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COMMUTANTS OF TOEPLITZ OPERATORS WITH POLYNOMIAL SYMBOLS ON THE DIRICHLET SPACE

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Abstract. We study commutants of Toeplitz operators acting on the Dirichlet space of the unit disk and prove that an operator in the Toeplitz algebra commuting with a Toeplitz operator with a nonconstant polynomial symbol must be a Toeplitz operator with an analytic symbol.

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. The Sobolev space $\mathscr{L}^{2,1}$ is the completion of the space of all smooth functions f on $\mathbb{D}$ for which

$$\|f\| = \left\{ \left| \int_{\mathbb{D}} f\, dA \right|^2 + \int_{\mathbb{D}} \left(\left| \frac{\partial f}{\partial z} \right|^2 + \left| \frac{\partial f}{\partial \bar{z}} \right|^2 \right) dA \right\}^{1/2} < \infty,$$

where dA denotes the normalized Lebesgue measure on $\mathbb{D}$. The space $\mathscr{L}^{2,1}$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f dA \int_{\mathbb{D}} \bar{g}\, dA + \int_{\mathbb{D}} \left(\frac{\partial f}{\partial z} \overline{\frac{\partial g}{\partial z}} + \frac{\partial f}{\partial \bar{z}} \overline{\frac{\partial g}{\partial \bar{z}}} \right) dA.$$

The Dirichlet space $\mathscr{D}$ is then a closed subspace consisting of all analytic functions in $\mathscr{L}^{2,1}$. Let P denote the orthogonal projection from $\mathscr{L}^{2,1}$ onto $\mathscr{D}$. Given a function $\varphi \in \mathscr{L}^{2,1}$, the Toeplitz operator T_φ with symbol φ is densely defined on $\mathscr{D}$ by

$$T_\varphi f = P(\varphi f)$$

whenever $f\varphi \in \mathscr{L}^{2,1}$.

In this paper, we are concerned with the commutants of Toeplitz operators on the Dirichlet space $\mathscr{D}$. For a bounded operator L on a Hilbert space H, we

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recall that the commutant of L is the set of all bounded operator S on H such that $SL = LS$ on H.

On the Hardy space of the unit disk, Brown and Halmos [2] first obtained a complete description of commuting Toeplitz operators asserting that two Toeplitz operators with general bounded symbols commute if and only if either both symbols are analytic, or both symbols are co-analytic, or a nontrivial linear combination of the symbols is constant. Also, the commutants of several kinds of Toeplitz operators have been described as in [5], [7] and [12].

Later, the corresponding problem for Toeplitz operators acting on the Bergman space of the unit disk has been also studied. Axler and Čučković [1] proved that the result of Brown and Halmos still holds for harmonic symbols. Also, Čučković [6] showed that operators belong to the intersection of the Toeplitz algebra and commutant of a Toeplitz operator with monomial symbol must be Toeplitz operators with analytic symbols. Later Tikaradze [11] extended the result of Čučković [6] to Toeplitz operators with polynomial symbols.

On the Dirichlet space $\mathscr{D}$ under consideration, Duistermaat and the second author [8] characterized harmonic symbols of Toeplitz operators belong to the commutant of a Toeplitz operator with harmonic symbol. More explicitly, for bounded harmonic symbols u, v whose their derivatives with respect to z and $\bar{z}$ are all bounded, it is shown that $T_u T_v = T_v T_u$ on $\mathscr{D}$ if and only if either u, v are holomorphic or u, v and 1 are linearly dependent. This result shows that the commuting property on the Dirichlet space $\mathscr{D}$ has a different phenomenon from that on the Hardy space or Bergman space.

Motivated by the results mentioned above, in this paper we continue to study the describing problem of commutants of Toeplitz operators on the Dirichlet space. We consider nonconstant polynomial symbols of Toeplitz operators and find their commutants in the norm closed subalgebra generated by Toeplitz operators on $\mathscr{D}$. More explicitly, we prove that an operator in such a subalgebra commuting with a Toeplitz operator with a nonconstant polynomial symbol must be a Toeplitz operator with an analytic symbol; see Theorem 9. In the course of the proof, we study the characterization problem of the multiplication operators with analytic symbols, which might be of independent interest; see Propositions 2.

In Section 2, we characterize boundedness and compactness of multiplication operators with analytic symbols. Such characterizations will be very useful in our proofs. In Section 3, we prove our main theorem.

2. Multiplication operators

In this section, we characterize the boundedness and compactness of the multiplication operator with analytic symbol in terms of certain Carleson measures. This will be useful in our characterizations.

A positive Borel measure μ on $\mathbb{D}$ is called an $\mathscr{D}$-Carleson measure if there exists a constant $C > 0$ for which

$$\left(\int_{\mathbb{D}} |f|^2 \, d\mu\right)^{\frac{1}{2}} \leq C||f||$$

for every $f \in \mathscr{D}$. See [10] or [13] for several characterizations for $\mathscr{D}$-Carleson measures. We let $\mathscr{M}$ be the space of all $u \in \mathscr{L}^{2,1}$ for which $u \in L^\infty(\mathbb{D})$ and the measures $|\frac{\partial u}{\partial z}|^2 dA$, $|\frac{\partial u}{\partial \bar{z}}|^2 dA$ are $\mathscr{D}$-Carleson measures. Here $L^p(\mathbb{D}) = L^p(\mathbb{D}, dA)$ denotes the usual Lebesgue space on $\mathbb{D}$.

In our characterization of multiplication operators, certain kernel functions will play an important role. We introduce two kernel functions. For $a \in \mathbb{D}$, we let

$$E_a(z) = \frac{z}{1 - \bar{a}z}, \qquad S_a(z) = \frac{1}{(1 - \bar{a}z)^2}, \qquad z \in \mathbb{D}.$$

Put $e_a = (1 - |a|^2)E_a$ and $s_a = (1 - |a|^2)S_a$. Then we check $e'_a = s_a$ and $||e_a|| = 1$. Also, it is well known that

$$f(a) = \langle f, S_a\rangle_{L^2} \tag{1}$$

for analytic $f \in L^1(\mathbb{D})$; see Section 4 of [15] for details. Here and in what follows, we use the notations

$$\langle \varphi, \psi\rangle_{L^2} = \int_{\mathbb{D}} \varphi\bar{\psi} \, dA, \qquad ||\psi||_{L^2} = \langle \psi, \psi\rangle_2$$

for functions $\varphi, \psi \in L^2(\mathbb{D})$. For any analytic function $f \in L^2(\mathbb{D})$, it is also well known that $(1 - |a|^2)|f(a)| \to 0$ as $|a| \to 1$; see Theorem 2.1 of [14] and its remark. In particular, we have

$$\lim_{|a|\to 1} (1 - |a|^2)|f'(a)| = 0 \tag{2}$$

for every $f \in \mathscr{D}$.

Proposition 1. *e_a converges weakly to 0 in $\mathscr{D}$ as $|a| \to 1$.*

Proof. Note $e_a(0) = 0$ for all $a \in \mathbb{D}$. It follows from (1) and (2) that

$$\lim_{|a|\to 1} \langle f, e_a\rangle = \lim_{|a|\to 1} (1 - |a|^2)\langle f', S_a\rangle_{L^2} = \lim_{|a|\to 1} (1 - |a|^2)f'(a) = 0$$

for every $f \in \mathscr{D}$, which shows that e_a converges weakly to 0 as $|a| \to 1$, as desired. The proof is complete. □

Given $u \in \mathscr{L}^{2,1}$, we let M_u denote the multiplication operator with symbol u defined by $M_u f = uf$.

We now characterize the boundedness and compactness of multiplication operators with analytic symbol. Proposition 2(b) below is known; for example see Theorem 5.1.7 of [9] where a different argument was used.

Proposition 2. *Let $u : \mathbb{D} \to \mathbb{C}$ be a function. Then the following statements hold.*

(a) *If* $u \in \mathscr{M}$*, then* $M_u : \mathscr{D} \to \mathscr{L}^{2,1}$ *is bounded.*
(b) *If* u *is analytic on* $\mathbb{D}$*, then* $M_u : \mathscr{D} \to \mathscr{D}$ *is bounded if and only if* $u \in \mathscr{M}$*.*
(c) $M_u : \mathscr{D} \to \mathscr{D}$ *is compact if and only if* $u = 0$ *on* $\mathbb{D}$*.*

Proof. First we prove (a). Note that $||\varphi||_{L^2} \le ||\varphi||$ for every $\varphi \in \mathscr{D}$. Since $u \in \mathscr{M}$, it follows that there exist constants C_1, C_2 for which

$$\begin{aligned}
||M_u h||^2 &= ||uh||^2 \\
&= |\int_{\mathbb{D}} uh\, dA|^2 + \int_{\mathbb{D}} \left(|\frac{\partial u}{\partial z} h + uh'|^2 + |\frac{\partial u}{\partial \bar{z}} h|^2 \right) dA \\
&\le ||u||_\infty^2 \int_{\mathbb{D}} |h|^2\, dA + C_1 ||h||^2 + ||u||_\infty^2 \int_{\mathbb{D}} |h'|^2\, dA + C_2 ||h||^2 \\
&\le (||u||_\infty^2 + C_1 + ||u||_\infty^2 + C_2) ||h||^2
\end{aligned}$$

for every $h \in \mathscr{D}$, which implies the boundedness of $M_u : \mathscr{D} \to \mathscr{L}^{2,1}$.

Now, assume u is analytic and prove (b). First suppose $M_u : \mathscr{D} \to \mathscr{D}$ is bounded. Since $u = M_u 1$, we note $u \in \mathscr{D}$. Fix $a \in \mathbb{D}$. Since $u' e_a$ is analytic in $L^2(\mathbb{D})$, we have $\langle u' e_a, S_a \rangle_{L^2} = a u'(a)$ by (1). It follows from (1) again that

$$\begin{aligned}
\langle M_u e_a, e_a \rangle &= \langle u e_a, e_a \rangle \\
&= \langle u' e_a, s_a \rangle_{L^2} + \langle u s_a, s_a \rangle_{L^2} \\
&= (1 - |a|^2) \Big[\langle u' e_a, S_a \rangle_{L^2} + \langle u s_a, S_a \rangle_{L^2} \Big] \\
&= (1 - |a|^2) \Big[u'(a) a + u(a) s_a(a) \Big] \\
&= (1 - |a|^2) u'(a) a + u(a).
\end{aligned} \tag{3}$$

Since $||e_a|| = 1$, it follows that

$$|u(a)| \le |\langle M_u e_a, e_a \rangle| + |a(1 - |a|^2) u'(a)| \le ||M_u|| + (1 - |a|^2) |u'(a)|$$

for each $a \in \mathbb{D}$. On the other hand, (2) implies that $(1 - |a|^2)|u'(a)|$ is bounded in $\mathbb{D}$. Thus, the above observation shows $u \in L^\infty(\mathbb{D})$. Now, it remains to show that $|u'|^2 dA$ is an $\mathscr{D}$-Carleson measure on $\mathbb{D}$. Given $k \in \mathscr{D}$, we put

$$\psi(z) = \int_0^z u'(\zeta) k(\zeta)\, d\zeta, \qquad z \in \mathbb{D}.$$

Then, since $\psi(0) = 0$, we have

$$\langle M_u k, \psi \rangle = \langle uk, \psi \rangle = \langle uk', u'k \rangle_{L^2} + \langle u'k, u'k \rangle_{L^2}$$

and hence

$$\begin{aligned}
||u'k||_{L^2}^2 &= |\langle u'k, u'k \rangle_{L^2}| \\
&\le |\langle M_u k, \psi \rangle| + |\langle uk', u'k \rangle_{L^2}| \\
&\le ||M_u|| ||k|| ||\psi|| + ||uk'||_{L^2} ||u'k||_{L^2} \\
&\le ||M_u|| ||k|| ||u'k||_{L^2} + ||u||_\infty ||k|| ||u'k||_{L^2}
\end{aligned}$$

for all $k \in \mathscr{D}$. It follows that

$$||u'k||_{L^2} \leq (||M_u|| + ||u||_\infty)||k||$$

for all $k \in \mathscr{D}$, which implies that $|u'|^2\, dA$ is an $\mathscr{D}$-Carleson measure. The converse implication follows from (a).

Finally, to prove (c), suppose $M_u : \mathscr{D} \to \mathscr{D}$ is compact. Recall $u \in \mathscr{D}$ because $u = M_u 1$ as before. Note that the normalized kernel e_a converges weakly to 0 in $\mathscr{D}$ as $|a| \to 1$ by Proposition 1. Thus, (3) together with (2) shows that $u(a) \to 0$ as $|a| \to 1$, which implies $u = 0$ as desired. Since the converse is clear, we complete the proof. □

3. Commutants of Toeplitz operators

Each point evaluation is easily verified to be a bounded linear functional on $\mathscr{D}$. Hence, for each $z \in \mathbb{D}$, there exists a unique kernel $K_z \in \mathscr{D}$ which has the following reproducing property

$$f(z) = \langle f, K_z \rangle \tag{4}$$

for every $f \in \mathscr{D}$. Then, it is easy to see that the kernel function K_z is given by

$$K_z(w) = 1 + \log\left(\frac{1}{1 - \bar{z}w}\right), \qquad w \in \mathbb{D}.$$

By the explicit formula for K_z, one can see that the projection P can be represented by the integral formula as follows:

$$\begin{aligned} P\psi(z) &= \langle P\psi, K_z \rangle \\ &= \langle \psi, K_z \rangle \\ &= \int_D \psi\, dA + \int_D \frac{z}{1 - z\bar{w}} \frac{\partial \psi}{\partial w}(w)\, dA(w), \qquad z \in \mathbb{D} \end{aligned} \tag{5}$$

for every function $\psi \in \mathscr{L}^{2,1}$. See [8] for details and related facts. Put

$$P_0\psi(z) = \int_D \frac{z}{1 - z\bar{w}} \frac{\partial \psi}{\partial w}(w)\, dA(w)$$

for notational simplicity. Then, by a simple calculation using the above formula for P_0, we can easily see that for integers $m, n \geq 1$

$$P_0[\bar{z}^m z^n](z) = \begin{cases} z^{n-m} & \text{if } \ n > m, \\ 0 & \text{if } \ n \leq m. \end{cases} \tag{6}$$

For the proof of the main theorem, we need the compactness of semi-commutators of certain Toeplitz operators. To do this, we use a decomposition of the Sobolev space $\mathscr{L}^{2,1}$.

For $\psi \in \mathscr{L}^{2,1}$, it turns out that $\psi(re^{i\theta})$ is absolutely continuous on $r \in [0, 1)$ for almost every $\theta \in [0, 2\pi]$ and absolutely continuous on $\theta \in [0, 2\pi]$ for almost every $r \in [0, 1)$. In particular, the radial limit $\psi|_{\partial\mathbb{D}}$ defined by

$$\psi|_{\partial\mathbb{D}}(e^{i\theta}) = \lim_{r \to 1} \psi(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi]$. See [3] or [4] for details and related facts. We let

$$\Delta_0 = \{\psi \in \mathscr{L}^{2,1} : \psi|_{\partial\mathbb{D}} = 0\}$$

and $\mathscr{D}_0$ be the space of all functions $f \in \mathscr{D}$ for which $f(0) = 0$. Also, put $\Delta = \Delta_0 + \mathbb{C}$. Then it turns out that the spaces Δ, $\mathscr{D}_0$ and $\overline{\mathscr{D}_0}$ are mutually orthogonal. It is also known that the Sobolev space $\mathscr{L}^{2,1}$ admits the following useful decomposition:

$$\mathscr{L}^{2,1} = \Delta \oplus \mathscr{D}_0 \oplus \overline{\mathscr{D}_0}; \tag{7}$$

see [4] for details and related facts.

Given a function $u \in \mathscr{L}^{2,1}$, we let $S_u : \mathscr{D}^\perp \to \mathscr{D}$ be the dual Hankel operator with symbol u defined by

$$S_u\varphi = P(u\varphi)$$

whenever $u\varphi \in \mathscr{L}^{2,1}$.

The following lemma shows that the dual Hankel operators with monomial symbol are finite rank operators.

Lemma 3. *For an integer $k \geq 0$, the rank of S_{z^k} is at most k.*

Proof. Let $\varphi \in \mathscr{D}^\perp$ be any function. By (7), we can write $\varphi = f + c + g + h$ where $f \in \Delta_0$, $c \in \mathbb{C}$, $g \in \mathscr{D}_0$ and $h \in \overline{\mathscr{D}_0}$. Recall that Δ, $\mathscr{D}_0$ and $\overline{\mathscr{D}_0}$ are mutually orthogonal. Since $\varphi \in \mathscr{D}^\perp$, we have $g = 0$ and hence $\varphi = f + c + h$. Note $P_0(\mathscr{L}^{2,1}) \subset \mathscr{D}_0$. Since $z^k f \in \Delta_0$ and Δ_0 is orthogonal to $\mathscr{D}_0$, we see $P_0(z^k f) = 0$ and then

$$P(w^k f) = \int_{\mathbb{D}} w^k f(w)\, dA(w).$$

Also, if we write $h(z) = \sum_{j=1}^{\infty} a_n \overline{z^j}$ for the Taylor series expansion of h, we see from (6)

$$\begin{aligned}
P(w^k h)(z) &= \sum_{j=1}^{\infty} a_j P(w^k \overline{w^j}) \\
&= \sum_{j=1}^{\infty} a_j \left[\int_{\mathbb{D}} w^k \overline{w^j}\, dA(w) + P_0(w^k \overline{w^j}) \right] \\
&= a_k \int_{\mathbb{D}} |w^k|^2\, dA(w) + \sum_{j=1}^{k-1} a_j z^{k-j} \\
&= \frac{a_k}{k+1} + \sum_{j=1}^{k-1} a_{k-j} z^j
\end{aligned}$$

for all $z \in \mathbb{D}$. It follows that

$$S_{z^k}\varphi(z) = P(w^k f + w^k c + w^k h)(z)$$

$$= \int_{\mathbb{D}} w^k f(w)\, dA(w) + cz^k + \frac{a_k}{k+1} + \sum_{j=1}^{k-1} a_{k-j} z^j$$

which implies that S_{z^k} has at most rank k. The proof is complete. $\square$

We let the notation $\mathcal{B}$ denote the algebra consisting of all bounded operators on $\mathscr{D}$ and $\mathcal{K}$ be the algebra of all compact operators on $\mathscr{D}$.

For $u \in \mathscr{M}$, the multiplication operator $M_u : \mathscr{D} \to \mathscr{L}^{2,1}$ is bounded by Proposition 2(b). So the Toeplitz operator T_u is also bounded because $T_u = PM_u$. The following will be very useful in our proof.

Proposition 4. *For an integer $k \geq 0$ and $u \in \mathscr{M}$, we have $T_{z^k u} - T_{z^k} T_u \in \mathcal{K}$.*

Proof. Note that

$$\begin{aligned}
[T_{z^k u} - T_{z^k} T_u] f &= P(w^k uf) - z^k P(uf) \\
&= P[w^k (uf - P(uf))] \\
&= S_{z^k}(I - P) M_u f
\end{aligned}$$

for all $f \in \mathscr{D}$, which shows that $T_{z^k u} - T_{z^k} T_u = S_{z^k}(I - P)M_u$ on $\mathscr{D}$. Note M_u is bounded by Proposition 2(a) and S_{z^k} is compact because it is a finite rank operator by Lemma 3. Thus $T_{z^k u} - T_{z^k} T_u$ is compact on $\mathscr{D}$. The proof is complete. $\square$

The following simple lemma shows that a bounded operator commuting with T_z is a Toeplitz operator with an analytic symbol.

Lemma 5. *Let $S \in \mathcal{B}$. If $ST_z = T_z S$ on $\mathscr{D}$, then $S = T_\varphi$ for some analytic $\varphi \in \mathscr{M}$.*

Proof. Put $g = S1$. Since $ST_z = T_z S$ by the assumption, we have $Sz^k = z^k g$ for every $k = 0, 1, \ldots$. Then, by the similar argument as in the proof of Theorem 1.4 of [6], we see $Sf = gf = M_g f$ for every $f \in \mathscr{D}$. Note $g \in \mathscr{M}$ by Proposition 2(b) because S is bounded. Thus $S = T_g$ on $\mathscr{D}$. The proof is complete. $\square$

Before we prove the main result, we first consider special cases of polynomial symbols as preliminary steps.

Proposition 6. *Let p be a nonconstant polynomial which is not of the form $q(z^\ell)$ where q is a polynomial and $\ell > 1$ is an integer. If $S \in \mathscr{B}$ commutes with T_p on $\mathscr{D}$, then $S = T_\varphi$ for some analytic $\varphi \in \mathscr{M}$.*

Proof. By Proposition 0.1 of [11], there exists an open set $U \subset \mathbb{D}$ such that

$$\frac{p(z) - p(w)}{z - w} \neq 0$$

for all $z \in \bar{\mathbb{D}}$ and $w \in U$, which implies that

$$[p - p(w)]\mathscr{D} = (z - w)\mathscr{D}, \qquad w \in U. \tag{8}$$

On the other hand, by (4), we can see

$$(9)\qquad T_u^* K_w(z) = \langle T_u^* K_w, K_z\rangle = \langle K_w, uK_z\rangle = \overline{u(w)K_z(w)} = \overline{u(w)}K_w(z)$$

for all $u \in \mathscr{D}$ and $z, w \in \mathbb{D}$. Since $T_pS = ST_p$, we have $T_{p-p(w)}S = ST_{p-p(w)}$ and hence $S^*T^*_{p-p(w)} = T^*_{p-p(w)}S^*$ for all $w \in U$. It follows from (9) that $T^*_{p-p(w)}S^*K_w = 0$ and then S^*K_w is orthogonal to the range of $T_{p-p(w)}$ for all $w \in U$. Note that for $w \in U$, the range of $T_{p-p(w)}$ is $(z-w)\mathscr{D}$ by (8) and K_w is orthogonal to $(z-w)\mathscr{D}$. Thus, we can find a function χ on U such that $S^*K_w(z) = \chi(w)K_w(z)$ for all $z \in \mathbb{D}$ and $w \in U$. It follows from (4) that

$$Sf(w) = \langle Sf, K_w\rangle = \langle f, S^*K_w\rangle = \langle f, \chi(w)K_w\rangle = \overline{\chi(w)}f(w)$$

and then $[ST_zf - T_zSf](w) = 0$ for all $w \in U$ and $f \in \mathscr{D}$. Thus S commutes with T_z and the result follows from Lemma 5. The proof is complete. $\square$

Moreover, if S is compact, we have some more as in the following proposition.

Proposition 7. *Let $S \in \mathcal{K}$ and $m \geq 1$ be an integer. Let p be a nonconstant polynomial which is not of the form $q(z^\ell)$ where q is a polynomial and $\ell > 1$ is an integer. If S commutes with $T_{p(z^m)}$ on $\mathscr{D}$, then $S = 0$.*

Proof. For each $j = 0, 1, \ldots, m-1$, we put

$$E_j := \overline{\text{span}}\{z^{nm+j} :\ n = 0, 1, 2, \ldots\}.$$

Then we can check that $z^mE_j \subset E_j$ and $\mathscr{D} = \sum_{j=0}^{m-1} E_j$. Let $P_j : \mathscr{D} \to E_j$ be the orthogonal projection and define $e_j : \mathscr{D} \to E_j$ by $e_j(z^n) = z^{nm+j}$ for $n = 0, 1, \ldots$. It's clear that e_j is invertible. Set $f_j = e_j^{-1}P_j$. Then we see $e_jT_p = T_{p(z^m)}e_j$ and $f_jT_{p(z^m)} = T_pf_j$ for each $j = 0, 1, \ldots, m-1$. For $0 \leq i, j < m$, we put $S_{i,j} := f_jSe_i$. Then, since $ST_{p(z^m)} = T_{p(z^m)}S$, we have

$$S_{i,j}T_p = f_jSe_iT_p = f_jST_{p(z^m)}e_i = f_jT_{p(z^m)}Se_i = T_pf_jSe_i = T_pS_{i,j}$$

and hence $S_{i,j}$ commutes with T_p for each i, j. By Proposition 6, $S_{i,j} = T_{\varphi_{i,j}}$ for some analytic $\varphi_{i,j} \in \mathscr{M}$. But, since S is compact, so is each $S_{i,j}$. By Proposition 2(c), we have $\varphi_{i,j} = 0$ and then $S_{i,j} = 0$ for all i, j. Note that

$$Se_j(z^n) = \sum_{k=0}^{m-1} P_kSe_j(z^n) = \sum_{k=0}^{m-1} e_kf_kSe_j(z^n) = \sum_{k=0}^{m-1} e_kS_{j,k}(z^n) = 0$$

for all $n = 0, 1, \ldots$ and $j = 0, 1, \ldots, m-1$. Hence $Se_j = 0$ for all j and then $S = 0$ as desired. The proof is complete. $\square$

Recall that the Toeplitz algebra $\mathscr{T}$ is the norm closed subalgebra of $\mathscr{B}$ generated by all Toeplitz operators with symbol in $\mathscr{M}$. The following will be useful in the proof of the main theorem.

Lemma 8. *Let $S \in \mathscr{T}$. Then $ST_z - T_zS \in \mathcal{K}$.*

Proof. If we first assume $S = T_u$ for some $u \in \mathscr{M}$, then $ST_z - T_zS = T_{uz} - T_zT_u$ is compact by Proposition 4 (with $k = 1$). Then, by using the cannonical homomorphism $\Psi : \mathscr{B} \to \mathscr{B}/\mathscr{K}$ given by $\Psi(T) = T + \mathscr{K}$, we have

$$\Psi(T_u)\Psi(T_z) = \Psi(T_z)\Psi(T_u)$$

for every $u \in \mathscr{M}$. If we assume $S = T_{u_1} \cdots T_{u_n}$ where $u_j \in \mathscr{M}$, the above implies that $\Psi[ST_z - T_zS] = 0$ and hence $ST_z - T_zS \in \mathscr{K}$. Now, for an arbitrary operator S in $\mathscr{T}$, we have $S = \lim S_k$ where each S_k is a finite sum of the form $T_{u_1} \cdots T_{u_n}$. It follows that $ST_z - T_zS$ is the limit of the compact operators $S_kT_z - T_zS_k$ and hence compact as desired. The proof is complete. □

Now, we are ready to prove our main theorem.

Theorem 9. *Let $S \in \mathscr{T}$ and p be a nonconstant polynomial. Then $ST_p = T_pS$ on $\mathscr{D}$ if and only if $S = T_\varphi$ for some analytic $\varphi \in \mathscr{M}$.*

Proof. First suppose $ST_p = T_pS$. Put $\mathbb{S} = ST_z - T_zS$ for simplicity. Note $\mathbb{S} \in \mathcal{K}$ by Lemma 8. Since $ST_p = T_pS$ by the assumption, we can easily see

$$\mathbb{S}T_p = ST_pT_z - T_zT_pS = T_pST_z - T_pT_zS = T_p\mathbb{S}$$

and hence $\mathbb{S}$ commutes with T_p. Now, by an application of Proposition 7, we see $\mathbb{S} = 0$. Then, Lemma 5 gives the desired result. The converse implication is clear. The proof is complete. □

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