be the solution obtained from Theorem 2.1 and u defined by (2.3), then u belongs to the class

$$u \in L^\infty(0,T;L^2(I_t)) \cap L^2(0,T;H^1(I_t)).$$

Denoting by $u(x,t) = v(y,t) = (v\circ\tau)(x,t)$ then from (2.2) it is easy to see that u satisfies equations (1.1) in the sense of $L^\infty(0,\infty;L^2(I_t))$. Let u_1 and u_2 be two solutions to (1.1) and v_1 and v_2 be the functions obtained through the diffeomorphism τ given by (2.1). Then v_1 and v_2 are solutions to (2.4). By the uniqueness result of Theorem 2.1, we have $v_1 = v_2$, so $u_1 = u_2$. Thus the proof of Theorem 2.2 is completed.

4. Stability

In this section we will prove Theorem 2.3. To prove the exponential decay of L^2-norm of solution, we are going to obtain time independent a priori estimate for this norm. Multiplying (2.4) by v, we get

$$\frac{d}{dt}||v||^2_{L^2(0,1)} \leq \Big(\frac{1}{\gamma} - \frac{\gamma'}{\gamma}\Big)||v||^2_{L^2(0,1)}.$$

By Gronwall's lemma this imply

$$||v||^2_{L^2(0,1)} \leq ||v_0||^2_{L^2(0,1)} \exp\Big\{\int_0^t \Big(\frac{1}{\gamma(s)} - \frac{\gamma'(s)}{\gamma(s)}\Big)ds\Big\} \tag{4.1}$$

for all $t \geq 0$.

Now we define

$$\phi(y) = 1 + 4y - y^3, \qquad y \in [0,1].$$

Then we can easily check that

$$\min_{y\in[0,1]}\phi(y) = 1, \qquad \max_{y\in[0,1]}\phi(y) = 4, \qquad \phi_y(y) \geq 1, \qquad \max_{y\in[0,1]}\phi_y(y) = 4. \tag{4.2}$$

Multiplying (2.4) by $\phi(y)v$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}(\phi,v^2) - \Big(\frac{\alpha'}{\gamma} - \frac{1}{\gamma}\Big)(v_y,\phi v) - \frac{\gamma'}{\gamma}(v_y,y\phi v) + \frac{1}{\gamma^3}(v_{yyy},\phi v)\\ + \frac{1}{\gamma}(v_y,\phi v^2) + (a(\alpha(t)+\gamma(t)y)v^2,\phi) = 0.\end{aligned} \tag{4.3}$$

We note that

$$(v_y, \phi v) = -\frac{1}{2}(\phi_y, v^2),$$
$$(v_y, y\phi v) = -\frac{1}{2}(\phi, v^2) - \frac{1}{2}(y\phi_y, v^2),$$
$$(v_{yyy}, \phi v) = -\frac{1}{2}(\phi_{yyy}, v^2) + (\phi_y, v_y^2) + \frac{1}{2}v_y^2(0,t) + \frac{1}{2}(\phi_y, y_y^2).$$

And using (4.1) and (4.2), we get

$$\begin{aligned}(v_y, \phi v^2) = -\frac{1}{3}(\phi_y, v^3) &\geq -\frac{4}{3}(1, v_y^2)||v||_{L^2(0,1)} \\ &\geq -\frac{4}{3}||v_0||_{L^2(0,1)} \exp\left\{\int_0^t \Big(\frac{1}{\gamma(s)} - \frac{\gamma'(s)}{\gamma(s)}\Big)ds\right\}(\phi_y, v_y^2).\end{aligned}$$

Substituting above calculations into (4.3) and using (2.6) and (4.2), we have

$$\begin{aligned}\frac{d}{dt}(\phi, v^2) + \left(\frac{3}{\gamma^3} - \frac{8}{3\gamma}||v_0||_{L^2(0,1)} \exp\left\{\frac{1}{2}\int_0^t \Big(\frac{1}{\gamma(s)} - \frac{\gamma'(s)}{\gamma(s)}\Big)ds\right\}\right)(\phi_y, v_y^2) \\ + \frac{3}{2\delta_1}(\phi, v^2) \leq M(t),\end{aligned} \tag{4.4}$$

where $M(t) = \frac{4}{\delta_0}(|\beta'| + |\gamma'| + 1)||v_0||_{L^2(0,1)} \exp\Big\{\frac{1}{2}\int_0^t \big(\frac{1}{\gamma(s)} - \frac{\gamma'(s)}{\gamma(s)}\big)ds\Big\}$.

Taking into account the condition Theorem 2.3 and using the fact $(\phi_y, v_y^2) \geq ||v_y||^2_{L^2(0,1)} \geq ||v||^2_{L^2(0,1)} \geq \frac{1}{4}(\phi, v^2)$, we can rewrite (4.4) as follows

$$\frac{d}{dt}(\phi, v^2) + \frac{1 + 3\delta_1^2}{2\delta_1^3}(\phi, v^2) \leq M(t).$$

We denote $||\phi^{\frac{1}{2}} v||_{L^2(0,1)} = z(t)$. Then above inequality becomes

$$\frac{d}{dt}z(t) + \frac{1 + 3\delta_1^2}{2\delta_1^3}z(t) \leq M(t).$$

This implies

$$z(t) \leq e^{-\frac{1+3\delta_1^2}{2\delta_1^3}t}\Big(z(0) + \int_0^t e^{\frac{1+3\delta_1^2}{2\delta_1^3}\tau} M(\tau)d\tau\Big).$$

Returning to the original value $u(x,t)$, the proof of Theorem 2.3 is completed.

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