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SEMI-CUBICALLY HYPONORMAL WEIGHTED SHIFTS WITH STAMPFLI'S SUBNORMAL COMPLETION

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Abstract. Let $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ be a weight sequence with Stampfli's subnormal completion and let W_α be its associated weighted shift. In this paper we discuss some properties of the region $\mathcal{U} : =\{(x, y) : W_\alpha$ is semi-cubically hyponormal$\}$ and describe the shape of the boundary of $\mathcal{U}$. In particular, we improve the results of [19, Theorem 4.2].

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A bounded operator T is said to be *subnormal* if it is the restriction of a normal operator to an invariant subspace ([15]). An operator T in $\mathcal{L}(\mathcal{H})$ is called *hyponormal* if $T^*T \geq TT^*$. In [5], Curto defined some classes of weak subnormality between hyponormality and subnormality in $\mathcal{L}(\mathcal{H})$, for examples, k-hyponormality and weak k-hyponormality. The weakly k-hyponormal weighted shift (whose definition will be defined below) is the main model in this paper. For a positive integer k, an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *weakly k-hyponormal* if for every polynomial p of degree k or less, $p(T)$ is hyponormal ([5,8,11,12]). An operator $T \in \mathcal{L}(\mathcal{H})$ is called *semi-weakly k-hyponormal* if $T + sT^k$ is hyponormal for $s \in \mathbb{C}$ ([13]). It is obvious that a weakly k-hyponormal operator is semi-weakly k-hyponormal. In particular, weak 2-hyponormality is equivalent to semi-weak 2-hyponormality. The weak 2-hyponormality (weak 3-hyponormality, semi-weak 3-hyponormality, resp.) is referred to as the *quadratic hyponormality* (*cubic hyponormality, semi-cubic hyponormality,* resp.). In particular, the quadratic hyponormality makes an important role in the study of gap on operator properties such as flatness, completion, and backward extension theory since 1990 (see, for instance, [1, 4, 6, 9, 10, 14, 16, 20]). In [6], Curto proved that a

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2-hyponormal weighted shift with two equal weights $\alpha_n = \alpha_{n+1}$ for some nonnegative integer n has the flatness property, i.e., $\alpha_1 = \alpha_2 = \cdots$. Moreover, he obtained a quadratically hyponormal weighted shift with first two equal weights which does not satisfy flatness ([6]). Also in [17], they showed that the weighted shift W_α with $\alpha : \sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}, \sqrt{\frac{n+1}{n+2}}$ $(n \geq 2)$ is not cubically hyponormal. Hence the following question arises naturally ([7]):

Problem 1.1. Describe all quadratically hyponormal weighted shifts with first two equal weights.

Recently Li-Cho-Lee in [18] proved that if a weighted shift W_α is cubically hyponormal with first two equal weights, then W_α has the flatness property. The structure of semi-cubically hyponormal weighted shifts has been studied by several authors (cf. [2, 3, 19]). To detect the structure of semi-cubically hyponormal weighted shifts, the following problem arises naturally:

Problem 1.2. Describe all semi-cubically hyponormal weighted shifts with first two equal weights.

As a study of Problem 1.2 it is worthwhile to describe the region $\mathcal{U} = \{(x,y) : W_\alpha \text{ is semi-cubically hyponormal}\}$ for weighted shifts W_α with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$, where $(1, \sqrt{x}, \sqrt{y})^\wedge$ is Stampfli's subnormal completion. Recall that Curto-Jung studied the shape of the region $\{(x, y) : W_\alpha$ is quadratically hyponormal$\}$ in [9]. In this paper we describe the region $\mathcal{U}$ in detail as a parallel study.

This note consists of four sections. In Section 2 we recall characterizations for semi-cubic hyponormality of a weighted shift W_α with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$. In Section 3, we describe the geometric shapes of the region $\mathcal{U}$ above. In Section 4, we discuss some remarks concerning the extremality of the region $\mathcal{U}$.

Throughout this note we denote $\mathbb{R}_+$ for the set of nonnegative real numbers. For a region $\mathcal{V}$ in $\mathbb{R}^2 (:= \mathbb{R} \times \mathbb{R})$, we denote the boundary of $\mathcal{V}$ by $\partial\mathcal{V}$.

Some of the calculations in this paper were aided by using the software tool *Mathematica* ([22]).

2. Preliminaries

We recall Stampfli's subnormal completion of three values ([21]). Let α_0, α_1, α_2 be the first three weights in $\mathbb{R}_+$ such that $\alpha_0 < \alpha_1 < \alpha_2$ (to avoid the flatness)[1]. Define

$$\widehat{\alpha}_n = \left(\Psi_1 + \frac{\Psi_0}{\alpha_{n-1}^2} \right)^{1/2}, \qquad n \geq 3, \tag{2.1}$$

[1]If W_α is a subnormal weighted shift such that $\alpha_0 = \alpha_1$ or $\alpha_1 = \alpha_2$, then $\alpha_1 = \alpha_2 = \cdots$.

where

$$\Psi_0 = -\frac{\alpha_0^2\alpha_1^2(\alpha_2^2-\alpha_1^2)}{\alpha_1^2-\alpha_0^2}, \qquad \Psi_1 = \frac{\alpha_1^2(\alpha_2^2-\alpha_0^2)}{\alpha_1^2-\alpha_0^2}.$$

This produces a bounded sequence $\widehat{\alpha} := \{\widehat{\alpha}_i\}_{i=0}^{\infty}$, where $\widehat{\alpha}_i = \alpha_i$ $(0 \le i \le 2)$ such that its associated weighted shift $W_{\widehat{\alpha}}$ is subnormal. As usual, we write $(\alpha_0, \alpha_1, \alpha_2)^\wedge$ for this weight sequence $\widehat{\alpha}$ induced by (2.1).

We now recall a characterization of the semi-cubic hyponormality of weighted shifts W_α with weight sequence $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$.

Lemma 2.1 ([19, Th. 4.1])**.** *Let $\alpha : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 < x < y$ be a weight sequence and let W_α be its associated weighted shift. Then W_α is semi-cubically hyponormal if and only if $f(x,y) := \sum_{i=0}^{9} \zeta_i y^i \ge 0$, where*

$$\begin{aligned}
&\zeta_0 = x^8,\ \zeta_1 = -x^5 + 8x^6 - 18x^7 + 2x^8,\\
&\zeta_2 = x^2 - 8x^3 + 39x^4 - 108x^5 + 131x^6 - 20x^7 + x^8,\\
&\zeta_3 = -3x + 32x^2 - 151x^3 + 338x^4 - 298x^5 - 12x^6 + 10x^7,\\
&\zeta_4 = 4 - 42x + 169x^2 - 274x^3 + 40x^4 + 276x^5 - 43x^6 - 4x^7,\\
&\zeta_5 = 16x - 130x^2 + 359x^3 - 330x^4 - 75x^5 + 34x^6,\\
&\zeta_6 = -2x + 38x^2 - 172x^3 + 260x^4 - 34x^5 - 6x^6,\\
&\zeta_7 = -x + 4x^2 + 17x^3 - 74x^4 + 18x^5,\\
&\zeta_8 = -2x^2 + 6x^3 + 7x^4 - 2x^5,\ \zeta_9 = -x^3.
\end{aligned}$$

Let $\alpha(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 < x < y$ be a weight sequence with Stampfli's subnormal completion tail and let $W_{\alpha(x,y)}$ be the associated weighted shift. For our convenience, we denote $x = 1+h$ and $y = 1+h+k$ $(h,\ k \in \mathbb{R}_+)$. Then we can rewrite the polynomials in Lemma 2.1 as following

$$p(h,k) := f(1+h, 1+h+k) = -\sum_{i=0}^{9} \xi_i(h) k^i \ge 0, \tag{2.2}$$

where

$$\begin{aligned}
&\xi_0(h) = 2h^9 (h+1)^4,\ \xi_1(h) = h^8 (16h+7)(h+1)^3,\\
&\xi_2(h) = 4h^6 \left(3h + 14h^2 + 14h^3 - 1\right)(h+1)^2,\\
&\xi_3(h) = h^5 (h+1)\left(3h + 98h^2 + 190h^3 + 112h^4 - 4\right),\\
&\xi_4(h) = h^4 \left(2h + 109h^2 + 322h^3 + 356h^4 + 140h^5 - 5\right),\\
&\xi_5(h) = 2h^3 (h+1)\left(5h + 46h^2 + 88h^3 + 56h^4 - 1\right),\\
&\xi_6(h) = h^2 (h+1)\left(13h + 64h^2 + 104h^3 + 56h^4 - 1\right),\\
&\xi_7(h) = h^2 (h+1)\left(34h + 42h^2 + 16h^3 + 9\right),\\
&\xi_8(h) = 2h\left(4h + h^2 + 2\right)(h+1)^2,\ \xi_9(h) = (h+1)^3.
\end{aligned}$$

For $\alpha(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $x = 1 + h$ and $y = 1 + h + k$ $(h, k \in \mathbb{R}_+)$, we denote

$$\mathcal{R} := \{(h,k) : W_{\alpha(x,y)} \text{ is semi-cubically hyponormal}\}$$

and

$$\mathcal{R}_{\mathrm{q}} := \{(h,k) : W_{\alpha(x,y)} \text{ is quadratically hyponormal}\}.$$

Then it follows from [19, Theorem 4.2] that both $\mathcal{R} \setminus \mathcal{R}_{\mathrm{q}}$ and $\mathcal{R}_{\mathrm{q}} \setminus \mathcal{R}$ are nonempty sets, *indeed*, a line segment $\{(\frac{1}{100}, k) : \beta_1 \le k < \alpha_1\}$ (or $\{(\frac{1}{100}, k) : \beta_2 < k \le \alpha_2\}$) contains in $\mathcal{R} \setminus \mathcal{R}_{\mathrm{q}}$ (or $\mathcal{R}_{\mathrm{q}} \setminus \mathcal{R}$, respectively), where $\alpha_1 \approx 0.000787776068\ldots$, $\alpha_2 \approx 0.0422764016\ldots$, $\beta_1 \approx 0.000786885627\ldots$, and $\beta_2 \approx 0.0402782805\ldots$; see the proof of [19, Theorem 4.2]. In the next section, the polynomial in (2.2) can be used to describe the shape of $\mathcal{R}$ as a crucial parts.

3. The shape of the region with semi-cubic hyponormality

Let $\alpha(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ with $1 < x < y$ be a weight sequence as usual and let $W_{\alpha(x,y)}$ be the associated weighted shift with $x = 1+h$ and $y = 1+h+k$ $(h,\ k \in \mathbb{R}_+)$. We may replace k by th, where t is a positive real number. Then $p(h,k)$ in (2.2) can be represented by

$$\begin{aligned} p(h,k) &= p(h,th) \\ &= h^8 \left(\phi_0(t) + \phi_1(t)h + \phi_2(t)h^2 + \phi_3(t)h^3 + \phi_4(t)h^4 + \phi_5(t)h^5\right), \end{aligned}$$

where

$$\begin{aligned}
\phi_0(t) &= 4t^2 + 4t^3 + 5t^4 + 2t^5 + t^6, \\
\phi_1(t) &= -2 - 7t - 4t^2 + t^3 - 2t^4 - 8t^5 - 12t^6 - 9t^7 - 4t^8 - t^9, \\
\phi_2(t) &= -8 - 37t - 76t^2 - 101t^3 - 109t^4 - 102t^5 - 77t^6 - 43t^7 - 16t^8 - 3t^9, \\
\phi_3(t) &= -12 - 69t - 180t^2 - 288t^3 - 322t^4 - 268t^5 - 168t^6 - 76t^7 - 22t^8 - 3t^9, \\
\phi_4(t) &= -8 - 55t - 168t^2 - 302t^3 - 356t^4 - 288t^5 - 160t^6 - 58t^7 - 12t^8 - t^9, \\
\phi_5(t) &= -2 - 16t - 56t^2 - 112t^3 - 140t^4 - 112t^5 - 56t^6 - 16t^7 - 2t^8.
\end{aligned}$$

For brevity, we set

$$\rho(h,t) = \phi_0(t) + \phi_1(t)h + \phi_2(t)h^2 + \phi_3(t)h^3 + \phi_4(t)h^4 + \phi_5(t)h^5. \tag{3.1}$$

Then $W_{\alpha(x,y)}$ is semi-cubically hyponormal if and only if $\rho(h,t) \ge 0$ for all positive numbers h and t. We will detect the set

$$\mathcal{C} := \{(h,th) | \rho(h,t) = 0 \text{ and } h > 0, t > 0\} \cup \{(0,0)\}$$

to consider the region of semi-cubic hyponormality of $W_{\alpha(x,y)}$ below. In fact, the set $\mathcal{C}$ will be a curve (see Lemma 3.1).

Lemma 3.1. *The set $\mathcal{C}$ is a loop with polar form of $r = f(\theta)$, $0 \le \theta \le \frac{\pi}{2}$. Therefore $\mathcal{R}$ is a starlike region with nonempty interior and $\mathcal{C} = \partial\mathcal{R}$.*

Proof. First we fix $t = t_0 > 0$. Since $\phi_i(t_0)$ $(i = 1, \ldots, 5)$ is negative obviously,

$$\frac{\partial}{\partial h}\rho(h, t_0) = \phi_1(t_0) + 2\phi_2(t_0)h + 3\phi_3(t_0)h^2 + 4\phi_4(t_0)h^3 + 5\phi_5(t_0)h^4$$

is negative for $h > 0$. Then it follows that $\rho(h, t_0)$ is decreasing in h. Since $\phi_0(t_0)$ is positive, the equation $\rho(h, t_0) = 0$ of h has a unique solution. $\square$

The following corollary improves the results of [19, Theorem 4.2], and its proof follows from the fact that the boundaries of $\mathcal{R}_{\mathrm{q}}$ and $\mathcal{R}$ are loops with polar forms of $r = f(\theta)$, $0 \le \theta \le \frac{\pi}{2}$.

Corollary 3.2. *Under the above notation, we have the following assertions*:
(i) $\mathcal{R}_q \cap \mathcal{R}$ *is a starlike region with nonempty interior,*
(ii) $\mathcal{R} \setminus \mathcal{R}_q$ *and* $\mathcal{R}_q \setminus \mathcal{R}$ *are regions with nonempty interiors.*

We now consider the tangent line to the closed curve $\mathcal{C}$ near the origin.

Lemma 3.3. *The tangent line to* $\mathcal{C}$ *converges to the* x*-axis as* $(h, t) \to (0^+, 0^+)$ *and it converges to the* y*-axis as* $(k, t) \to (0^+, \infty)$.

Proof. We mimic the proof of [9, Lemma 4.7]. From $k = th$, we have

$$\frac{dk}{dh} = \frac{dt}{dh}h + t. \tag{3.2}$$

Since $\rho(h, t) = 0$ on $\mathcal{C}$, we get

$$\frac{dt}{dh} = -\frac{\frac{\partial\rho}{\partial h}}{\frac{\partial\rho}{\partial t}} = -\frac{\phi_1(t) + 2\phi_2(t)h + 3\phi_3(t)h^2 + 4\phi_4(t)h^3 + 5\phi_5(t)h^4}{\phi_0'(t) + \phi_1'(t)h + \phi_2'(t)h^2 + \phi_3'(t)h^3 + \phi_4'(t)h^4 + \phi_5'(t)h^5}.$$

Furthermore, we have

$$\lim_{(h,t)\to(0^+,0^+)} \frac{dk}{dh} = \lim_{(h,t)\to(0^+,0^+)} \frac{dt}{dh}h.$$

Since $\lim_{(h,t)\to(0^+,0^+)} \frac{dt}{dh} = \infty$, by using the L'Hospital's rule and some elementary computations, we can obtain

$$\begin{aligned}
\lim_{(h,t)\to(0^+,0^+)} \frac{dt}{dh}h &= \lim_{(h,t)\to(0^+,0^+)} \frac{h}{\frac{dh}{dt}} = \lim_{(h,t)\to(0^+,0^+)} \frac{1}{\frac{d}{dh}\left(\frac{dh}{dt}\right)} \\
&= \lim_{(h,t)\to(0^+,0^+)} \frac{1}{\frac{\partial}{\partial t}\left(\frac{dh}{dt}\right)\frac{dt}{dh} + \frac{\partial}{\partial h}\left(\frac{dh}{dt}\right)} \\
&= \lim_{(h,t)\to(0^+,0^+)} \frac{F_1(h,t)}{F_2(h,t)} = 0
\end{aligned}$$

for some polynomials F_1 and F_2 of h and t (see http://arxiv.org/pdf/1803.03349 .pdf for expressions of F_1 and F_2) such that

$$\lim_{(h,t)\to(0^+,0^+)} F_1(h,t) = 0 \text{ and } \lim_{(h,t)\to(0^+,0^+)} F_2(h,t) = 32.$$

Similarly, we have $\lim_{(k,t)\to(0^+,\infty)} \frac{dk}{dh} = \infty$. Hence the proof is complete. $\square$

We now set

(3.3) $h_M = \max\{h : (h,k) \in \mathcal{R}, k \in \mathbb{R}_+\},\ k_M = \max\{k : (h,k) \in \mathcal{R}, h \in \mathbb{R}_+\}.$

Obviously two maximum values h_M and k_M are well defined. Recall that the problem [9, Problem 5.1] which is finding the values or expressions of $\max\{h : (h,k) \in \mathcal{R}_{\mathrm{q}}, k \in \mathbb{R}_+\}$ and $\max\{k : (h,k) \in \mathcal{R}_{\mathrm{q}}, h \in \mathbb{R}_+\}$ are not solved yet. Hence it is worthwhile finding extremal values h_M and k_M. We discuss the values of h_M and k_M below.

Lemma 3.4. *Under the same notation in* (3.3), *we have* $0 < h_M < \frac{14}{100}$.

Proof. We can obtain that

$$\rho\left(\frac{14}{100}, t\right) = \frac{1}{156250000}\sum_{k=0}^{9} c_k t^k,$$

where

$$\begin{aligned}
&c_0 = -73892007,\ c_1 = -299457081,\ c_2 = 217020204,\ c_3 = 195013758,\\
&c_4 = 243084610,\ c_5 = -308008392,\ c_6 = -424167096,\ c_7 = -364763406,\\
&c_8 = -146669607,\ c_9 = -32408775.
\end{aligned}$$

This can be represented by

$$\begin{aligned}
\sum_{k=0}^{9} c_k t^k &< 10^7(-29t + 30t^2 + 20t^3 + 25t^4 - 30t^5 - 40t^6 - 35t^7 - 10t^8 - 3t^9)\\
&= 10^7 t(-29 + 30t + 20t^2 + 25t^3 - 30t^4 - 40t^5 - 35t^6 - 10t^7 - 3t^8)\\
&= 10^7 t((-4 + 20t^2 - 30t^4) + (-5 + 25t^3 - 35t^6) - 10t^7 - 3t^8 - \eta(t))\\
&= 10^7 t\left(-A^2 - 5B^2 - \frac{101}{84} - 10t^7 - 3t^8 - \eta(t)\right),
\end{aligned}$$

where $A = \sqrt{30}t^2 - \sqrt{\frac{10}{3}}$, $B = \sqrt{7}t^3 - \frac{5}{\sqrt{28}}$ and $\eta(t) = 20 - 30t + 40t^5$. Here, since $\eta(t)$ has exactly one critical number $\sqrt[4]{\frac{3}{20}}$ on $\mathbb{R}_+$ and $\eta''(t) > 0$ on $\mathbb{R}_+$, $\eta(t)$ has a positive minimum at $\sqrt[4]{\frac{3}{20}}$. So, $\rho\left(\frac{14}{100}, t\right)$ is negative and since $\mathcal{C}$ is a loop in the first quadrant, $\mathcal{C}$ lies on the left side of a line $h = \frac{14}{100}$. $\square$

Recall Descartes' rule of signs that if $p(x)$ is a polynomial with real coefficients, then the number of positive roots either is equal to the number of variations in sign of $p(x)$ or is less than that number by an even number; and the number of negative roots either is equal to the number of variations in sign of $p(-x)$ or is less than that number by an even number.

Lemma 3.5. *Given* $h > 0$, *there exist at most two roots* (*possibly a double root*) $k_0 > 0$ *such that* $p(h, k_0) = 0$.

Proof. According to Lemma 3.4, it is sufficient to consider $h < \frac{14}{100}$. Recall that

$$p(h,k) = -\sum_{i=0}^{9} \xi_i(h) k^i,$$

where $\xi_i(h)$ are shown in (2.2). Here, all of the coefficients of $\xi_i(h)$, $i = 0,1,7,8,9$ are positive and $\xi_i(h)$, $i = 2,3,4,5,6$ has one variation in sign, so it has exactly one positive root ϵ_i for $i = 2,3,4,5,6$, respectively. Especially, $\epsilon_6 \approx 0.0584537$ and $\epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5 > \frac{14}{100}$, so $\xi_2(h), \xi_3(h), \xi_4(h), \xi_5(h)$ are negative for $h < \frac{14}{100}$. Hence the signs of the coefficients of $p(h,k)$ change twice as a polynomial in k for $h < \frac{14}{100}$. By Descartes' rule of signs, it follows that for fixed $h > 0$, the equation $p(h,k) = 0$ of k has no or two roots. $\square$

We may obtain the following lemma similarly.

Lemma 3.6. *Given $k > 0$, there exist at most two roots (possibly a double root) $h_0 > 0$ such that $p(h_0, k) = 0$.*

Note that $\mathcal{C}$ consists of two functions $k = f_1(h)$ and $k = f_2(h)$ on the interval $(0, h_M]$. Similarly, $\mathcal{C}$ consists of two functions $h = g_1(k)$ and $h = g_2(k)$ on the interval $(0, k_M]$.

Combining above lemmas, we obtain the main theorem of this paper.

Theorem 3.7. *The region $\mathcal{R}$ is a simply connected with boundary $\partial\mathcal{R}$ such that*

(i) *$\partial\mathcal{R}$ is a loop with polar form $r = f(\theta)$, $0 \le \theta \le \frac{\pi}{2}$,*

(ii) *the tangent lines of $\partial\mathcal{R}$ near origin $(0,0)$ converge to the x- and y-axes,*

(iii) $\operatorname{card}(\partial\mathcal{R} \cap \{(a,k) : k \in \mathbb{R}\}) = 2$, *where $0 < a < h_M$,*

(iv) $\operatorname{card}(\partial\mathcal{R} \cap \{(h,b) : h \in \mathbb{R}\}) = 2$, *where $0 < b < k_M$.*[2]

4. Further remarks

Let $\alpha(x,y) : 1, (1, \sqrt{x}, \sqrt{y})^\wedge$ be a weight sequence with $x = 1 + h$ and $y = 1 + h + k$ ($h,\ k \in \mathbb{R}_+$). Recall that h_M and k_M are well defined (see Section 3). The problems of expressions about the extremal values h_M and k_M are a parallel ones which were suggested as a question in [9, Prob. 5.1]. So it is worth describing the extremal values h_M and k_M. For this purpose, we denote

$$Q := Q(h,t) = \frac{\partial \rho}{\partial t} = \sum_{i=0}^{5} \phi_i'(t) h^i. \tag{4.1}$$

From (3.2), we obtain

$$\frac{dk}{dh} = \frac{dt}{dh} h + t = \frac{S}{Q},$$

[2] card$(\cdot)$ denotes for the cardinality of $\cdot$.

for a polynomial

$$S := S(h,t) = \sum_{j=0}^{4} \nu_j(t) h^j, \tag{4.2}$$

where

$$\begin{aligned}
\nu_0(t) &= 8t^2 + 12t^3 + 20t^4 + 10t^5 + 6t^6,\\
\nu_1(t) &= 2 - 4t^2 + 2t^3 - 6t^4 - 32t^5 - 60t^6 - 54t^7 - 28t^8 - 8t^9,\\
\nu_2(t) &= 16 + 37t - 101t^3 - 218t^4 - 306t^5 - 308t^6 - 215t^7 - 96t^8 - 21t^9,\\
\nu_3(t) &= 36 + 138t + 180t^2 - 322t^4 - 536t^5 - 504t^6 - 304t^7 - 110t^8 - 18t^9,\\
\nu_4(t) &= 32 + 165t + 336t^2 + 302t^3 - 288t^5 - 320t^6 - 174t^7 - 48t^8 - 5t^9,\\
\nu_5(t) &= 10 + 64t + 168t^2 + 224t^3 + 140t^4 - 56t^6 - 32t^7 - 6t^8.
\end{aligned}$$

Hence we arrive at the following proposition.

Proposition 4.1. *Under the notation as in* (3.3), *we have that*
(i) $h_M = \max\{h : \rho(h,t) = 0 \text{ and } Q(h,t) = 0\}$,
(ii) $k_M = \max\{th : \rho(h,t) = 0 \text{ and } S(h,t) = 0\}$,
where $\rho(h,t)$, $Q(h,t)$ *and* $S(h,t)$ *are as in* (3.1), (4.1) *and* (4.2), *respectively.*

Before closing this note, we describe the curvature of $\partial\mathcal{R}$ for the further information above the shape of $\mathcal{R}$. Since $k = th$,

$$\frac{d^2k}{dh^2} = \frac{d^2t}{dh^2}h + 2\frac{dt}{dh},$$

and since $\rho(h,t) = 0$ on $\mathcal{C}$,

$$\frac{\partial\rho}{\partial t}\frac{dt}{dh} + \frac{\partial\rho}{\partial h} = 0.$$

By differentiation with respect to h, we obtain that

$$\left[\frac{\partial^2\rho}{\partial t^2}\frac{dt}{dh} + \frac{\partial}{\partial h}\left(\frac{\partial\rho}{\partial t}\right)\right]\frac{dt}{dh} + \frac{\partial\rho}{\partial t}\frac{d^2t}{dh^2} + \frac{\partial}{\partial t}\left(\frac{\partial\rho}{\partial h}\right)\frac{dt}{dh} + \frac{\partial^2\rho}{\partial h^2} = 0.$$

Then

$$\frac{d^2t}{dh^2} = -\frac{\left(\frac{\partial^2\rho}{\partial t^2}\frac{dt}{dh} + \frac{\partial^2\rho}{\partial h\partial t}\right)\frac{dt}{dh} + \frac{\partial^2\rho}{\partial t\partial h}\frac{dt}{dh} + \frac{\partial^2\rho}{\partial h^2}}{\frac{\partial\rho}{\partial t}}. \tag{4.3}$$

It follows from (4.3) that

$$\frac{d^2k}{dh^2} = \frac{2(t+1)P}{Q^3},$$

where a polynomial $P := P(h,t)$ as follows:

$$P(h,t) = \sum_{j=0}^{14} \mu_j(t) h^j \tag{4.4}$$

(see https://arxiv.org/pdf/1803.03349.pdf for detail expression). Hence the curvature κ of $\mathcal{C}$ can be represented by

$$\kappa = \frac{\left|\frac{d^2k}{dh^2}\right|}{\left(1+\left(\frac{dk}{dh}\right)^2\right)^{\frac{3}{2}}} = \frac{2(t+1)\,|P|}{(Q^2+S^2)^{\frac{3}{2}}}.$$

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