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A VALUE DISTRIBUTION RESULT RELATED TO HAYMAN'S ALTERNATIVE

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Abstract. Motivated by Bloch's principle, we prove a value distribution result for meromorphic functions which is related to Hayman's alternative in certain sense.

1. Introduction and main result

The reader is assumed to be familiar with the standard notations of Nevanlinna value distribution theory of meromorphic functions (one may refer to [4,5]) such as $T(r,f)$, $m(r,f)$, $N(r,f)$, etc. We shall denote the class of all meromorphic functions on a domain D in $\mathbb{C}$ by $\mathcal{M}(D)$ and we shall write, '$\langle f, D\rangle \in \mathcal{P}$' for '$f \in \mathcal{M}(D)$ satisfies the property $\mathcal{P}$ on D' .

We say that $\phi \in \mathcal{M}(\mathbb{C})$ is a small function of $f \in \mathcal{M}(\mathbb{C})$ if $T(r,\phi) = S(r,f)$ as $r \to \infty$ possibly outside a set of r of finite linear measure.

W. K. Hayman proved the following 'Picard type' theorem, also known as Hayman's alternative:

Theorem 1.1 ([6]). *Let $f \in \mathcal{M}(\mathbb{C})$ and let $l \geq 1$. Suppose that $f(z) \neq 0$, and $f^{(l)}(z) - 1 \neq 0$ for all $z \in \mathbb{C}$. Then f is constant.*

A subfamily $\mathcal{F}$ of $\mathcal{M}(D)$ is said to be normal in D if every sequence of members of $\mathcal{F}$ contains a subsequence that converges locally uniformly (with respect to the spherical metric) in D. Recall Bloch's principle (see [9,10]): *A subfamily $\mathcal{F}$ of $\mathcal{M}(D)$ with $\langle f, D\rangle \in \mathcal{P}$ for each $f \in \mathcal{F}$ is likely to be normal on D if $\mathcal{P}$ reduces every $f \in \mathcal{M}(\mathbb{C})$ to a constant.* Neither Bloch's principle nor its converse is true (see [1–3,8,9]).

According to Bloch's principle, to every 'Picard type' theorem there corresponds a normality criterion. A normality criterion corresponding to Theorem 1.1 was proved by Y. Gu as follows:

Theorem 1.2 ([7]). *Let $\mathcal{F} \subseteq \mathcal{M}(D)$ and let $l \geq 1$. Suppose that $f(z) \neq 0$, and $f^{(l)}(z) - 1 \neq 0$ for all $z \in D$ and $f \in \mathcal{F}$. Then $\mathcal{F}$ is normal in D.*

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The constants 0 and 1 in Theorem 1.1 and Theorem 1.2 can be replaced by arbitrary constants a and $b \neq 0$:

Theorem 1.3 ([6]). *Let $f \in \mathcal{M}(D)$ and let $l \geq 1$. Suppose that $f(z) \neq a$, and $f^{(l)}(z) - b \neq 0$ for all $z \in \mathbb{C}$, where $a, b \in \mathbb{C}$, $b \neq 0$. Then f is constant.*

Theorem 1.4 ([7]). *Let $\mathcal{F} \subseteq \mathcal{M}(D)$ and let $l \geq 1$. Suppose that $f(z) \neq a$, and $f^{(l)}(z) - b \neq 0$ for all $z \in D$, $f \in \mathcal{F}$, where $a, b \in \mathbb{C}$ with $b \neq 0$. Then $\mathcal{F}$ is normal in D.*

Note that if $l \geq 1$ and $b \in \mathbb{C} \setminus \{0\}$, then there is a polynomial $P(z)$ such that $P^{(l)}(z) = b$. Using this observation, Theorem 1.3 and Theorem 1.4 can be restated as:

Theorem 1.5. *Suppose that $P(z)$ is a polynomial of degree $l \geq 1$ and $a \in \mathbb{C}$. If $f \in \mathcal{M}(\mathbb{C})$ is such that $f(z) \neq a$ and $(f(z) - P(z))^{(l)} \neq 0$, then f is constant.*

Theorem 1.6. *Suppose that $P(z)$ is a polynomial of degree $l \geq 1$ and $a \in \mathbb{C}$. If $\mathcal{F} \subseteq \mathcal{M}(D)$ is such that each $f \in \mathcal{F}$ satisfies:*

$$f(z) \neq a \text{ and } (f(z) - P(z))^{(l)} \neq 0,$$

then $\mathcal{F}$ is normal in D.

Remark 1.7. Put $g = f - P$ and $R = Q - P$, where P and Q are polynomials with $\deg(P - Q) = \deg(Q) = l$ and Q is non-constant. If $f(z) - P(z) \neq 0$ and $(f(z) - Q(z))^{(l)} \neq 0$, then by using Theorem 1.5, we find that $f(z) = P(z) + c$, for some constant $c \neq 0$.

Remark 1.7 shows that Theorem 1.3 does not hold if a is replaced by some non-constant function.

Remark 1.8. Suppose f is an entire function such that $f - g$ has only finitely many zeros in the plane, where g is some non-constant entire function. Further, let

$$F(z) = \sum_{k=1}^{n} a_k(z)\,(f - g)^{(k)}$$

omits 1, where $a_k(z)$ are small functions of f. Then by using Theorem 3.2 in [5], we find that $f(z) = g(z) + p(z)$ for some polynomial $p(z)$. Indeed,

$$T(r, f - g) < \overline{N}(r, f - g) + N(r, \frac{1}{f - g}) + \overline{N}\left(r, \frac{1}{F - 1}\right) - N_0\left(r, \frac{1}{F'}\right) + S(r, f),$$

where $N_0\left(r, \frac{1}{F'}\right)$ is the counting function of the zeros of F' which are not zeros of $F - 1$.

Since $f - g$ is entire and has only finitely many zeros, it follows that

$$T(r, f - g) < \overline{N}\left(r, \frac{1}{F - 1}\right) - N_0\left(r, \frac{1}{F'}\right) + S(r, f)$$

$$\Rightarrow T(r, f-g) < \overline{N}\left(r, \frac{1}{F-1}\right) + S(r, f).$$

If $f - g$ is transcendental, then $F(z) = 1$ must have infinitely many roots, which is a contradiction and hence $f - g$ must be a polynomial, say $p(z)$; that is, $f(z) = g(z) + p(z)$.

Let P and Q be polynomials with $1 \leq \deg(P) < \deg(Q) = l$ and $\mathcal{P}$ be the property defined as follows: "$\langle f, D\rangle \in \mathcal{P} \Leftrightarrow f - P \neq 0$ and $(f-Q)^{(l)} \neq 0$ on D". That is, f satisfies the property $\mathcal{P}$ if, and only if $f - P$ and $(f-Q)^{(l)}$ have no zeros in D. With this $\mathcal{P}$, Theorem 1.6 immediately yields:

Theorem 1.9. *The family* $\mathcal{F} := \{f \in \mathcal{M}(D) : \langle f, D\rangle \in \mathcal{P}\}$ *is normal in* D.

Note that Remark 1.7 and Theorem 1.9 provide a counterexample to the converse of Bloch's principle.

W. Schwick generalized Theorem 1.2:

Theorem 1.10 ([11])**.** *Let* $g \not\equiv 0$ *be in* $\mathcal{M}(D)$ *and let* $l \in \mathbb{N}$. *Let* $\mathcal{F} \subseteq \mathcal{M}(D)$ *be such that* $f \neq 0$, $f^{(l)} \neq g$, *and* f *and* g *have no common poles for each* $f \in \mathcal{F}$. *Then* $\mathcal{F}$ *is normal in* D.

According to the converse of Bloch's principle, one may find a 'Picard type' theorem corresponding to Theorem 1.10, and this is the purpose of this paper. In fact, we prove the following value distribution result corresponding to Theorem 1.10 which is related to Hayman's alternative in certain sense:

Theorem 1.11. *Suppose that* $f \in \mathcal{M}(\mathbb{C})$ *is transcendental and* ϕ *is a small function of* f *such that* f *and* ϕ *have no common poles. Let* $l \in \mathbb{N}$ *and* $\psi(z) = f^{(l)}(z)$. *If* $f(z) \neq 0$ *and* $\psi(z) \neq \phi(z)$ *for all* $z \in \mathbb{C}$, *then* $\psi'(z) = \phi(z)$ *and* $\psi'(z) = \phi'(z)$ *have infinitely many solutions.*

2. Proof of Theorem 1.11

Since the proof of Theorem 1.11 is based on Milloux techniques (see [5, p. 60]), we need to prove some key lemmas for the proof of Theorem 1.11. Throughout this paper, we shall denote $f^{(l)}(z)$ by $\psi(z)$, where $l \in \mathbb{N}$.

Lemma 2.1. *Let* $f \in \mathcal{M}(\mathbb{C})$ *and let* ϕ *be a small function of* f. *Then for* $r \to \infty$ *outside a set of finite linear measure,*

$$T(r, f) \leq \bar{N}(r, f) + N\left(r, \frac{1}{f}\right) + \bar{N}\left(r, \frac{1}{\psi - \phi}\right) - N_2^0(r, \psi) + S(r, f), \tag{2.1}$$

where $N_2^0(r, \psi) = N\left(r, \frac{1}{\psi'}\right) - N_0\left(r, \frac{1}{\psi'}\right)$ *and* $N_0\left(r, \frac{1}{\psi'}\right)$ *is the counting function of zeros of* ψ' *which are not zeros of* ψ.

Proof. By the second fundamental theorem of Nevanlinna for three small functions (see [5, Theorem 2.5], also see [4, Theorem 5.9.1]) with $a_1 = 0$, $a_2 = \infty$ and $a_3 = \phi$, we have

$$T(r,f) \leq \bar{N}\left(r, \frac{1}{f}\right) + \bar{N}\left(r, f\right) + \bar{N}\left(r, \frac{1}{f-\phi}\right) + S(r,f) \tag{2.2}$$

as $r \to \infty$ outside a set of r of finite linear measure.

Since

$$\bar{N}\left(r, \frac{1}{f}\right) = N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{f'}\right),$$
$$\bar{N}\left(r, \frac{1}{f-\phi}\right) = N\left(r, \frac{1}{f-\phi}\right) - N\left(r, \frac{1}{f'-\phi'}\right) + N_0\left(r, \frac{1}{f'-\phi'}\right)$$

and

$$\bar{N}\left(r, f\right) = N\left(r, f'\right) - N\left(r, f\right),$$

therefore (2.2) yields (after adding $m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-\phi}\right)$ to both sides)

$$\begin{aligned} & T\left(r, f\right) + m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-\phi}\right) \\ \leq\ & m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-\phi}\right) \\ & + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-\phi}\right) + N\left(r, f'\right) - N\left(r, f\right) \\ & - N\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{f'-\phi'}\right) \\ & - N\left(r, \frac{1}{f'-\phi'}\right) + S(r,f) \end{aligned}$$

$$\begin{aligned} \Rightarrow\quad & m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-\phi}\right) + m\left(r, f\right) \\ \leq\ & 2T\left(r, f\right) - N_1\left(r, f\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{f'-\phi'}\right) - N\left(r, \frac{1}{f'-\phi'}\right) \\ & + S(r,f), \end{aligned} \tag{2.3}$$

where $N_1\left(r, f\right) = N\left(r, \frac{1}{f'}\right) + 2N\left(r, f\right) - N\left(r, f'\right)$.

Applying (2.3) to $\psi = f^{(l)}$ and put $g = \psi - \phi$, we have

$$\begin{aligned} m\left(r, \frac{1}{\psi}\right) + m\left(r, \frac{1}{g}\right) + m\left(r, \psi\right) \leq 2T\left(r, \psi\right) - N_1\left(r, \psi\right) + N_0\left(r, \frac{1}{\psi'}\right) \\ + N_0\left(r, \frac{1}{g'}\right) - N\left(r, \frac{1}{g'}\right) + S(r,\psi) \end{aligned} \tag{2.4}$$

as $r \to \infty$ outside a set of r of finite linear measure.

Since

$$N(r,\psi') - N(r,\psi) = \bar{N}(r,f)$$

and

$$N\left(r,\frac{1}{g}\right) - N\left(r,\frac{1}{g'}\right) = \bar{N}\left(r,\frac{1}{g}\right) - N_0\left(r,\frac{1}{g'}\right)$$

and using the first fundamental theorem of Nevanlinna, we have

$$\begin{aligned}
2T(r,\psi) - N_1(r,\psi) &= m(r,\psi) + m\left(r,\frac{1}{g}\right) + N(r,\psi) + N\left(r,\frac{1}{g}\right)\\
&\quad - N\left(r,\frac{1}{\psi'}\right) - 2N(r,\psi) + N(r,\psi') + S(r,f)\\
&= m(r,\psi) + m\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{g}\right) - N\left(r,\frac{1}{\psi'}\right)\\
&\quad + \bar{N}(r,f) + S(r,f)
\end{aligned}$$

and hence (2.4) reduces to

$$m\left(r,\frac{1}{\psi}\right) \le \bar{N}\left(r,\frac{1}{g}\right) - N\left(r,\frac{1}{\psi'}\right) + \bar{N}(r,f) + N_0\left(r,\frac{1}{\psi'}\right) + S(r,f). \tag{2.5}$$

Also

$$\begin{aligned}
T(r,f) &= T\left(r,\frac{1}{f}\right) + O(1)\\
&= m\left(r,\frac{\psi}{f\psi}\right) + N\left(r,\frac{1}{f}\right) + O(1)\\
&\le m\left(r,\frac{\psi}{f}\right) + m\left(r,\frac{1}{\psi}\right) + N\left(r,\frac{1}{f}\right) + O(1)\\
&= m\left(r,\frac{1}{\psi}\right) + N\left(r,\frac{1}{f}\right) + S(r,f). \qquad (2.6)
\end{aligned}$$

Now by using (2.5) in (2.6), we get

$$\begin{aligned}
T(r,f) &\le \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{g}\right) - N\left(r,\frac{1}{\psi'}\right) + N_0\left(r,\frac{1}{\psi'}\right)\\
&\quad + S(r,f)\\
&= \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right) - N_2^0(r,\psi) + S(r,f) \qquad (2.7)
\end{aligned}$$

as $r \to \infty$ outside a set of r of finite linear measure, where

$$N_2^0(r,\psi) = N\left(r,\frac{1}{\psi'}\right) - N_0\left(r,\frac{1}{\psi'}\right) = N\left(r,\frac{1}{\psi}\right) - \bar{N}\left(r,\frac{1}{\psi}\right)$$

counts only repeated zeros of ψ with multiplicity reduced by 1. □

Lemma 2.2. *Let $f \in \mathcal{M}(\mathbb{C})$ and let ϕ be a small function of f such that f and ϕ have no common poles. Then*

$$lN_1(r,f) \le \bar{N}_2(r,f) + \bar{N}\left(r, \frac{1}{\psi - \phi}\right) + \bar{N}\left(r, \frac{1}{\psi' - \phi}\right) + S(r,f), \tag{2.8}$$

where $N_1(r,f)$ is the counting function of simple poles of $f$and $\bar{N}_2(r,f)$ is the counting function of multiple poles of f counted once.

The proof of Lemma 2.2 is carried out by following the proof of Lemma 3.1 in [5] with certain modifications.

Proof of Lemma 2.2. Put

$$G(z) = \frac{\{\psi'(z) - \phi(z)\}^{l+1}}{\{\phi(z) - \psi(z)\}^{l+2}}.$$

If z_0 is a simple pole of $f(z)$, then near z_0, we have

$$f(z) = \frac{a}{z - z_0} + O(1),$$

and

$$\psi(z) = \frac{(-1)^l a(l!)}{(z - z_0)^{l+1}} + O(1)$$

near z_0, and hence

$$\psi' = \frac{(-1)^{l+1} a(l+1)!}{(z - z_0)^{l+2}} \left\{1 + O(z - z_0)^{l+2}\right\}.$$

Since f and ϕ have no common poles, therefore near z_0, we have

$$G(z) = \frac{(-1)^{l+1}(l+1)^{l+1}}{al!} \left\{1 + O(z - z_0)^{l+1}\right\}$$

which implies that $G(z_0) \neq 0, \infty$, and $G'(z)$ has a zero of order at least l at z_0 and so

$$lN_1(r,f) \le N_0\left(r, \frac{1}{G'}\right). \tag{2.9}$$

Applying Jensen's formula to G'/G, we get

$$N\left(r, \frac{G}{G'}\right) - N\left(r, \frac{G'}{G}\right) = m\left(r, \frac{G'}{G}\right) - m\left(r, \frac{G}{G'}\right) + O(1). \tag{2.10}$$

Since the only zeros of G'/G are the zeros of G' which are not zeros of G, we have

$$N\left(r, \frac{G}{G'}\right) = N_0\left(r, \frac{1}{G'}\right). \tag{2.11}$$

Also, G'/G has only simple poles at the zeros and poles of G, so

$$N\left(r, \frac{G'}{G}\right) = \bar{N}(r,G) + \bar{N}\left(r, \frac{1}{G}\right). \tag{2.12}$$

Using (2.11) and (2.12) in (2.10), we obtain

$$N_0\left(r,\frac{1}{G'}\right)-\bar{N}\left(r,\frac{1}{G}\right)-\bar{N}\left(r,G\right)=m\left(r,\frac{G'}{G}\right)-m\left(r,\frac{G}{G'}\right)+O(1). \tag{2.13}$$

Now, from (2.9), (2.10) and (2.13), we have

$$\begin{aligned} lN_1\left(r,f\right) &\le N_0\left(r,\frac{1}{G'}\right) \\ &= \bar{N}\left(r,\frac{1}{G}\right)+\bar{N}\left(r,G\right)+m\left(r,\frac{G'}{G}\right)-m\left(r,\frac{G}{G'}\right)+O(1) \\ &\le \bar{N}\left(r,\frac{1}{G}\right)+\bar{N}\left(r,G\right)+m\left(r,\frac{G'}{G}\right)+O(1). \end{aligned} \tag{2.14}$$

Let z_0 be a pole of $\psi(z)-\phi(z)$ of order m, say. Then near z_0

$$\psi(z)-\phi(z)=\frac{s_0(z)}{(z-z_0)^m}$$

for some function $s_0(z)$ analytic in a neighborhood of z_0 such that $s_0(z_0)\neq 0$.

Now, there are two cases:

Case 1: z_0 is a pole of $f(z)$. Then $m=k+l$, where $k>1$ is the multiplicity of z_0 as a pole of f. Since z_0 is not a pole of ϕ, we see that z_0 is a pole of $\psi'(z)-\phi(z)$ of multiplicity $m+1$. Therefore, near z_0

$$G(z)=t_0(z)\,(z-z_0)^{k-1}$$

for some function $t_0(z)$ analytic in a neighborhood of z_0 such that $t_0(z_0)\neq 0$. So z_0 is a zero of order $k-1$ of $G(z)$.

Case 2: z_0 is a pole of $\phi(z)$. Then z_0 is also a pole of $\psi'(z)-\phi(z)$ of multiplicity m. Therefore, near z_0

$$G(z)=t_1(z)\,(z-z_0)^m$$

for some function $t_1(z)$ analytic in a neighborhood of z_0 such that $t_1(z_0)\neq 0$. This shows that z_0 is a pole of $G(z)$ of the same multiplicity as that of $\phi(z)$.

Similarly, looking at the poles of $\psi'(z)-\phi(z)$, we obtain the same conclusion as in the case of poles of $\psi(z)-\phi(z)$.

Next, corresponding to the zeros of $\psi(z)-\phi(z)$ and $\psi'(z)-\phi(z)$, we have the following three cases:

Case 1: z_0 is a zero of $\psi(z)-\phi(z)$ but it is not a zero of $\psi'(z)-\phi(z)$. Then z_0 is a pole of $G(z)$.

Case 2: z_0 is zero of $\psi'(z)-\phi(z)$ but it is not a zero of $\psi(z)-\phi(z)$. Then z_0 is a zero of $G(z)$.

Case 3: z_0 is a common zero of $\psi'(z)-\phi(z)$ and $\psi(z)-\phi(z)$. Let j and k be the multiplicities of z_0 as a zero of $\psi'(z)-\phi(z)$ and $\psi(z)-\phi(z)$, respectively. Then near z_0,

$$G(z)=t_2(z)\,(z-z_0)^{(l+1)j-(l+2)k}$$

for some function $t_2(z)$ analytic in a neighborhood of z_0 such that $t_2(z_0)\neq 0$.

Thus z_0 is a pole of $G(z)$ if $k>\frac{l+1}{l+2}j$ and z_0 is a zero of $G(z)$ if $k<\frac{l+1}{l+2}j$.

Let $N(r,\frac{1}{f},\frac{1}{g^0})$ be the counting function of zeros of f which are not zeros of g, $N(r,\frac{1}{f},\frac{1}{g})$ be the counting function corresponding to the common zeros of f and g and $N^{(\alpha)}(r,\frac{1}{f},\frac{1}{g})$ be the counting function corresponding to the common zeros of f and g, such that the $m(f,z_0)>\alpha m(g,z_0)$, where by $m(f,z_0)$ we denote the multiplicity of z_0 as a zero of f. With these notations and the preceding arguments, we find that

$$\bar{N}\left(r,\frac{1}{G}\right)\leq \bar{N}_2(r,f)+\bar{N}(r,\phi)+\bar{N}\left(r,\frac{1}{\psi'-\phi},\frac{1}{(\psi-\phi)^0}\right)$$
$$+\bar{N}^{\left(\frac{l+2}{l+1}\right)}\left(r,\frac{1}{\psi'-\phi},\frac{1}{\psi-\phi}\right) \tag{2.15}$$

and

$$\bar{N}(r,G)\leq \bar{N}\left(r,\frac{1}{\psi-\phi},\frac{1}{(\psi'-\phi)^0}\right)+\bar{N}^{\left(\frac{l+1}{l+2}\right)}\left(r,\frac{1}{\psi-\phi},\frac{1}{\psi'-\phi}\right). \tag{2.16}$$

Note that

$$\begin{aligned}&\bar{N}\left(r,\frac{1}{\psi-\phi},\frac{1}{(\psi'-\phi)^0}\right)+\bar{N}^{\left(\frac{l+1}{l+2}\right)}\left(r,\frac{1}{\psi-\phi},\frac{1}{\psi'-\phi}\right)\\ &\quad+\bar{N}\left(r,\frac{1}{\psi'-\phi},\frac{1}{(\psi-\phi)^0}\right)+\bar{N}^{\left(\frac{l+2}{l+1}\right)}\left(r,\frac{1}{\psi'-\phi},\frac{1}{\psi-\phi}\right)\\ &\leq \bar{N}\left(r,\frac{1}{\psi-\phi}\right)+\bar{N}\left(r,\frac{1}{\psi'-\phi}\right).\end{aligned}$$

Therefore, using (2.15) and (2.16) in (2.14), we get

$$lN_1(r,f)\leq \bar{N}_2(r,f)+\bar{N}(r,\phi)+\bar{N}\left(r,\frac{1}{\psi-\phi}\right)+\bar{N}\left(r,\frac{1}{\psi'-\phi}\right)$$
$$+m\left(r,\frac{G'}{G}\right)+O(1). \tag{2.17}$$

Since $T(r,\phi)=S(r,f)$ and $S(r,\psi)=S(r,f)$, by Theorem 3.1 in [5], we have

$$m\left(r,\frac{G'}{G}\right)=S(r,f).$$

Thus from (2.17), it follows that

$$lN_1\left(r,f\right) \le \bar{N}_2(r,f) + \bar{N}\left(r, \frac{1}{\psi - \phi}\right) + \bar{N}\left(r, \frac{1}{\psi' - \phi}\right) + S(r,f). \tag{2.18}$$

□

Lemma 2.3. *Let $f \in \mathcal{M}(\mathbb{C})$ and let ϕ be a small function of f such that f and ϕ have no common poles. Then*

$$lN_1\left(r,f\right) \le \bar{N}_2\left(r,f\right) + \bar{N}\left(r, \frac{1}{\psi - \phi}\right) + N_0\left(r, \frac{1}{\psi' - \phi'}\right) + S(r,f), \tag{2.19}$$

where $N_1\left(r,f\right)$ is the counting function of simple poles of f, $\bar{N}_2\left(r,f\right)$ is the counting function of multiple poles of f counted once and $N_0\left(r, \frac{1}{\psi'-\phi'}\right)$ is the counting function of zeros of $\psi' - \phi'$ which are not repeated zeros of $\psi - \phi$.

Proof. Define

$$G(z) = \frac{\{\psi'(z) - \phi'(z)\}^{l+1}}{\{\phi(z) - \psi(z)\}^{l+2}}.$$

Then as in the proof of Lemma 2.2 above, we again arrive at (2.14).

Next, to find the distribution of poles and zeros of $G(z)$, we proceed as follows:

Put

$$h(z) = \psi(z) - \phi(z).$$

If z_0 is a pole of $h(z)$ of order m, then near z_0,

$$h(z) = \frac{s(z)}{(z-z_0)^m} \text{ and } h'(z) = \frac{t(z)}{(z-z_0)^{m+1}},$$

where $s(z)$ and $t(z)$ are functions analytic in a neighborhood of z_0 and both have no zeros at z_0. So,

$$G(z) = \frac{w(z)}{(z-z_0)^{l+1-m}} \tag{2.20}$$

for some function $w(z)$ analytic in a neighborhood of z_0 such that $w(z_0) \neq 0$.

Next if z_0 is a zero of $h(z)$, then near z_0, $h(z) = l(z)\left(z-z_0\right)^m$ and so

$$G(z) = \frac{m(z)}{(z-z_0)^{l+1+m}}, \tag{2.21}$$

where $l(z)$ and $m(z)$ are functions analytic in a neighborhood of z_0 and both have no zeros at z_0.

From (2.21) and (2.20) we see that the only poles of $G(z)$ occur at

(i) the roots of $h(z) = 0$ and
(ii) the poles of $\phi(z)$ of multiplicity less than $l+1$.

Therefore,

$$\bar{N}(r,G) \le \bar{N}\left(r,\frac{1}{\psi-\phi}\right) + \bar{N}(r,\phi) - \bar{N}_{l+1}(r,\phi), \tag{2.22}$$

where $\bar{N}_k(r,\phi)$ is the counting function of poles of $\phi(z)$ which have multiplicity at least k, each pole is counted once.

Since the zeros of $G(z)$ occur at

(i) the roots of $h'(z)=0$ which are not the roots of $h(z)=0$
(ii) multiple poles of $f(z)$ and
(iii) poles of ϕ of multiplicity greater than $l+1$,

therefore,

$$\bar{N}\left(r,\frac{1}{G}\right) \le \bar{N}_0\left(r,\frac{1}{\psi'-\phi'}\right) + \bar{N}_2(r,f) + \bar{N}_{l+1}(r,\phi). \tag{2.23}$$

Adding (2.22) and (2.23), we have;

$$\begin{aligned}\bar{N}(r,G)+\bar{N}\left(r,\frac{1}{G}\right) \le \bar{N}_2(r,f) + \bar{N}(r,\phi) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right)\\ + N_0\left(r,\frac{1}{\psi'-\phi'}\right).\end{aligned} \tag{2.24}$$

Since $T(r,\phi)=S(r,f)$ and $S(r,\psi)=S(r,f)$, by Theorem 3.1 in [5], we have

$$m\left(r,\frac{G'}{G}\right)=S(r,f).$$

Thus, from (2.14) and (2.24), it follows that

$$lN_1(r,f) \le \bar{N}_2(r,f) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right) + N_0\left(r,\frac{1}{\psi'-\phi'}\right) + S(r,f). \tag{2.25}$$ □

Lemma 2.4. *Let f,ψ and ϕ be as in Lemma 2.2. Then*

(a) $$\begin{aligned}(T(r,f) \le \left(2+\frac{1}{l}\right)N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{l}\right)\bar{N}\left(r,\frac{1}{\psi-\phi}\right)\\ + \frac{1}{l}\bar{N}\left(r,\frac{1}{\psi'-\phi}\right) - \left(2+\frac{1}{l}\right)N_2^0(r,\psi) + S(r,f).\end{aligned}$$

(b) $$\begin{aligned}T(r,f) \le \left(2+\frac{1}{l}\right)N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{l}\right)\bar{N}\left(r,\frac{1}{\psi-\phi}\right)\\ + \frac{1}{l}N_0\left(r,\frac{1}{\psi'-\phi'}\right) - \left(2+\frac{1}{l}\right)N_2^0(r,\psi) + S(r,f).\end{aligned}$$

Proof. By Lemma 2.1, we have

$$\begin{aligned}N_1(r,f)+2\bar{N}_2(r,f) &\le N(r,f) \le T(r,f)\\ &\le \bar{N}(r,f) + N\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right)\\ &\quad - N_2^0(r,\psi) + S(r,f).\end{aligned} \tag{2.26}$$

Since $\bar{N}(r,f) = N_1(r,f) + \bar{N}_2(r,f)$, from (2.26) we have

$$\bar{N}_2(r,f) \leq N\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right) - N_2^0(r,\psi) + S(r,f). \tag{2.27}$$

Using (2.27) in Lemma 2.2, we obtain

$$\begin{aligned} N_1(r,f) \leq \frac{1}{l}N\left(r,\frac{1}{f}\right) + \frac{2}{l}\bar{N}\left(r,\frac{1}{\psi-\phi}\right) - \frac{1}{l}N_2^0(r,\psi) \\ + \frac{1}{l}\bar{N}\left(r,\frac{1}{\psi'-\phi}\right) + S(r,f). \end{aligned} \tag{2.28}$$

Now from (2.27) and (2.28) it follows that

$$\begin{aligned} \bar{N}(r,f) &= N_1(r,f) + \bar{N}_2(r,f) \\ &\leq N_1(r,f) + N\left(r,\frac{1}{f}\right) + \bar{N}\left(r,\frac{1}{\psi-\phi}\right) - N_2^0(r,\psi) + S(r,f) \\ &\leq \left(1+\frac{1}{l}\right)N\left(r,\frac{1}{f}\right) + \left(1+\frac{2}{l}\right)\bar{N}\left(r,\frac{1}{\psi-\phi}\right) - \left(1+\frac{1}{l}\right)N_2^0(r,\psi) \\ &\quad + \frac{1}{l}\bar{N}(r,\phi) + \frac{1}{l}\bar{N}\left(r,\frac{1}{\psi'-\phi}\right) + S(r,f). \end{aligned} \tag{2.29}$$

Now in view of (2.29), Lemma 2.1 yields

$$\begin{aligned} T(r,f) \leq \left(2+\frac{1}{l}\right)N\left(r,\frac{1}{f}\right) + \left(2+\frac{2}{l}\right)\bar{N}\left(r,\frac{1}{\psi-\phi}\right) \\ + \frac{1}{l}\bar{N}\left(r,\frac{1}{\psi'-\phi}\right) - \left(2+\frac{1}{l}\right)N_2^0(r,\psi) + S(r,f), \end{aligned}$$

which proves (a).

The conclusion (b) follows by using Lemma 2.3 instead of Lemma 2.2 in the proof of (a), above. □

Proof of Theorem 1.11. Since $N_2^0(r,\psi) \geq 0$, by Lemma 2.4(a) we have

$$T(r,f) \leq 3N\left(r,\frac{1}{f}\right) + 4\bar{N}\left(r,\frac{1}{\psi-\phi}\right) + \bar{N}\left(r,\frac{1}{\psi'-\phi}\right) + S(r,f). \tag{2.30}$$

Since f and $\psi-\phi$ have no zeros, $N\left(r,\frac{1}{f}\right) = 0$ and $\bar{N}\left(r,\frac{1}{\psi-\phi}\right) = 0$. Therefore, (2.30) reduces to

$$T(r,f) \leq \bar{N}\left(r,\frac{1}{\psi'-\phi}\right) + S(r,f). \tag{2.31}$$

Since $f \in \mathcal{M}(\mathbb{C})$ is transcendental, (2.31) implies that $\psi'(z) = \phi(z)$ has infinitely many solutions.

Similarly, Lemma 2.4(b) implies that $\psi'(z) = \phi'(z)$ has infinitely many solutions. □

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