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RESEARCH ON NORMAL STRUCTURE IN A BANACH SPACE VIA SOME PARAMETERS IN ITS DUAL SPACE

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Abstract. Let X be a Banach space and X^* be its dual. In this paper, we give relationships among some parameters in X^*: ε-nonsquareness parameter, $J(\varepsilon, X^*)$; ε-boundary parameter, $Q(\varepsilon, X^*)$; the modulus of smoothness, $\rho_{X^*}(\varepsilon)$; and ε-Pythagorean parameter, $E(\varepsilon, X^*)$, and weak orthogonality parameter, $\omega(X)$ in X that imply uniform norm structure in X. Some existing results are extended or approved.

Let X be a Banach space with $\dim X \geq 2$, and let $S(X) = \{x \in X : \|x\| = 1\}$ and $B(X) = \{x \in X : \|x\| \leq 1\}$ be the unit sphere and unit ball of X, respectively.

Let C be a subset of a Banach space X. A mapping $T : C \to C$ is called a nonexpansive mapping if $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. Many sufficient conditions for guaranteeing the existence of fixed points of a nonexpansive mapping are widely investigated by many authors. Such conditions are usually expressed in terms of geometric properties or geometric parameters.

Definition 1 ([13])**.** A Banach space X is called uniformly non-square if there exists $\delta > 0$ such that for $x, y \in S(X)$, either $\frac{\|x+y\|}{2} \leq 1 - \delta$ or $\frac{\|x-y\|}{2} \leq 1 - \delta$.

Definition 2 ([1])**.** A bounded convex subset K of a Banach space X is said to have normal structure if for every convex subset H of K that contains more than one point there exists a point $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

A Banach space X is said to have normal structure if every bounded convex subset of X has normal structure. A Banach space X is said to have weak normal structure if each weakly compact convex set K of X that contains more than one point has normal structure. X is said to have uniform normal structure if there exists $0 < c < 1$ such that for any bounded convex subset K of X there exists $x_0 \in K$ such that

$$\sup\{\|x_0 - y\| : y \in K\} < c \cdot \sup\{\|x - y\| : x, y \in K\}.$$

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It is clear that for a reflexive Banach space, normal structure and weak normal structure coincide.

The normal structure is not a property which is invariant under passing to its dual space.

Kirk [16] proved that if a weakly compact convex subset K of X has normal structure, then any nonexpansive mapping on K has a fixed point. Whether or not a Banach space has normal structure depends on the geometry of the unit ball.

Definition 3 ([3], [4])**.** Let X and Y be Banach spaces. We say that Y is *finitely representable in* X if for any $\varepsilon > 0$ and any finite dimensional subspace $N \subseteq Y$ there is an isomorphism $T : N \to X$ such that for any $y \in N$, $(1-\varepsilon)\|y\| \leq \|Ty\| \leq (1+\varepsilon)\|y\|$.

The Banach space X is called *super-reflexive* if any space Y which is finitely representable in X is reflexive.

Theorem 4. *If X is uniformly non-square, then X is supper-reflexive and therefore X is reflexive.*

Theorem 5. *X is supper-reflexive if and only if X^* is supper-reflexive.*

Many parameters have been used to study and describe the shape and structure of unit balls of Banach spaces. The measures of their values are used to determine the existence of fixed points of nonexpansive mappings in Banach spaces.

Let a and b be two real numbers, we use $a \bigwedge b$ to denote the smaller number between a and b.

Gao and Lau ([6], [11]) defined four functions via antipodal points x and $-x$ in $S(X)$, and

$$J(X) = \sup\{\|x+y\| \bigwedge \|x-y\| : x, y \in S(X)\}$$

is one of them.

Gao and Lau proved that:

Theorem 6 ([6], [11])**.** *X is a uniformly non-square if and only if either $J(X) < 2$.*

Gao [7] extended the above concepts to the following parameter:

$$J(\varepsilon, X) = \sup\{\|x+\varepsilon y\| \bigwedge \|x-\varepsilon y\|\} : x, y \in S(X)\},$$

where $0 < \varepsilon \leq 1$.

It is easy to see that:

Theorem 7. *X is a uniformly non-square if $J(\varepsilon, X) < 1+\varepsilon$.*

Theorem 8 ([19])**.** *For a Banach space X if $J(\varepsilon, X) < \frac{\varepsilon+\sqrt{\varepsilon^2+4}}{2}$, then X has uniform normal structure.*

Gao [8] introduced the parameter:

$$Q(\varepsilon, X) = \sup\{\|x + \varepsilon y\| + \|x - \varepsilon y\| : x, y \in S(X)\},$$

where $0 < \varepsilon \leq 1$, and proved:

Theorem 9 ([8]). *A Banach space X with $Q(\varepsilon, X) < 2(1+\varepsilon)$ for some $0 < \varepsilon \leq 1$ is uniformly non-square.*

Since $Q(\varepsilon, X) \geq 2J(\varepsilon, X)$, we have:

Theorem 10. *A Banach space X with $Q(\varepsilon, X) < \varepsilon + \sqrt{\varepsilon^2 + 4}$ for some $0 < \varepsilon \leq 1$ has normal structure.*

Lindenstrauss [18] introduced the modulus of smoothness:

$$\rho_X(\epsilon) = \sup\{\frac{\|x + \varepsilon y\| + \|x - \varepsilon y\| - 2}{2},\ x, y \in S(X)\},\ \text{where } \epsilon \geq 0.$$

The following results were proved:

Theorem 11 ([8]). *A Banach space X with $\rho_X(\varepsilon) < \varepsilon$ for some $0 < \varepsilon \leq 1$ is uniformly non-square.*

Theorem 12 ([19]). *A Banach space X with $\rho_X(\varepsilon) < \frac{\varepsilon - 2 + \sqrt{\varepsilon^2+4}}{2}$ for some $0 < \varepsilon \leq 1$ has normal structure.*

In fact, $Q(\varepsilon, X) = 2 + 2\rho_X(\epsilon)$.

To compare the unit balls between Banach spaces and Hilbert spaces, Gao [9] also introduced the Pythagorean parameter:

$$E(\varepsilon, X) = \sup\{\|x + \varepsilon y\|^2 + \|x - \varepsilon y\|^2 : x, y \in S(X)\},$$

where $0 < \varepsilon \leq 1$, and proved:

Theorem 13 ([9]). *A Banach space X with $E(\varepsilon, X) < 2(1+\varepsilon)^2$ for some $0 < \varepsilon \leq 1$ is uniformly non-square.*

Theorem 14 ([12]). *Let X be a Banach space.*

(a) *If $E(\varepsilon, X) < 2 + \varepsilon^2 + \varepsilon\sqrt{4 + \varepsilon^2}$ for some $0 < \varepsilon \leq 1$, then X has uniform normal structure*;

(b) *If $E(\varepsilon, X) < \frac{(1+\varepsilon)^2 + (1+\varepsilon)\sqrt{(1+\varepsilon)^2 + 8\varepsilon}}{2}$ for some $0 < \varepsilon \leq 1$, then X^* has uniform normal structure.*

Since $\frac{(1+\varepsilon)^2 + (1+\varepsilon)\sqrt{(1+\varepsilon)^2 + 8\varepsilon}}{2} \leq 2 + \varepsilon^2 + \varepsilon\sqrt{4 + \varepsilon^2} \leq 2(1+\varepsilon)^2$, we actually have:

Theorem 15. *If $E(\varepsilon, X) < \frac{(1+\varepsilon)^2 + (1+\varepsilon)\sqrt{(1+\varepsilon)^2 + 8\varepsilon}}{2}$ for some $0 < \varepsilon \leq 1$, then both X and X^* have uniform normal structure.*

Example. Let X be the space either l_p or L_p where $1 < p < \infty$.

Then [7]

(a) $J(\varepsilon, X) = (1 + \varepsilon^p)^{\frac{1}{p}},\ 1 < p \leq 2,$

(b) $J(\varepsilon, X) = 2^{-\frac{1}{p}}((1+\varepsilon)^p + (1-\varepsilon)^p)^{\frac{1}{p}},\ p > 2$, where $\frac{1}{p} + \frac{1}{q} = 1$;

and [8]

(c) $Q(\varepsilon, X) = 2(1+\varepsilon^p)^{\frac{1}{p}},\ 1 < p \leq 2$,

(d) $Q(\varepsilon, X) = 2^{\frac{1}{q}}((1+\varepsilon)^p + (1-\varepsilon)^p)^{\frac{1}{p}},\ p > 2$, where $\frac{1}{p} + \frac{1}{q} = 1$;

and [9]

(e) $E(\varepsilon, X) \geq 2(1+\varepsilon^p)^{\frac{2}{p}},\ 1 < p \leq 2$,

(f) $E(\varepsilon, X) = 2^{1-\frac{2}{p}}((1+\varepsilon)^p + (1-\varepsilon)^p)^{\frac{2}{p}},\ p > 2$, where $\frac{1}{p} + \frac{1}{q} = 1$.

In [23], Sims introduced the property WORTH: A Banach space X has the property WORTH whenever

$$\lim_{n\to\infty} \sup\{\|x_n + x\| - \|x_n - x\|\} = 0$$

for all the weakly null sequence $\{x_n\}$ in X and all the element x of X.

In [24], Sims introduced the following parameter:

$$\omega(X) \equiv \sup\{\lambda > 0 : \lambda \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_{n\to\infty} \|x_n - x\|\},$$

where the supremum is taken over all the weakly null sequence $\{x_n\}$ in X and all the element x of X.

It was proved in [24] that $\frac{1}{3} \leq \omega(X) \leq 1$ for all Banach space X, and X has the property WORTH if and only if $w(X) = 1$.

It was also proved in [14] that if X is a reflexive Banach space, then $\omega(X) = \omega(X^*)$.

The following results were proved:

Theorem 16 ([14])**.** *If X is a Banach space with $J(X) < 1 + \omega(X)$, then X has normal structure.*

But it was also proved in [14]: $J(X) < 1 + \omega(X)$ does not imply that X^* has normal structure.

Theorem 17 ([25])**.** *If X is a Banach space with $J(X) < 2\omega(X)$, then both X and X^* have normal structure.*

Theorem 18 ([12])**.** *If X is a Banach space with $J(\varepsilon, X) < 1 + \varepsilon\omega(X)$ for some $0 < \varepsilon \leq 1$, then X has normal structure.*

Since $Q(\varepsilon, X) \geq 2J(\varepsilon, X)$, we have:

Theorem 19. *If X is a Banach space with $Q(\varepsilon, X) < 2(1 + \varepsilon\omega(X))$ for some $0 < \varepsilon \leq 1$, then X has normal structure.*

Since $Q(\varepsilon, X) = 2 + 2\rho_X(\epsilon)$, we have:

Theorem 20. *If X is a Banach space with $\rho_X(\epsilon) < \varepsilon\omega(X)$ for some $0 < \varepsilon \leq 1$, then X has normal structure.*

Theorem 21 ([10], [12])**.** (a) *A Banach space X with $E(\varepsilon, X) < 2(1 + \varepsilon\omega(X))^2$ implies that X has normal structure*;

(b) *A Banach space X with $E(\varepsilon, X) < (1+\varepsilon)^2(1+\omega^2(X))$ implies that X^* has normal structure.*

Since $(1+\varepsilon)^2(1+\omega^2(X)) \leq 2(1+\varepsilon\omega(X))^2 \leq 2(1+\varepsilon)^2$, we actually have:

Theorem 22. *A Banach space X with $E(\varepsilon, X) < (1+\varepsilon)^2(1+\omega^2(X))$ implies that both X and X^* have normal structure.*

Definition 23 ([17])**.** Let X be a Banach space. The norm-separation

$$\mu(X) \equiv \sup\{\varepsilon > 0 : \text{there is a sequence } \{x_n\} \subseteq S(X) \\ \text{with } sep(x_n) \equiv \inf\{\|x_n - x_m\| : n \neq m\} \geq \varepsilon\}.$$

Lemma 24 ([5])**.** *Let X be a Banach space without weak normal structure. Then for any $0 < \epsilon < 1$, there exists a sequence $\{x_n\} \subseteq S(X)$ with $x_n \rightarrow^w 0$, such that*

$$1 - \epsilon < \|x_{n+1} - x\| < 1 + \epsilon$$

for sufficiently large n, and any $x \in co\{x_k\}_{k=1}^n$.

Lemma 24 can be used to get the following result for the norm-separation $\mu(X^*)$:

Lemma 25 ([20])**.** *If X is a Banach space with $B(X^*)$ is weak* sequentially compact (for example, X is reflexive or separable, or has an equivalent smooth norm) and fails to have weak normal structure, then for any $\varepsilon > 0$ there is a sequence $\{x_n\} \subseteq S(X)$ and a sequence $\{f_n\} \subseteq S(X^*)$ such that*

(a) $|\|x_i - x_j\| - 1| < \varepsilon$, *where $i \neq j$;*
(b) $\langle x_i, f_i\rangle = 1$, *where $1 \leq i \leq \infty$;*
(c) $|\langle x_j, f_i\rangle| < \varepsilon$, *where $i \neq j$; and*
(d) $\|f_i - f_j\| > 2 - \varepsilon$, *where $i \neq j$.*

The following result regarding the relationship between separation measure in X^* and normal structure in X was proved:

Theorem 26 ([21])**.** *If X is a Banach space with $B(X^*)$ is weak* sequentially compact, and $\mu(X^*) < 2$, then X has weak normal structure.*

Remark 27. In general, the values of these parameters in X are different from the corresponding values in $X^* : J(\varepsilon, X) \neq J(\varepsilon, X^*)$, $Q(\varepsilon, X) \neq Q(\varepsilon, X^*)$, $\rho_X(\varepsilon) \neq \rho_{X^*}(\varepsilon)$, and $E(\varepsilon, X) \neq E(\varepsilon, X^*)$,

Let $X = l_{\frac{3}{2}}$, then $X^* = l_3$, and let $\varepsilon = \frac{1}{2}$.

It is easy to see that

$$J(\frac{1}{2}, l_{\frac{3}{2}}) = (1 + (\frac{1}{2})^{\frac{3}{2}})^{\frac{2}{3}} = 1.223630407\ldots; \text{ and}$$

$$J(\frac{1}{2}, l_3) = 2^{\frac{-1}{3}}((1+\frac{1}{2})^3 + (1-\frac{1}{2})^3)^{\frac{1}{3}} \\ = 2^{\frac{-1}{3}}((\frac{3}{2})^3 + (\frac{1}{2})^3)^{\frac{1}{3}} = 1.205071132\ldots.$$

This implies $J(\frac{1}{2}, l_{\frac{3}{2}}) \neq J(\frac{1}{2}, l_3)$.

$$Q(\frac{1}{2}, l_{\frac{3}{2}}) = 2(1 + (\frac{1}{2})^{\frac{3}{2}})^{\frac{2}{3}} = 2.447260815\ldots; \text{ and}$$
$$Q(\frac{1}{2}, l_3) = 2^{\frac{2}{3}}((1+\frac{1}{2})^3 + (1-\frac{1}{2})^3)^{\frac{1}{3}}$$
$$= 2^{\frac{2}{3}}((\frac{3}{2})^3 + (\frac{1}{2})^3)^{\frac{1}{3}} = 2.410142264\ldots.$$

This implies $Q(\frac{1}{2}, l_{\frac{3}{2}}) \neq Q(\frac{1}{2}, l_3)$, and $\rho_X(\frac{1}{2}) \neq \rho_{X^*}(\frac{1}{2})$.

$$E(\frac{1}{2}, l_{\frac{3}{2}}) \geq 2(1 + (\frac{1}{2})^{\frac{3}{2}})^{\frac{4}{3}} = 2.994542748\ldots; \text{ and}$$
$$E(\frac{1}{2}, l_3) = 2^{\frac{1}{3}}((1+\frac{1}{2})^3 + (1-\frac{1}{2})^3)^{\frac{2}{3}}$$
$$= 2^{\frac{1}{3}}((\frac{3}{2})^3 + (\frac{1}{2})^3)^{\frac{2}{3}} = 2.904392867\ldots.$$

This implies $E(\frac{1}{2}, l_{\frac{3}{2}}) \neq E(\frac{1}{2}, l_3)$.

We now consider the relationship between the values of these parameters in X^* and geometric properties in X.

Theorem 28. *For a Banach space X with $B(X^*)$ is weak* sequentially compact, if*

$$J(\varepsilon, X^*) < (1+\varepsilon)\omega(X),$$

or

$$Q(\varepsilon, X^*) < (1+\varepsilon)(1+\omega(X)),$$

or

$$\rho_{X^*}(\varepsilon) < \frac{\omega(X) + \varepsilon + \varepsilon\omega(X) - 1}{2},$$

or

$$E(\varepsilon, X^*) < (1+\varepsilon)^2(1+\omega^2(X))$$

for some $0 < \varepsilon \leq 1$, then X has weak normal structure.

Proof. If X fails to have weak normal structure, from Lemma 25, for any $\eta > 0$, there are two sequences $\{x_n\} \subseteq S(X)$ with $x_n \to^w 0$, and $\{f_n\} \subseteq S(X^*)$, satisfies four conditions there.

From (a) of Lemma 25, we have $\|x_n - x_1\| \leq 1 + \eta$, so

$$\liminf_{n\to\infty} \|x_n - x_1\| \leq 1 + \eta.$$

On the other hand, from definition of $\omega(X)$, we have

$$(\omega(X) - \eta) \liminf_{n\to\infty} \|x_n + x_1\| \leq \liminf_{n\to\infty} \|x_n - x_1\|.$$

Therefore exists an n such that

$$(\omega(X) - \eta)\|x_n + x_1\| \leq \|x_n - x_1\| \leq 1 + \eta.$$

That is

$$\|x_n + x_1\| \le \frac{1+\eta}{\omega(X)-\eta}.$$

From

$$\begin{aligned}\|f_n - \varepsilon f_1\| &= \|(f_n - f_1) + (f_1 - \varepsilon f_1)\| \\ &\ge 2 - \eta - (1-\varepsilon) = 1 + \varepsilon - \eta,\end{aligned}$$

and

$$\begin{aligned}\|f_n + \varepsilon f_1\| &\ge \frac{\omega(X)-\eta}{1+\eta}\langle x_n + x_1, f_n + \varepsilon f_1\rangle \\ &= \frac{\omega(X)-\eta}{1+\eta}(1 + \langle x_1, f_n\rangle + \varepsilon\langle x_n, f_1\rangle + \varepsilon\langle x_1, f_1\rangle) \\ &\ge \frac{\omega(X)-\eta}{1+\eta}(1+\varepsilon-\eta-\varepsilon\eta) \\ &= \frac{\omega(X)-\eta}{1+\eta}(1+\varepsilon)(1-\eta),\end{aligned}$$

we have

$$\begin{aligned}\|f_n + \varepsilon f_1\| \bigwedge \|f_n - \varepsilon f_1\| &= \frac{\omega(X)-\eta}{1+\eta}(1+\varepsilon)(1-\eta) \bigwedge (1+\varepsilon-\eta), \\ \|f_n + \varepsilon f_1\| + \|f_n - \varepsilon f_1\| &\ge \frac{\omega(X)-\eta}{1+\eta}(1+\varepsilon)(1-\eta) + (1+\varepsilon-\eta)), \\ \frac{\|f_n + \varepsilon f_1\| + \|f_n - \varepsilon f_1\| - 2}{2} &\ge \frac{1}{2}(\frac{\omega(X)-\eta}{1+\eta}(1+\varepsilon)(1-\eta) + (1+\varepsilon-\eta)) - 1,\end{aligned}$$

and

$$\|f_n + \varepsilon f_1\|^2 + \|f_n - \varepsilon f_1\|^2 \ge \frac{(\omega(X)-\eta)^2}{(1+\eta)^2}(1+\varepsilon)^2(1-\eta)^2 + (1+\varepsilon-\eta)^2.$$

Since η can be arbitrarily small, from definition of $J(\varepsilon, X^*)$, $Q(\varepsilon, X^*)$, $\rho_{X^*}(\varepsilon)$ and $E(\varepsilon, X^*)$, we have

$$\begin{aligned}J(\varepsilon, X^*) &\ge (1+\varepsilon)w(X), \\ Q(\varepsilon, X^*) &\ge (1+\varepsilon)(1+w(X)), \\ \rho_{X^*}(\varepsilon) &\ge \frac{\omega(X)+\varepsilon+\varepsilon\omega(X)-1}{2},\end{aligned}$$

and

$$E(\varepsilon, X^*) \ge (1+\varepsilon)^2(1+w^2(X)). \qquad \square$$

Theorem 29. *For a Banach space X, if $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$, or $Q(\varepsilon, X^*) < (1+\varepsilon)(1+\omega(X))$, or $\rho_{X^*}(\varepsilon) < \frac{\omega(X)+\varepsilon+\varepsilon\omega(X)-1}{2}$, or $E(\varepsilon, X^*) < (1+\varepsilon)^2(1+\omega^2(X))$ for some $0 < \varepsilon \le 1$, then X has normal structure.*

Proof. Since $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$ implies $J(\varepsilon, X^*) < 1+\varepsilon$, from Theorem 7, X^* is uniformly non-square, so X^* and therefore X is reflexive. From Theorem 28, X has normal structure.

Since $Q(\varepsilon, X^*) < (1+\varepsilon)(1+\omega(X))$ implies $Q(\varepsilon, X^*) < 2(1+\varepsilon)$, from Theorem 9 and Theorem 28, X has normal structure.

Since $\rho_{X^*}(\varepsilon) < \frac{\omega(X)+\varepsilon+\varepsilon\omega(X)-1}{2}$ implies $\rho_{X^*}(\varepsilon) < \varepsilon$, from Theorem 11 and Theorem 28, X has normal structure.

Since $E(\varepsilon, X^*) < (1+\varepsilon)^2(1+\omega^2(X))$ implies $E(\varepsilon, X^*) < 2(1+\varepsilon)^2$, from Theorem 13 and Theorem 28, X has normal structure. $\square$

Theorem 30. *For a Banach space X, if $J(\varepsilon, X) < (1+\varepsilon)\omega(X)$ for some $0 < \varepsilon \leq 1$, then both X and X^* have normal structure.*

Proof. Since X is reflexive, we have $\omega(X) = \omega(X^*)$.

From Theorem 29, $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$ for some $0 < \varepsilon \leq 1$, implies X has normal structure, and from Theorem 18, $J(\varepsilon, X^*) < 1+\varepsilon\omega(X)$, for some $0 < \varepsilon \leq 1$, implies X^* has normal structure. Since $(1+\varepsilon)\omega(X) \leq 1+\varepsilon\omega(X)$, $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$ implies both X and X^* have normal structure. So, $J(\varepsilon, X) < (1+\varepsilon)\omega(X)$ for some $0 < \varepsilon \leq 1$, implies both X and X^* have normal structure. $\square$

When $\omega(X) \geq \frac{\varepsilon+\sqrt{\varepsilon^2+4}}{2(1+\varepsilon)}$, $(1+\varepsilon)\omega(X) \geq \frac{\varepsilon+\sqrt{\varepsilon^2+4}}{2}$, Theorem 30 improves Theorem 8 for $\omega(X) \geq \frac{\varepsilon+\sqrt{\varepsilon^2+4}}{2(1+\varepsilon)}$.

Theorem 31. *For a Banach space X, if $Q(\varepsilon, X) < (1+\varepsilon)(1+\omega(X))$ for some $0 < \varepsilon \leq 1$, then both X and X^* have normal structure.*

Proof. Since X is reflexive and $(1+\varepsilon)(1+\omega(X)) \leq 2(1+\varepsilon\omega(X))$, the proof is similar to the proof of Theorem 30. $\square$

When $\omega(X) \geq \frac{\sqrt{\varepsilon^2+4}-1}{1+\varepsilon}$, $(1+\varepsilon)(1+\omega(X)) \geq \varepsilon + \sqrt{\varepsilon^2+4}$, Theorem 31 improves Theorem 10 for $\omega(X) \geq \frac{\sqrt{\varepsilon^2+4}-1}{1+\varepsilon}$.

Since $Q(\varepsilon, X^*) = 2 + 2\rho_{X^*}(\epsilon)$. We have:

Theorem 32. *For a Banach space X, if $\rho_{X^*}(\varepsilon) < \frac{\varepsilon+\omega(X)+\varepsilon\omega(X)-1}{2}$ for some $0 < \varepsilon \leq 1$, then both X and X^* have normal structure.*

When $\omega(X) \geq \frac{\sqrt{\varepsilon^2+4}-1}{1+\varepsilon}$, $\frac{\varepsilon+\omega(X)+\varepsilon\omega(X)-1}{2} \geq \frac{\varepsilon-2+\sqrt{\varepsilon^2+4}}{2}$, Theorem 32 improves Theorem 12 for $\omega(X) \geq \frac{\sqrt{\varepsilon^2+4}-1}{1+\varepsilon}$.

From Theorem 13 and Theorem 21 and similar to the proof of Theorem 30, we have:

Theorem 33. *For a Banach space X, if $E(\varepsilon, X) < (1+\varepsilon)^2(1+\omega^2(X))$ for some $0 < \varepsilon \leq 1$, then both X and X^* have normal structure.*

The Theorem 33 is the same as Theorem 22.

We consider the uniform normal structure. To discuss this result, let us recall the concept of the "ultra"-technique.

Let $\mathcal{F}$ be a filter of an index set I, and let $\{x_i\}_{i\in I}$ be a subset in a Hausdorff topological space X, $\{x_i\}_{i\in I}$ is said to *converge to x with respect to* $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$, if for each neighborhood U of x, $\{i \in I : x_i \in U\} \in \mathcal{F}$. A filter $\mathcal{U}$ on I is called an *ultrafilter* if it is maximal with respect to the ordering of the set inclusion. An ultrafilter is called *trivial* if it is of the form $\{A : A \subseteq I, i_0 \in A\}$ for some $i_0 \in I$. We will use the fact that if $\mathcal{U}$ is an ultrafilter, then

(i) for any $A \subseteq I$, either $A \subseteq U$ or $I - A \subseteq U$;
(ii) if $\{x_i\}_{i\in I}$ has a cluster point x, then $\lim_{\mathcal{U}} x_i$ exists and equals to x.

Let $\{X_i\}_{i\in I}$ be a family of Banach spaces and let $l_\infty(I, X_i)$ denote the subspace of the product space equipped with the norm $\|(x_i)\| = \sup_{i\in I} \|x_i\| < \infty$.

Definition 34 ([2, 22])**.** Let $\mathcal{U}$ be an ultrafilter on I and let $N_U = \{(x_i) \in l_\infty(I, X_i) : \lim_{\mathcal{U}} \|x_i\| = 0\}$. The *ultra-product* of $\{X_i\}_{i\in I}$ is the quotient space $l_\infty(I, X_i)/N_{\mathcal{U}}$ equipped with the quotient norm.

We will use $(x_i)_{\mathcal{U}}$ to denote the element of the ultra-product. It follows from remark (ii) above, and the definition of quotient norm that

$$\|(x_i)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_i\|. \tag{1}$$

In the following we will restrict our index set I to be $\mathbb{N}$, the set of natural numbers, and let $X_i = X, i \in \mathbb{N}$ for some Banach space X. For an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we use $X_{\mathcal{U}}$ to denote the ultra-product. Note that if $\mathcal{U}$ is nontrivial, then X can be embedded into $X_{\mathcal{U}}$ isometrically.

Lemma 35 ([22])**.** *Suppose that $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and X is a Banach space. Then $(X^*)_{\mathcal{U}} \cong (X_{\mathcal{U}})^*$ if and only if X is super-reflexive*; *and in this case, the mapping J is defined by*

$$\langle (x_i)_{\mathcal{U}}, J((f_i)_{\mathcal{U}})\rangle = \lim_{\mathcal{U}} \langle x_i, f_i\rangle \quad \text{for all } (x_i)_{\mathcal{U}} \in X_{\mathcal{U}}$$

is the canonical isometric isomorphism from $(X^*)_{\mathcal{U}}$ onto $(X_{\mathcal{U}})^*$.

Theorem 36. *Let X be a super-reflexive Banach space. Then for any nontrivial ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and for all $n \in \mathbb{N}$ and $0 < \varepsilon \le 1$, we have $J(\varepsilon, X^*_{\mathcal{U}}) = J(\varepsilon, X^*)$, $Q(\varepsilon, X^*_{\mathcal{U}}) = Q(\varepsilon, X^*)$, $\rho_{X^*_{\mathcal{U}}}(\varepsilon) = \rho_{X^*}(\varepsilon)$, $E(\varepsilon, X^*_{\mathcal{U}}) = E(\varepsilon, X^*)$, and $\omega(X_{\mathcal{U}}) = \omega(X)$.*

Proof. The proof is similar to the proof of Theorem 6 in [8]. □

Lemma 37 ([15])**.** *If X is a super-reflexive Banach space, then X has uniform normal structure if and only if $X_{\mathcal{U}}$ has normal structure.*

Theorem 38. *For a Banach space X, if $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$, or $Q(\varepsilon, X^*) < (1+\varepsilon)(1+\omega(X))$, or $\rho_{X^*}(\varepsilon) < \frac{\omega(X)+\varepsilon+\varepsilon\omega(X)-1}{2}$, or $E(\varepsilon, X^*) < (1+\varepsilon)^2(1+\omega^2(X))$ for some $0 < \varepsilon \le 1$, then X has uniform normal structure.*

Proof. It follows directly from Theorems 29, Theorem 36 and Lemma 37. □

Finally, we have:

Theorem 39. *For a Banach space X, if $J(\varepsilon, X^*) < (1+\varepsilon)\omega(X)$, or $Q(\varepsilon, X^*) < (1+\varepsilon)(1+\omega(X))$, or $\rho_{X^*}(\varepsilon) < \frac{\varepsilon+\omega(X)+\varepsilon\omega(X)-1}{2}$, or $E(\varepsilon, X^*) < (1+\varepsilon)^2(1+\omega^2(X))$ for some $0 < \varepsilon \leq 1$, then both X and X^* have uniform normal structure.*

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